A NOTE ON THE GAPS BETWEEN CONSECUTIVE ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. Assuming the Riemann Hypothesis, we show that infinitely often consecutive non-trivial zeros of the Riemann zeta-function differ by at most 0.5155 times the average spacing and infinitely often they differ by at least 2.69 times the average spacing.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function. Assuming the Riemann Hypothesis, we denote the non-trivial zeros of $\zeta(s)$ as $\rho = \frac{1}{2} \pm i\gamma$ where $\gamma \in \mathbb{R}$. It is well known that, for $T \geq 10$,

$$N(T) := \sum_{0<\gamma\leq T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O\big(\log T\big).$$

Hence, if we let $\gamma \leq \gamma'$ denote consecutive ordinates of the zeros of $\zeta(s)$, we see that the average size of $\gamma' - \gamma$ is $2\pi/\log\gamma$. Normalizing, we let

$$\lambda := \limsup_{\gamma\geq 0} \frac{(\gamma'-\gamma)\log\gamma}{2\pi}$$

and

$$\mu := \liminf_{\gamma\geq 0} \frac{(\gamma'-\gamma)\log\gamma}{2\pi}$$

and we observe that $\mu \leq 1 \leq \lambda$. It is expected that there are arbitrarily large and arbitrarily small (normalized) gaps between consecutive zeros of the Riemann zeta-function on the critical line; in other words, that $\mu = 0$ and $\lambda = +\infty$. In this note, we prove the following theorem.

Theorem 1.1. *Assume the Riemann Hypothesis. Then $\lambda > 2.69$ and $\mu < 0.5155$.*

We briefly describe the history of the problem. Very little is known unconditionally; however, Selberg (unpublished, but announced in [11]) has shown that $\mu < 1 < \lambda$. Assuming the Riemann Hypothesis, numerous authors [2, 5, 6, 7, 9] have obtained explicit bounds for μ and λ. Theorem 1.1 improves the previously best known results under this assumption which were $\mu < 0.5172$ due to Conrey,

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Ghosh & Gonek [2] and $\lambda > 2.63$ due to R. R. Hall[1] [5]. Assuming the generalized Riemann Hypothesis for the zeros of Dirichlet L-functions, Conrey, Ghosh & Gonek [3] have shown that $\lambda > 2.68$. Their method can be modified (see [10] and [1]) to show that $\lambda > 3$.

Understanding the distribution of the zeros of the zeta-function is important for a number of reasons. One reason, in particular, is the connection between the spacing of the zeros of $\zeta(s)$ and the class number problem for imaginary quadratic fields. This is described by Conrey & Iwaniec in [4]; see also Montgomery & Weinberger [8]. Studying this connection led Montgomery [6] to investigate the pair correlation of the ordinates of the zeros of the zeta-function. He conjectured that, for any fixed $0 < \alpha < \beta$,

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ \frac{2\pi\alpha}{\log T}\leq \gamma'-\gamma \leq \frac{2\pi\beta}{\log T}}} 1 \ \sim \ N(T) \int_\alpha^\beta \Big(1 - \Big(\frac{\sin \pi u}{\pi u}\Big)^2\Big)\, du.$$

Here γ and γ' run over two distinct sets of ordinates of the non-trivial zeros of $\zeta(s)$. Clearly, Montgomery's conjecture implies that $\mu = 0$. Moreover, F. J. Dyson observed that the eigenvalues of large, random complex Hermitian or unitary matrices have the same pair correlation function. This observation (among other things) has led to a stronger conjecture that the zeros of the zeta-function should behave, asymptotically, like the eigenvalues of large random matrices from the Gaussian Unitary Ensemble. These ideas lead to the conjecture that $\lambda = +\infty$.

2. Montgomery & Odlyzko's method for exhibiting irregularity in the gaps between consecutive zeros of $\zeta(s)$

Throughout the remainder of this note, we assume the truth of the Riemann Hypothesis.

Let T be large and put $K = T(\log T)^{-2}$. Further, let

$$h(c) := c - \frac{\mathrm{Re}\,\left(\sum_{nk\leq K} a_k \overline{a_{nk}} g_c(n)\Lambda(n) n^{-1/2}\right)}{\sum_{k\leq K}|a_k|^2}$$

where

$$g_c(n) = \frac{2\sin\left(\pi c \frac{\log n}{\log T}\right)}{\pi \log n}$$

and $\Lambda(\cdot)$ is von Mangoldt's function defined by $\Lambda(n) = \log p$ if $n = p^k$ for a prime p and by $\Lambda(n) = 0$, otherwise. In [7], by an argument using the Guinand-Weil explicit formula for the zeros of $\zeta(s)$, Montgomery & Odlyzko show that if $h(c) < 1$ for some choice of $c > 0$ and a sequence $\{a_n\}$ then $\lambda \geq c$ and that if $h(c) > 1$ for a choice of $c > 0$ and a sequence $\{a_n\}$ then $\mu \leq c$. In particular, for any such choices of c and $\{a_n\}$, their method proves the existence of a pair of consecutive zeros of $\zeta(s)$ with ordinates $\gamma \leq \gamma'$ in the interval $[T/2, 2T]$ which satisfy $\gamma' - \gamma \geq \frac{2\pi c}{\log T}$ and $\gamma' - \gamma \leq \frac{2\pi c}{\log T}$, respectively.

[1]The results in Hall's paper are actually unconditional, but a lower bound for λ can only be obtained if the Riemann Hypothesis is assumed.

Conrey, Ghosh & Gonek [2] have given an alternative, and much simpler, way of viewing this problem.[2] Let

$$A(t) = \sum_{k \le K} a_k k^{-it}$$

be a Dirichlet polynomial and set

$$M_1 = \int_T^{2T} \big|A(t)\big|^2 \, dt \quad \text{and} \quad M_2(c) = \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{T \le \gamma \le 2T} \big|A(\gamma+\alpha)\big|^2 \, d\alpha.$$

Then, clearly, $M_2(c)$ is monotonically increasing and $M_2(\mu) \le M_1 \le M_2(\lambda)$. Therefore, if it can be shown that $M_2(c) < M_1$ for some choice of $A(t)$ and c, then $\lambda > c$. Similarly, if $M_2(c) > M_1$ for some choice of $A(t)$ and c, then $\mu < c$. Using standard techniques to estimate M_1 and $M_2(c)$, it can be shown that

$$M_2(c)/M_1 = h(c) + o(1).$$

Hence, this argument is seen to be equivalent to Montgomery & Odlyzko's method, described above. Moreover, we note that this formulation of the method suggests that we should choose a test function $A(t)$ which is small near the zeros of $\zeta(s)$ to exhibit large gaps between the zeros of the zeta-function and a test function $A(t)$ which is large near the zeros of $\zeta(s)$ to exhibit small gaps.

In [7], Montgomery & Odlyzko make the choices of

$$a_k = \frac{1}{\sqrt{k}} f\big(\tfrac{\log k}{\log K}\big) \quad \text{and} \quad a_k = \frac{\lambda(k)}{\sqrt{k}} f\big(\tfrac{\log k}{\log K}\big)$$

where f is a continuous function of bounded variation on $[0,1]$ normalized so that $\int_0^1 |f|^2 = 1$ and $\lambda(k)$, the Liouville function, equals $(-1)^{\Omega(k)}$; here, $\Omega(k)$ denotes the total number of primes dividing k. By optimizing over such functions f, the values $\mu < 0.5179$ and $\lambda > 1.97$ are obtained. The authors of [7] choose f to be a certain modified Bessel function and they mention this is close to an optimal choice.

In [2], Conrey, Ghosh & Gonek choose the coefficients

$$a_k = \frac{d_r(k)}{\sqrt{k}} \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}}$$

where $d_r(k)$ is a multiplicative function defined on integral powers of a prime p by

$$d_r(p^k) = \frac{\Gamma(k+r)}{\Gamma(r) k!}.$$

In this context, this becomes an optimization in the variable r. The choice $r = 1.1$ yields $\mu < 0.5172$ and the choice $r = 2.2$ yields $\lambda > 2.337$.

In order to prove Theorem 1.1, we combine the approaches of [7] and [2]. We choose the coefficients

$$a_k = \frac{d_r(k)}{\sqrt{k}} f\big(\tfrac{\log K/k}{\log K}\big) \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f\big(\tfrac{\log K/k}{\log K}\big)$$

for sufficiently smooth functions f. This variant allows us to optimize over both r and f, rather than over just r or f.

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[2] Mueller [9] was the first to observe that this method gave a lower bound for λ. In [2], it was noticed that the method also gave an upper bound for μ and was equivalent to the method in [7].

We now provide further insight into the choice of these coefficients. For simplicity, suppose f is a polynomial. Since, for Re $s>1$,

$$\sum_{k\geq 1}\frac{d_r(k)}{k^s}=\zeta(s)^r \quad \text{and} \quad \sum_{k\geq 1}\frac{\lambda(k)d_r(k)}{k^s}=\Big(\frac{\zeta(2s)}{\zeta(s)}\Big)^r,$$

with our choice of coefficients we see that the test function $A(t)$ approximates $\zeta(\frac12+it)^r$ and $\zeta(1+2it)^r/\zeta(\frac12+it)^r$, respectively, and should have the desired effect of making $A(t)$ small (respectively large) near the zeros of $\zeta(s)$. Moreover, when we multiply $d_r(k)$ by $f\big(\frac{\log K/k}{\log K}\big)$ then $A(t)$ behaves like a linear combination of $\zeta(\frac12+it)^r$ and its derivatives and an analogous comment applies to the other case. The presence of the function f leads to improved numerical results for bounds for μ and λ.

3. Large Gaps: a lower bound for λ

In this section, we establish a lower bound for λ by evaluating $h(c)$ with the coefficients

$$a_k=\frac{d_r(k)}{\sqrt{k}}f\big(\tfrac{\log K/k}{\log K}\big)$$

where f is a continuous, real-valued function of bounded variation on $L^2[0,1]$. In what follows, we assume that $r\geq 1$ so that $d_r(mn)\leq d_r(m)d_r(n)$ for $m,n\in\mathbb{N}$. It is well known that, for fixed $r\geq 1$,

$$\sum_{k\leq x}\frac{d_r(k)^2}{k}=A_r(\log x)^{r^2}+O\big((\log T)^{r^2-1}\big) \tag{3.1}$$

uniformly for $x\leq T$; here A_r is a certain arithmetical constant (the exact value is not important in our argument). By partial summation, recalling that $K=T(\log T)^{-2}$, we find that the denominator in the ratio of sums in the definition of $h(c)$ is

$$\begin{aligned}\sum_{k\leq K}|a_k|^2&=\int_{1^-}^{K}f\big(\tfrac{\log K/x}{\log K}\big)^2 d\Big(\sum_{k\leq x}\frac{d_r(k)^2}{k}\Big)\\&=A_r r^2\int_1^K f\big(\tfrac{\log K/x}{\log K}\big)^2(\log x)^{r^2-1}\frac{dx}{x}+O_{f,r}\big((\log T)^{r^2-1}\big)\end{aligned}$$

by (3.1). By the variable change $u=1-\frac{\log x}{\log K}$

$$\begin{aligned}\sum_{k\leq K}|a_k|^2&=A_r r^2(\log K)^{r^2}\int_0^1(1-u)^{r^2-1}f(u)^2\,du+O_{f,r}\big((\log T)^{r^2-1}\big)\\&=A_r r^2(\log T)^{r^2}\int_0^1(1-u)^{r^2-1}f(u)^2\,du+O_{f,r,\varepsilon}\big((\log T)^{r^2-1+\varepsilon}\big)\end{aligned}$$

where $\varepsilon>0$ is arbitrary.

We now evaluate the numerator in the ratio of sums in the definition of $h(c)$. If we let

$$N(c):=\sum_{nk\leq K}a_k\overline{a_{nk}}g_c(n)\Lambda(n)n^{-1/2}$$

then a straightforward argument shows that

$$\begin{aligned} N(c) &= \frac{2}{\pi} \sum_{nk \le K} \frac{d_r(k) d_r(kn) \Lambda(n)}{kn \log n} f\big(\tfrac{\log K/k}{\log K}\big) f\big(\tfrac{\log K/nk}{\log K}\big) \sin\big(\pi c \tfrac{\log n}{\log T}\big) \\ &= \frac{2}{\pi} \sum_{pk \le K} \frac{d_r(k) d_r(kp)}{kp} f\big(\tfrac{\log K/k}{\log K}\big) f\big(\tfrac{\log K/pk}{\log K}\big) \sin\big(\pi c \tfrac{\log p}{\log T}\big) + O_{f,r}\big((\log T)^{r^2-1}\big) \\ &= \frac{2r}{\pi} \sum_{p \le K} \frac{\sin\big(\pi c \frac{\log p}{\log T}\big)}{p} \sum_{k \le K/p} \frac{d_r(k)^2}{k} f\big(\tfrac{\log K/k}{\log K}\big) f\big(\tfrac{\log K/pk}{\log K}\big) + O_{f,r}\big((\log T)^{r^2-1}\big). \end{aligned}$$

By Stieltjes integration and a variable change, the inner sum in the main term of the last expression for $N(c)$ is

$$\begin{aligned} &\int_{1^-}^{K/p} f\big(\tfrac{\log K/x}{\log K}\big) f\big(\tfrac{\log K/px}{\log K}\big) d\Big(\sum_{k \le x} \frac{d_r(k)^2}{k}\Big) \\ &\quad = A_r r^2 \int_1^{K/p} f\big(\tfrac{\log K/x}{\log K}\big) f\big(\tfrac{\log K/px}{\log K}\big) (\log x)^{r^2-1} \frac{dx}{x} + O_{f,r}\big((\log T)^{r^2-1}\big) \\ &\quad = A_r r^2 (\log K)^{r^2} \int_{\frac{\log p}{\log K}}^1 (1-u)^{r^2-1} f(u) f\big(u - \tfrac{\log p}{\log K}\big)\, du + O_{f,r}\big((\log T)^{r^2-1}\big). \end{aligned}$$

By combining the above estimates and interchanging the order of summation and integration, we conclude that $N(c) = M(c) + O_{f,r}\big((\log T)^{r^2-1}\big)$ where

$$\begin{aligned} M(c) &= \frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_{\frac{\log 2}{\log K}}^1 (1-u)^{r^2-1} f(u) \sum_{2 \le p \le K^u} \frac{\sin\big(\pi c \frac{\log p}{\log T}\big)}{p} f\big(u - \tfrac{\log p}{\log K}\big)\, du \\ &= \frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f(u) \sum_{2 \le p \le K^u} \frac{\sin\big(\pi c \frac{\log p}{\log T}\big)}{p} f\big(u - \tfrac{\log p}{\log K}\big)\, du \\ &\qquad + O_{f,r,\varepsilon}\big((\log T)^{r^2-1+\varepsilon}\big). \end{aligned}$$

By the prime number theorem with remainder term, it follows that

$$\sum_{2 \le p \le K^u} \frac{\sin\big(\pi c \frac{\log p}{\log T}\big)}{p} f\big(u - \tfrac{\log p}{\log K}\big) = \int_2^{K^u} \frac{\sin\big(\pi c \frac{\log x}{\log T}\big)}{x \log x} f\big(u - \tfrac{\log x}{\log K}\big)\, dx + O_{f,r}\Big(\frac{1}{\log T}\Big).$$

By the variable change $v = \frac{\log x}{\log K}$, the integral is

$$\int_{\frac{\log 2}{\log K}}^u \frac{\sin\big(\pi c v \frac{\log K}{\log T}\big)}{v} f(u-v)\, dv = \int_0^u \frac{\sin(\pi c v)}{v} f(u-v) dv + O_{f,r,\varepsilon}\big((\log T)^{-1+\varepsilon}\big).$$

Hence,

$$\begin{aligned} M(c) = &\frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi c v)}{v} f(u-v)\, dv\, du \\ &+ O_{f,r,\varepsilon}\big((\log T)^{r^2-1+\varepsilon}\big). \end{aligned}$$

Consequently, we find that

$$h(c) = c - \frac{2r}{\pi}\frac{\int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi c v)}{v} f(u-v)\,dv\,du}{\int_0^1 (1-u)^{r^2-1} f(u)^2\,du} + O_{f,r,\varepsilon}\big((\log T)^{-1+\varepsilon}\big). \tag{3.2}$$

Choosing $r = 3.1$ and $f(x) = 1+10x+39x^2$, we obtain (by a numerical calculation) that $h(2.69) < 1$ when T is sufficiently large. This provides the lower bound for λ in Theorem 1.1.

4. Small Gaps: an upper bound for μ

Since $\lambda(n)^2 = 1$ and $\lambda(pn) = -\lambda(n)$ for every $n \in \mathbb{N}$ and every prime p, we can evaluate $h(c)$ using the coefficients

$$a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f\big(\tfrac{\log K/k}{\log K}\big)$$

in a similar manner to the calculations in the previous section. Here, as above, $f \in L^2[0,1]$ is a real-valued, continuous function of bounded variation. With this choice of coefficients, we obtain that

$$h(c) = c + \frac{2r}{\pi}\frac{\int_0^1 (1-u)^{r^2-1} f(u) \int_0^u \frac{\sin(\pi c v)}{v} f(u-v)\,dv\,du}{\int_0^1 (1-u)^{r^2-1} f(u)^2\,du} + O_{f,r,\varepsilon}\big((\log T)^{-1+\varepsilon}\big). \tag{4.1}$$

Choosing $r = 1.23$ and $f(x) = 1 + 0.99x - 0.42x^2$, a numerical calculation implies that $h(0.5155) > 1$. This provides the upper bound for μ in Theorem 1.1.

5. Some Concluding Remarks

Theorem 1.1 offers the best known bounds for λ and μ assuming the Riemann Hypothesis; however, we are still far from proving the conjectured values of $\mu = 0$ and $\lambda = \infty$. In fact, it known that this is not attainable using Montgomery and Odlyzko's method. Specifically, in [2], it is shown that $h(c) < 1$ if $c < \frac{1}{2}$ and $h(c) > 1$ if $c \geq 6.2$. It would be interesting to better understand the limitations of this method and, in particular, if it can be used to show that $\mu \leq \frac{1}{2}$.

We have not been able to prove that our bounds for λ and μ in Theorem 1.1 are the optimal bounds for our choice of coefficients

$$a_k = \frac{d_r(k)}{\sqrt{k}} f\big(\tfrac{\log K/k}{\log K}\big) \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f\big(\tfrac{\log K/k}{\log K}\big).$$

In the special case of $r = 1$, this optimization problem has been solved (in terms of prolate spheroidal wave functions). When $r \neq 1$, the analogous optimization problem seems considerably more difficult. Instead of trying to solve it explicitly, we have instead chosen f to be a polynomial of low degree (≤ 6). Using Mathematica, we numerically optimize (3.2) and (4.1) over the coefficients for each choice of c and r.

Our numerical calculations seem to suggest that degree 2 polynomials work well; it does not seem like there is much to gain by increasing the degree of f. To demonstrate this phenomenon, we observe that one can recover the bounds for λ and μ, in the case of $r = 1$, derived in [7] using degree 2 polynomials in place of prolate spheroidal wave functions (which are much more difficult to compute numerically). In this case, after some rearranging and a change of variables, our above calculations imply that

$$h(c) = c \pm \frac{\int_0^1 f(u) \int_0^1 f(v) \frac{\sin(\pi c(u-v))}{\pi(u-v)} \, dv \, du}{\int_0^1 f(u)^2 \, du} + o(1).$$

Letting $f(x) = 1 + 0.46526x - 0.46526x^2$, a numerical calculation shows that $h(0.5179) > 1$ and if we let $f(x) = 1 + 17.9426x - 17.9426x^2$, then it can be shown that $h(1.97) < 1$. These are essentially the (optimal) values obtained by Mongomery and Odlyzko in [7] when $r = 1$.

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