LOWER BOUNDS FOR MOMENTS OF $\zeta'(\rho)$

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Abstract. Assuming the Riemann Hypothesis, we establish lower bounds for moments of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$. Our proof is based upon a recent method of Rudnick and Soundararajan that provides analogous bounds for moments of L-functions at the central point, averaged over families.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. In this article we are interested in obtaining lower bounds for moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \tag{1}$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, we let the function

$$N(T) = \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{2}$$

denote the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity.

Independently, Gonek [3] and Hejhal [5] have conjectured that $J_k(T) \asymp (\log T)^{k(k+2)}$ for each $k \in \mathbb{R}$. By modeling the Riemann zeta-function and its derivative using characteristic polynomials of random matrices, Hughes, Keating, and O'Connell [6] have refined this conjecture to state that $J_k(T) \sim C_k(\log T)^{k(k+2)}$ for a precise constant C_k when $k \in \mathbb{C}$ and $\mathfrak{Re}k > -3/2$. However, we no longer believe this conjecture to be true for $\mathfrak{Re}k < -3/2$. This is since we expect there exist infinitely many zeros ρ such that $|\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon}$ for each $\varepsilon > 0$.

Results of the sort suggested by these conjectures are only known for a few small values of k. See, for instance, the results of Gonek [1] for the case $k = 1$ and Ng [8] for the case $k = 2$. Also, Gonek [3] obtained a lower bound in the case $k = -1$. Our main result is to obtain a lower bound for $J_k(T)$ for each $k \in \mathbb{N}$ of the order of magnitude that is suggested by these conjectures.

Theorem 1. *Assume the Riemann Hypothesis and let $k \in \mathbb{N}$. Then for sufficiently large T we have*

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \gg_k (\log T)^{k(k+2)}.$$

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Under the assumption of the Riemann Hypothesis, Milinovich [7] has recently shown that $J_k(T) \ll_{k,\varepsilon} (\log T)^{k(k+2)+\varepsilon}$ for $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. When combined with Theorem 1, this result lends strong support for the conjecture of Gonek and Hejhal for k a positive integer.

Theorem 1 can be used to exhibit large values of $\zeta'(\rho)$. For example, as an immediate corollary we have the following result.

Corollary 1.1. *Assume the Riemann Hypothesis and let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then for each $A > 0$ the inequality*

$$\big|\zeta'(\rho)\big| \geq (\log|\gamma|)^A \tag{3}$$

is satisfied infinitely often.

This result was previously proven by Ng [10] by an application of Soundararajan's resonance method [13]. The present proof is simpler and provides many more zeros ρ such that (3) is true. On the other hand, the resonance method is capable of detecting much larger values of $\zeta'(\rho)$ assuming a very weak form of the generalized Riemann hypothesis.

Our proof of Theorem 1 relies on combining a method of Rudnick and Soundararajan [11, 12] with a mean-value theorem of Ng (our Lemma 2) and a well-known lemma of Gonek (our Lemma 3). It is likely that our proof can be adapted to prove a lower bound for $J_k(T)$ of the conjectured order of magnitude for all rational k (with $k \geq 1$) in a manner analogous to that suggested in [11].

Let $k \in \mathbb{N}$ and define, for $\xi \geq 1$, the function $\mathcal{A}_\xi(s) = \sum_{n \leq \xi} n^{-s}$. Assuming the Riemann Hypothesis, we will estimate

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\overline{\mathcal{A}_\xi(\rho)}^k \quad \text{and} \quad \Sigma_2 = \sum_{0<\gamma\leq T} \big|\mathcal{A}_\xi(\rho)\big|^{2k}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Hölder's inequality implies that

$$\sum_{0<\gamma\leq T} \big|\zeta'(\rho)\big|^{2k} \geq \frac{\big|\Sigma_1\big|^{2k}}{\big(\Sigma_2\big)^{2k-1}},$$

and so we see that Theorem 1 will follow from the estimates

$$\Sigma_1 \gg T(\log T)^{k^2+2} \quad \text{and} \quad \Sigma_2 \ll T(\log T)^{k^2+1}. \tag{4}$$

It is convenient to express Σ_1 and Σ_2 slightly differently. Assuming the Riemann Hypothesis, $1-\rho = \bar{\rho}$ for any non-trivial zero ρ of $\zeta(s)$. Thus, $\overline{\mathcal{A}_\xi(\rho)} = \mathcal{A}_\xi(1-\rho)$. This allows us to re-write the sums in (1) as

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\mathcal{A}_\xi(1-\rho)^k \quad \text{and} \quad \Sigma_2 = \sum_{0<\gamma\leq T} \mathcal{A}_\xi(\rho)^k\mathcal{A}_\xi(1-\rho)^k. \tag{5}$$

It is with these representations of Σ_1 and Σ_2 that we establish the bounds in (4).

2. Some preliminary estimates

For each real number $\xi \geq 1$ and each $k \in \mathbb{N}$, we define the arithmetic sequence of real numbers $\tau_k(n;\xi)$ by

$$\sum_{n \leq \xi^k} \frac{\tau_k(n;\xi)}{n^s} = \Big(\sum_{n \leq \xi} \frac{1}{n^s} \Big)^k = \mathcal{A}_\xi(s)^k. \tag{6}$$

The function $\tau_k(n;\xi)$ is a truncated approximation to the arithmetic function $\tau_k(n)$ (the k-th iterated divisor function) which is defined by

$$\zeta^k(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n^s} \Big)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} \tag{7}$$

for $\mathfrak{Res} > 1$. We require a few estimates for sums involving the functions $\tau_k(n)$ and $\tau_k(n;\xi)$ in order to establish the bounds for Σ_1 and Σ_2 in (4).

We use repeatedly that, for $x \geq 3$ and $k, \ell \in \mathbb{N}$,

$$\sum_{n \leq x} \frac{\tau_k(n)\tau_\ell(n)}{n} \asymp_{k,\ell} (\log x)^{k\ell} \tag{8}$$

where the implied constants depend on k and ℓ. These bounds are well-known.

From (6) and (7) we notice that $\tau_k(n;\xi)$ is non-negative and $\tau_k(n;\xi) \leq \tau_k(n)$ with equality holding when $n \leq \xi$. In particular, choosing $k = \ell$ in (8) we find that, for $\xi \geq 3$,

$$(\log \xi)^{k^2} \ll_k \sum_{n \leq \xi} \frac{\tau_k(n)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n;\xi)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n)^2}{n} \ll_k (\log \xi)^{k^2}. \tag{9}$$

3. A Lower Bound for Σ_1

In order to establish a lower bound for Σ_1, we require a mean-value estimate for sums of the form

$$S(X,Y;T) = \sum_{0 < \gamma \leq T} \zeta'(\rho) X(\rho) Y(1-\rho)$$

where

$$X(s) = \sum_{n \leq N} \frac{x_n}{n^s} \qquad \text{and} \qquad Y(s) = \sum_{n \leq N} \frac{y_n}{n^s}$$

are Dirichlet polynomials. For $X(s)$ and $Y(s)$ satisfying certain reasonable conditions, a general formula for $S(X,Y;T)$ has been established by the second author [9]. Before stating the formula, we first introduce some notation. For T large, we let $\mathscr{L} = \log \frac{T}{2\pi}$ and $N = T^\vartheta$ for some fixed $\vartheta \geq 0$. The functions $\mu(\cdot)$ and $\Lambda(\cdot)$ are used to denote the usual arithmetic functions of Möbius and von Mangoldt. Also, we define the arithmetic function $\Lambda_2(\cdot)$ by $\Lambda_2(n) = (\mu * \log^2)(n)$ for each $n \in \mathbb{N}$.

Lemma 2. *Let x_n and y_n satisfy $|x_n|, |y_n| \ll \tau_\ell(n)$ for some $\ell \in \mathbb{N}$ and assume that $0 < \vartheta < 1/2$. Then for any $A > 0$, any $\varepsilon > 0$, and sufficiently large T we have*

$$\begin{aligned} S(X,Y;T) &= \frac{T}{2\pi}\sum_{mn\le N}\frac{x_m y_{mn}}{mn}\Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L})\log n + (\Lambda * \log)(n)\Big) \\ &\quad -\frac{T}{4\pi}\sum_{mn\le N}\frac{y_m x_{mn}}{mn}\mathcal{Q}_2(\mathscr{L}-\log n) + \frac{T}{2\pi}\sum_{\substack{a,b\le N\\(a,b)=1}}\frac{r(a;b)}{ab}\sum_{g\le\min\left(\frac{N}{a},\frac{N}{b}\right)}\frac{y_{ag}x_{bg}}{g} \\ &\quad + O_A\big(T(\log T)^{-A} + T^{3/4+\vartheta/2+\varepsilon}\big) \end{aligned}$$

where $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{Q}_2$ are monic polynomials of degrees 1,2, and 2, respectively, and for $a, b \in \mathbb{N}$ the function $r(a;b)$ satisfies the bound

$$|r(a;b)| \ll \Lambda_2(a) + (\log T)\Lambda(a)\ . \tag{10}$$

Proof. This is a special case of Theorem 1.3 of Ng [9]. □

Letting $\xi = T^{1/(4k)}$, we find that the choices $X(s) = \mathcal{A}_\xi(s)^{k-1}$ and $Y(s) = \mathcal{A}_\xi(s)^k$ satisfy the conditions of Lemma 2 with $\vartheta = 1/4$, $N = \xi^k$, $x_n = \tau_{k-1}(n;\xi)$, and $y_n = \tau_k(n;\xi)$. Consequently, for this choice of ξ,

$$\begin{aligned} \Sigma_1 &= \frac{T}{2\pi}\sum_{\substack{mn\le\xi^k\\ m\le\xi^{k-1}}}\frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn}\Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L})\log n + (\Lambda*\log)(n)\Big) \\ &\quad -\frac{T}{4\pi}\sum_{mn\le\xi^{k-1}}\frac{\tau_k(m;\xi)\tau_{k-1}(mn;\xi)}{mn}\mathcal{Q}_2(\mathscr{L}-\log n) \\ &\quad +\frac{T}{2\pi}\sum_{\substack{a,b\le\xi^k\\(a,b)=1}}\frac{r(a;b)}{ab}\sum_{g\le\min\left(\frac{N}{a},\frac{N}{b}\right)}\frac{\tau_k(ag;\xi)\tau_{k-1}(bg;\xi)}{g} + O(T) \\ &= \mathcal{S}_{11} + \mathcal{S}_{12} + \mathcal{S}_{13} + O(T), \end{aligned}$$

say. To estimate $\mathcal{S}_{11}$, notice that, for T sufficiently large, $n \le \xi^k = T^{1/4}$ implies that

$$\Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L})\log n + (\Lambda*\log)(n)\Big) \gg \mathscr{L}^2$$

and moreover, by (9),

$$\sum_{\substack{mn\le\xi^k\\ m\le\xi^{k-1}}}\frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \ge \sum_{n\le\xi^k}\frac{\tau_k(n;\xi)^2}{n} \gg (\log T)^{k^2}.$$

Thus, $\mathcal{S}_{11} \gg T(\log T)^{k^2+2}$. Since $\mathcal{Q}_2(\mathscr{L}-\log n) \ll \mathscr{L}^2$, we can bound $\mathcal{S}_{12}$ by using the inequalities $\tau_k(n;\xi) \le \tau_k(n)$ and $\tau_k(mn) \le \tau_k(m)\tau_k(n)$. In particular, by twice using (8), we find that

$$\begin{aligned} \mathcal{S}_{12} &\ll T\mathscr{L}^2\sum_{mn\le\xi^k}\frac{\tau_k(m)\tau_{k-1}(m)\tau_k(n)}{mn} \le T\mathscr{L}^2\left(\sum_{m\le T}\frac{\tau_k(m)\tau_{k-1}(m)}{m}\right)\left(\sum_{n\le T}\frac{\tau_{k-1}(n)}{n}\right) \\ &\ll T(\log T)^{2+k(k-1)+k-1} \ll T(\log T)^{k^2+1}. \end{aligned}$$

It remains to consider the contribution from $\mathcal{S}_{13}$. Again using the inequalities $\tau_k(n;\xi) \le \tau_k(n)$ and $\tau_k(mn) \le \tau_k(m)\tau_k(n)$ along with (10), it follows that $\mathcal{S}_{13}$ is bounded by

$$\begin{aligned}\sum_{a,b\le\xi^k} \frac{(\Lambda_2(a)+(\log T)\Lambda(a))}{ab} \sum_{g\le\xi^k} \frac{\tau_k(a)\tau_k(g)\tau_{k-1}(b)\tau_{k-1}(g)}{g} \\ \ll \sum_{a\le T} \frac{(\Lambda_2(a)+(\log T)\Lambda(a))\tau_k(a)}{a} \sum_{b\le T} \frac{\tau_{k-1}(b)}{b} \sum_{g\le T} \frac{\tau_k(g)\tau_{k-1}(g)}{g} \\ \ll (\log T)^{2+(k-1)+k(k-1)} = (\log T)^{k^2+1}.\end{aligned}$$

Combining this with our estimates for $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$, we conclude that $\Sigma_1 \gg T(\log T)^{k^2+2}$.

4. An Upper Bound for Σ_2

Assuming the Riemann Hypothesis, we interchange the sums in (5) and find that

$$\Sigma_2 = N(T) \sum_{n\le\xi^k} \frac{\tau_k(n;\xi)^2}{n} + 2\mathfrak{Re} \sum_{m\le\xi^k} \sum_{m<n\le\xi^k} \frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n} \sum_{0<\gamma\le T} \left(\frac{n}{m}\right)^{\rho} \tag{11}$$

where $N(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to a height T. Recalling that $\xi = T^{1/(4k)}$ and using (2) and (9), it follows that

$$N(T) \sum_{n\le\xi^k} \frac{\tau_k(n;\xi)^2}{n} \ll T(\log T)^{k^2+1}. \tag{12}$$

In order to bound the second sum on the right-hand side of (11), we require the following version of the Landau-Gonek explicit formula.

Lemma 3. *Let $x, T > 1$ and let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then*

$$\begin{aligned}\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O\big(x\log(2xT)\log\log(3x)\big) \\ + O\Big(\log x \min\Big(T, \frac{x}{\langle x\rangle}\Big)\Big) + O\Big(\log(2T)\min\Big(T, \frac{1}{\log x}\Big)\Big)\end{aligned}$$

where $\langle x\rangle$ denotes the distance from x to the closest prime power other than x itself and $\Lambda(x) = \log p$ if x is a positive integral power of a prime p and $\Lambda(x) = 0$ otherwise.

Proof. This is a result of Gonek [2, 4]. □

Applying the lemma, we find that

$$\begin{aligned}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n}\sum_{0<\gamma\le T}\Big(\frac{n}{m}\Big)^{\rho} &= -\frac{T}{2\pi}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)\Lambda(\frac{n}{m})}{n}\\ &\quad+O\left(\mathscr{L}\log\mathscr{L}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\right)\\ &\quad+O\left(\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\frac{\log\frac{n}{m}}{\langle\frac{n}{m}\rangle}\right)\\ &\quad+O\left(\log T\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n\log\frac{n}{m}}\right)\\ &= \mathcal{S}_{21}+\mathcal{S}_{22}+\mathcal{S}_{23}+\mathcal{S}_{24},\end{aligned}$$

say. Since we only require an upper bound for Σ_2 (which, by definition, is clearly positive), we can ignore the contribution from $\mathcal{S}_{21}$ because all the non-zero terms in the sum are negative. In what follows, we use ε to denote a small positive constant which may be different at each occurrence. To estimate $\mathcal{S}_{22}$, we note that $\tau_k(n;\xi)\le\tau_k(n)\ll_\varepsilon n^\varepsilon$ which implies $\mathcal{S}_{22}\ll T^{1/4+\varepsilon}$. Turning to $\mathcal{S}_{23}$, we write n as $qm+\ell$ with $-\frac{m}{2}<\ell\le\frac{m}{2}$ and find that

$$\mathcal{S}_{23}\ll T^{\epsilon}\sum_{m\le\xi^k}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}\ \sum_{-\frac{m}{2}<\ell\le\frac{m}{2}}\frac{1}{\langle q+\frac{\ell}{m}\rangle}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Notice that $\langle q+\frac{\ell}{m}\rangle=\frac{|\ell|}{m}$ if q is a prime power and $\ell\ne0$, otherwise $\langle q+\frac{\ell}{m}\rangle$ is $\ge\frac{1}{2}$. Hence,

$$\begin{aligned}\mathcal{S}_{23}&\ll T^{\varepsilon}\Big(\sum_{m\le\xi^k}\frac{1}{m}\sum_{\substack{q\le\lfloor\frac{\xi^k}{m}\rfloor+1\\ \Lambda(q)\ne0}}\ \sum_{1\le\ell\le\frac{m}{2}}\frac{m}{\ell}+\sum_{m\le\xi^k}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}\ \sum_{1\le\ell\le\frac{m}{2}}1\Big)\\ &\ll T^{\varepsilon}\Big(\sum_{m\le\xi^k}\ \sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}1\Big)\ll T^{1/4+\epsilon}\ .\end{aligned}$$

It remains to consider $\mathcal{S}_{24}$. For integers $1\le m<n\le\xi^k$, let $n=m+\ell$. Then

$$\log\frac{n}{m}=-\log\big(1-\frac{\ell}{m}\big)>\frac{\ell}{m}.$$

Consequently,

$$\mathcal{S}_{24}\ll T^{\epsilon}\sum_{m\le\xi^k}\ \sum_{1\le\ell\le\xi^k}\frac{1}{(m+\ell)\frac{\ell}{m}}\ll T^{\epsilon}\xi^k=T^{1/4+\epsilon}. \tag{13}$$

Combining (12) with our estimates for $\mathcal{S}_{22}$, $\mathcal{S}_{23}$, and $\mathcal{S}_{24}$ we deduce that $\Sigma_2\ll T(\log T)^{k^2+1}$ which, when combined with our estimate for Σ_1, completes the proof of Theorem 1.

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