The distribution of the summatory function of the Möbius function

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Abstract

Let the summatory function of the Möbius function be denoted $M(x)$. We deduce in this article conditional results concerning $M(x)$ assuming the Riemann Hypothesis and a conjecture of Gonek and Hejhal on the negative moments of the Riemann zeta function. The main results shown are that the weak Mertens conjecture and the existence of a limiting distribution of $e^{-y/2}M(e^y)$ are consequences of the aforementioned conjectures. By probabilistic techniques, we present an argument that suggests $M(x)$ grows as large positive and large negative as a constant times $\pm\sqrt{x}(\log\log\log x)^{\frac{5}{4}}$ infinitely often, thus providing evidence for an unpublished conjecture of Gonek's.

1 Introduction

The Möbius function is defined for positive integers n by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is squarefree and } n = p_1 \dots p_k \end{cases} . \tag{1}$$

Its summatory function $M(x) = \sum_{n\leq x} \mu(n)$ is closely related to the reciprocal of the Riemann zeta function. This connection may be observed by the identities

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s\int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx$$

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valid for $\mathrm{Re}(s) > 1$ and

$$M(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\zeta(s)}\ ds \tag{2}$$

where $c > 1$ and $x \notin \mathbb{Z}$. In the theorems of this article, we assume the truth of the Riemann hypothesis (RH) which asserts that all non-real zeros of $\zeta(s)$ take the form $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. At times, we also assume that all zeros of the zeta function are simple. It is widely expected that all zeros of the zeta function are simple. Currently, the best unconditional result is that at least two-fifths of the zeros are simple [3]. In light of (2), sums of the form

$$J_{-k}(T) = \sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^{2k}}$$

where $k \in \mathbb{R}$ are important in obtaining information concerning $M(x)$. From different points of view Gonek [7] and Hejhal [11] independently conjectured that

$$J_{-k}(T) \asymp T(\log T)^{(k-1)^2}. \tag{3}$$

Gonek studied Dirichlet polynomial approximations of these moments, whereas Hejhal studied the value distribution of $\log \zeta'(\rho)$ employing ideas of Selberg's. Henceforth, the former assumption (3) will be referred to as the Gonek-Hejhal conjecture. For $k = 0$ we have $J_0(T) = N(T)$ where $N(T)$ is the number of zeros in the box with vertices $0, 1, 1 + iT$, and iT. Von Mangoldt (see [5] pp. 97-100) proved that

$$J_0(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)\ . \tag{4}$$

For $k = 1$ Gonek [8] conjectured the asymptotic formula

$$J_{-1}(T) \sim \frac{3}{\pi^3} T\ . \tag{5}$$

Moreover, he proved that $J_{-1}(T) \gg T$ (see [7]) subject to RH and all zeros of the Riemann zeta function are simple. Recently, Hughes et al. [13] using random matrix model techniques conjectured that

$$\sum_{0<\gamma\leq T} |\zeta'(\rho)|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a_k \frac{T}{2\pi} \left(\log \frac{T}{2\pi}\right)^{(k+1)^2} \tag{6}$$

for $k > -\frac{3}{2}$ where

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}\right)$$

and G is Barnes' function defined by

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{1}{2}(z^2 + \gamma z^2 + z)\right) \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/2n}\right)$$

where γ denotes Euler's constant. One should note that in the above definition of a_k, one must take an appropriate limit if $k = 0$ or $k = -1$. Furthermore, one may check that $G(1) = 1$ and $a_{-1} = \frac{6}{\pi^2}$ and hence (6) implies (5) and moreover it agrees with (4).
One notes that Gonek [8] arrives at conjecture (5) by pursuing ideas of Montgomery's concerning the zero spacings (pair-correlation) of the zeta function. On the other hand, the random matrix technique originated with the work of Keating and Snaith [17]. Their idea was to model the Riemann zeta function by the characteristic polynomial of a large random unitary matrix. They computed moments of these characteristic polynomials averaged over the group of unitary matrices. These moments are much simpler to evaluate since they may be transformed into the well-studied Selberg integral. Following the work of Keating and Snaith, other authors have used this analogy to speculate on the exact nature of certain families of L-functions. This analogy has been viewed as rather fruitful, since to date it has always produced conjectures that agree with known theorems.

In this article we deduce results about $M(x)$ assuming the Riemann Hypothesis and the conjectural bound

$$J_{-1}(T) = \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T\ . \tag{7}$$

By making assumption (7), we implicitly assume that all zeros are simple. If there were a multiple zero of $\zeta(s)$, $J_{-1}(T)$ would be undefined for sufficiently large T and (7) would fail to make sense. For a long time, number theorists were interested in $M(x)$ as RH was a consequence of the famous Mertens conjecture which states that

$$|M(x)| \leq x^{\frac{1}{2}} \text{ for } x \geq 1.$$

For an excellent historical account of work on this problem see [20]. An averaged version of this conjecture is the weak Mertens conjecture which asserts that

$$\int_2^X \left(\frac{M(x)}{x}\right)^2 dx \ll \log X \ . \tag{8}$$

The weak Mertens conjecture implies RH, all zeros of $\zeta(s)$ are simple, and that $\sum_{\gamma>0} \frac{1}{|\rho\zeta'(\rho)|^2}$ converges. These consequences are proven in Titchmarsh [23] pp. 376-380. Not surprisingly, the Mertens conjecture was disproven by Odlyzko and te Riele [20] as they showed that

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \text{ and } \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.06 \ .$$

However, they did not actually provide a specific counterexample to (12). In fact, the Mertens conjecture was put in serious doubt many years earlier when Ingham [15] proved

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} = -\infty \text{ and } \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} = \infty$$

assuming the following conjecture:

Linear independence conjecture (LI) Assume $\zeta(s)$ satisfies the Riemann Hypothesis. If its zeros are written as $\frac{1}{2} + i\gamma$, then the positive imaginary ordinates of the zeros are linearly independent over $\mathbb{Q}$.

Currently there is very little numerical or theoretical evidence supporting this conjecture. However, it is considered rather unlikely that the imaginary ordinates of the zeros of the zeta function satisfy any linear relations. The linear independence conjecture has been used in the past to get a handle on some very difficult problems in number theory (see [15], [18], [22]). For some modest numerical computations see [1]. Despite the above results, we would like to have a better understanding of what the upper and lower bounds of $x^{-\frac{1}{2}}M(x)$ should be. The true order of $M(x)$ is something of a mystery. In fact, Odlyzko and te Riele [20] p. 3 comment that "No good conjectures about the rate of growth of $M(x)$ are known." Motivated by this comment, we attempt to give an explanation of the true behaviour of $M(x)$ assuming reasonable conjectures about the zeta function.

We briefly mention some notation used throughout this article. We will denote a sequence of effectively computable positive constants as $c_1, c_2, c_3, \ldots$.

We will also employ the following notation. Let $f(x), g(x)$ be two real valued functions with $g(x) > 0$. Then the notation $f(x) = \Omega_+(g(x))$ means

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$$

and $f(x) = \Omega_-(g(x))$ means

$$\liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0 \ .$$

We now state our current knowledge of $M(x)$. The best known unconditional upper bound is

$$M(x) = O\left(x \exp\left(-c_1 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right)\right)$$

for $c_1 > 0$ (see Ivić [16] pp. 309-315) . However, the Riemann hypothesis is equivalent to the bound

$$M(x) = O\left(x^{\frac{1}{2}} \exp\left(\frac{c_2 \log x}{\log\log x}\right)\right)$$

for $c_2 > 0$ (see [23] p. 371). The best unconditional omega result for $M(x)$ is

$$M(x) = \Omega_{\pm}(x^{\frac{1}{2}}) \ .$$

It should also be noted that if $\zeta(s)$ had a multiple zero of of order $m \geq 2$ then

$$M(x) = \Omega_{\pm}(x^{\frac{1}{2}}(\log x)^{m-1}) \ .$$

However, if RH is false then

$$M(x) = \Omega_{\pm}(x^{\theta-\delta})$$

where

$$\theta = \sup_{\rho, \zeta(\rho)=0} \mathrm{Re}(\rho)$$

and δ is any positive constant (see Ingham [14] p. 90).

To better understand the behaviour of $M(x)$, it is useful to consider the closely related function

$$\psi(x) - x = \sum_{n\leq x} \Lambda(n) - x \qquad (9)$$

where $\Lambda(n)$ is Von-Mangoldt's function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^j \ , \ j \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{10}$$

Here we review what is known concerning $\psi(x) - x$. This may give us some better idea what type of upper and lower bounds we should expect for $M(x)$. Von Koch (see [5] p.116) showed that RH is equivalent to

$$\psi(x) - x = O(x^{\frac{1}{2}} \log^2 x) \ . \tag{11}$$

Moreover, Gallagher [6] showed that RH implies that

$$\psi(x) - x = O(x^{\frac{1}{2}} (\log\log x)^2)$$

except on a set of finite logarithmic measure. On the other hand, Littlewood demonstrated

$$\psi(x) - x = \Omega_{\pm} \left(x^{\frac{1}{2}} \log\log\log x \right)$$

under the assumption of RH (see [14] Chapter V) . Moreover, Montgomery [18] has given an unpublished probabilistic argument that suggests

$$\overline{\underline{\lim}} \frac{\psi(x) - x}{x^{\frac{1}{2}} (\log\log\log x)^2} = \pm \frac{1}{2\pi} \tag{12}$$

under the assumption of the Riemann hypothesis and the LI conjecture.

Although the Mertens conjecture is false, we can still obtain some averaged upper bounds for $M(x)$. We prove the following results:

Theorem 1 *The Riemann Hypothesis and $J_{-1}(T) \ll T$ imply:*
(i)

$$M(x) \ll x^{\frac{1}{2}} (\log x)^{\frac{3}{2}} \ ,$$

(ii)

$$M(x) \ll x^{\frac{1}{2}} (\log\log x)^{\frac{3}{2}}$$

except on a set of finite logarithmic measure,
(iii)

$$\int_2^X \frac{M(x)^2}{x} \, dx \ll X \ ,$$

(iv) and the weak Mertens conjecture (8)

$$\int_2^X \left(\frac{M(x)}{x} \right)^2 \, dx \ll \log X \ .$$

Theorem 1(i) is due to Gonek, but had never published. The proof of Theorem 1 (ii) follows an argument due to Gallagher [6] and the proofs of Theorem 1 (iii), (iv) follow an argument due to Cramér [4]. We note that by a more careful calculation we can obtain an asymptotic evaluation in (iv). However, since (iv) is easily deduced from Lemma 6, we include the argument.

Our study of $M(x)$ requires some notions from probability theory. Most importantly, we make use of distribution functions. A distribution function $F(x)$ on $\mathbb{R}$ satisfies F is non-decreasing, $F(-\infty)=0$, $F(\infty)=1$, F is right-continuous, and F has a limit on the left at each $x\in\mathbb{R}$. Recall that if P is a probability measure on $\mathbb{R}$, then $F_P(x) := P((-\infty,x])$ is a distribution function. On the other hand, given a distribution function $F(x)$, there is a theorem from probability theory which states there exists a probability measure P on $\mathbb{R}$ such that $F=F_P$.

In an attempt to better understand $M(x)$, we give a conditional proof of the existence of a limiting distribution function for $\phi(y)=e^{-\frac{y}{2}}M(e^y)$. The idea to prove such a theorem originated with Heath-Brown's comment: [10] "It appears to be an open question whether

$$x^{-\frac{1}{2}}M(x)=x^{-\frac{1}{2}}\sum_{n\le x}\mu(n)$$

has a distribution function. To prove this one would want to assume the Riemann Hypothesis and the simplicity of the zeros, and perhaps also a growth condition on $M(x)$." Applying techniques from Cramér [4] and Rubinstein-Sarnak [22] we establish the following result.

Theorem 2 *Assume the Riemann Hypothesis and $J_{-1}(T)\ll T$. Then $e^{-\frac{y}{2}}M(e^y)$ has a limiting distribution ν on $\mathbb{R}$, that is,*

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y f(e^{-\frac{y}{2}}M(e^y))\ dy=\int_{-\infty}^{\infty} f(x)\ d\nu(x) \tag{13}$$

for all bounded Lipschitz continuous functions f on $\mathbb{R}$.

We note that the above theorem may be extended to all bounded continuous functions $f(x)$ by standard approximation techniques. However, we omit these arguments to keep the exposition simple. Clearly Theorem 2 is useful in studying $M(x)$. To see this, suppose the above theorem remains valid for indicator functions. Let V be a fixed real number and define $f=1_V$ where

$$1_V(x)=\begin{cases}1 \text{ if } x\ge V\\ 0 \text{ if } x<V\end{cases}.$$

With the above choice of $f(x)$ (13) translates to

$$\lim_{Y\to\infty} \frac{1}{Y}\text{meas}\{y \in [0,Y] \mid M(e^y) \geq e^{\frac{y}{2}} V \ \} = \nu([V,\infty)). \tag{14}$$

As noted in [22] p. 174, the above identity would be true if $\nu(x)$ is absolutely continuous. Under the additional assumption of LI, one may show that ν is absolutely continuous. Moreover, the LI conjecture implies that the Fourier transform of ν may be computed explicitly.

Corollary 1 *Assume the Riemann Hypothesis, $J_{-1}(T) \ll T$, and LI. Then the Fourier transform $\widehat{\nu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t}\, d\nu(t)$ exists and equals*

$$\widehat{\nu}(\xi) = \prod_{\gamma>0} \tilde{J}_0\left(\frac{2\xi}{\left|(\frac{1}{2}+i\gamma)\zeta'(\frac{1}{2}+i\gamma)\right|} \right) \tag{15}$$

where $\tilde{J}_0(z)$ is the Bessel function $\tilde{J}_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}z)^{2m}}{(m!)^2}$.

Note that we have employed non-standard notation for the Bessel function, so as not to confuse it with the moments $J_{-k}(T)$. Under the same assumptions as Corollary 1, we observe that the set

$$S = \{ x \geq 1 \mid |M(x)| \leq \sqrt{x} \ \}$$

has a logarithmic density. Namely,

$$\delta(S) = \lim_{X\to\infty} \frac{1}{\log X} \int_{[2,X]\bigcap S} \frac{dt}{t}$$

exists and $0 < \delta(S) < 1$. Since no counterexamples to the Mertens conjecture have ever been found, we expect this logarithmic density to be very close to 1. In fact, preliminary calculations indicate this.

In the same spirit as Theorems 1 and 2 we prove a strong form of the weak Mertens conjecture is true. This follows Cramér's argument [4] subject to the same assumptions as the previous theorems.

Theorem 3 *Assume the Riemann hypothesis and $J_{-1}(T) \ll T$, then we have*

$$\int_0^Y \left(\frac{M(e^y)}{e^{\frac{y}{2}}} \right)^2 dy \sim \beta Y \tag{16}$$

where

$$\beta = \sum_{\gamma>0} \frac{2}{|\rho\zeta'(\rho)|^2} \ . \tag{17}$$

Note that the assumption $J_{-1}(T) \ll T$ implies (17) is convergent.

A change of variable transforms (16) to

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx \sim \beta \log X \ . \tag{18}$$

Also, note that Theorem 3 corresponds to Theorem 2 with $f(x) = x^2$. However, $f(x) = x^2$ is not a bounded function and does not fall under the assumptions of Theorem 2. We further note that the same techniques allow one to establish

$$\int_0^Y \frac{M(e^y)}{e^{\frac{y}{2}}} \, dy = o(Y) \tag{19}$$

under the same conditions as Theorem 3. Consequently, (16) and (19) reveal that the variance of the probability measure constructed in Theorem 2 is β.

As one can see by equation (14) and Theorem 3, the constructed limiting distribution of Theorem 2 reveals significant information concerning $M(x)$. The above formula (15) for the Fourier transform is crucial in studying the behaviour of $x^{-\frac{1}{2}}M(x)$. Upon proving Theorem 2, we realized that the constructed distribution could be used to study large values of $M(x)$. Using Montgomery's probabilistic methods we study the tail of this distribution and give a conditional proof that

$$\exp(-\exp(\tilde{c}_1 V^{\frac{4}{5}})) \ll \nu([V,\infty)) \ll \exp(-\exp(\tilde{c}_2 V^{\frac{4}{5}}))$$

for positive effective constants $\tilde{c}_1$ and $\tilde{c}_2$. We believe that $\tilde{c}_1 = \tilde{c}_2$, however it is not presently clear what this value should be. Nevertheless, these bounds seem to suggest the following version of an unpublished conjecture of Gonek's.

Gonek's Conjecture There exists a number $B > 0$ such that

$$\overline{\underline{\lim}}_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = \pm B \ . \tag{20}$$

After the completion of this work the author learned that Gonek had arrived at this conjecture at least ten years ago via Montgomery's techniques. He had annuciated this conjecture at several conferences in the early 1990's. We note that the exponent of the iterated triple logarithm is $\frac{5}{4}$ in (20) precisely because of the Gonek-Hejhal conjecture (3). Montgomery's conjecture (12) on $\frac{\psi(x)-x}{\sqrt{x}}$ shows that the corresponding exponent on the iterated triple logarithm is 2. The difference between these cases is due directly to the different discrete moments of

$$\sum_{\gamma \leq T} \frac{1}{|\rho|} \asymp (\log T)^2 \text{ and } \sum_{\gamma \leq T} \frac{1}{|\rho \zeta'(\rho)|} \asymp (\log T)^{\frac{5}{4}}$$

where the second inequality is currently conjectural.

Finally, we remark that many of the results in this paper may be extended to the summatory function of the Liouville function. The Liouville function is defined as $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ denotes the total number of prime factors of n. Pólya was interested in the summatory function

$$L(x) = \sum_{n \leq x} \lambda(n) \tag{21}$$

since if the inequality $L(x) \leq 0$ always persisted then the Riemann hypothesis would follow. Haselgrove [9] showed that this statement cannot be true. By the methods of this article, we can prove that $e^{-\frac{y}{2}} L(e^y)$ has a limiting distribution under the same conditions as Theorems 1-3. The reason we can extend the work to this case is because

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{22}$$

and thus the only difference is the term $\zeta(2s)$ in the numerator. Nevertheless, this can be treated easily since we understand the zeta function on the $\mathrm{Re}(s) = 1$ line.

The majority of this article constitutes the last chapter of my Ph.D. thesis. However, Theorem 3 was proven during a stay at the Institute for Advanced Study during Spring 2002. I would like to thank my Ph.D. supervisor, Professor David Boyd, for providing me with academic and financial support during the writing of the thesis and throughout my graduate studies. I also thank the I.A.S. for its support and excellent working conditions. Finally, thanks to Professor Steve Gonek for allowing me include Theorem 1(i) and also for informing me of his earlier unpublished work.

2 Proof of Theorem 1

Various estimates throughout this article require estimates for averages of sums containing the expression $|\zeta^{'}(\rho)|^{-1}$. This lemma establishes such estimates, subject to various special cases of the Gonek-Hejhal conjecture (3).

Lemma 1 *(i)* $J_{-\frac{1}{2}}(T) = \sum_{0<\gamma<T} |\zeta^{'}(\rho)|^{-1} \ll T(\log T)^v$ *implies*

$$\sum_{0<\gamma<T} \frac{1}{|\rho\zeta'(\rho)|} \ll (\log T)^{v+1}.$$

(ii) $J_{-1}(T) = \sum_{0<\gamma<T} |\zeta^{'}(\rho)|^{-2} \ll T$ *implies*

$$\sum_{T<\gamma<2T} \frac{1}{|\rho\zeta'(\rho)|^2} \ll \frac{1}{T}.$$

(iii) $J_{-\frac{1}{2}}(T) = \sum_{0<\gamma<T} |\zeta^{'}(\rho)|^{-1} \ll T^u(\log T)^v$ *implies*

$$\sum_{\gamma>T} \frac{(\log\gamma)^a}{\gamma^b|\zeta'(\rho)|} \ll \frac{(\log T)^{a+v}}{T^{b-u}}$$

subject to $b > u \geq 1$.

Proof. For part(i) note that

$$\begin{aligned}\sum_{0<\gamma<T} \frac{1}{|\rho\zeta'(\rho)|} \ll \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|\gamma} &= \left[\frac{J_{-\frac{1}{2}}(t)}{t}\right]_{14}^{T} + \int_{14}^{T} \frac{J_{-\frac{1}{2}}(t)}{t^2}\,dt \\ &= O\left((\log T)^v + \int_{14}^{T} \frac{t(\log t)^v}{t^2}\,dt\right) \\ &= O((\log T)^{v+1})\ .\end{aligned} \tag{23}$$

Observe that we have made use of fact that all non-trivial zeros $\rho = \beta + i\gamma$ satisfy $|\gamma| \geq 14$. Part (ii) is proven in an analogous fashion. For part (iii) let $\phi(t) = (\log t)^a t^{-b}$ and note that its derivative is $\phi^{'}(t) = (a(\log t)^{a-1} - b(\log t)^a)/t^{b+1}$. Partial summation implies

$$\sum_{\gamma>T} \frac{(\log\gamma)^a}{\gamma^b|\zeta'(\rho)|} = \left[\phi(t)J_{-\frac{1}{2}}(t)\right]_T^{\infty} - \int_T^{\infty} J_{-\frac{1}{2}}(t)\phi^{'}(t)\,dt.$$

The first term is $\ll \phi(T)J_{-\frac{1}{2}}(T) = (\log T)^{a+v}/T^{b-u}$. Assuming the bound on $J_{-\frac{1}{2}}(T)$, the second term is

$$\ll \int_T^\infty \frac{(t^u(\log t)^v)(\log t)^a}{t^{b+1}}\,dt = \int_T^\infty \frac{(\log t)^{a+v}}{t^{b-u+1}}\,dt \ll \frac{(\log T)^{a+v}}{T^{b-u}}$$

where the last integral is computed by an integration by parts.

We require Perron's formula in order to express $M(x)$ as the sum of a complex integral and an error term.

Lemma 2 *Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be absolutely convergent for $\sigma = \mathrm{Re}(s) > 1$, $a_n \ll \Phi(n)$ where $\Phi(x)$ is positive and non-decreasing, and*

$$\sum_{n=1}^\infty \frac{|a_n|}{n^\sigma} = O\left(\frac{1}{(\sigma-1)^\alpha}\right) \text{ as } \sigma \to 1^+ \ .$$

Then if $w = u + iv$ with $c > 0$, $u + c > 1$, $T > 0$, we have for all $x \geq 1$

$$\begin{aligned}\sum_{n\leq x} \frac{a_n}{n^w} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(w+s)\frac{x^s}{s}ds \\ + O\left(\frac{x^c}{T(u+c-1)^\alpha} + \frac{\Phi(2x)x^{1-u}\log(2x)}{T} + \Phi(2x)x^{-u}\right)\end{aligned} \tag{24}$$

Proof. This is a well-known theorem and is proven in [21] pp. 376-379.

We need the following technical lemma in order to choose a good contour for the complex integral obtained by Perron's formula.

Lemma 3 *There exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^\infty$ which satisfies*

$$n \leq T_n \leq n+1 \text{ and } \frac{1}{\zeta(\sigma+iT)} = O(T^\epsilon)$$

for all $-1 \leq \sigma \leq 2$.

Proof. The above fact is proven in Titchmarsh [23] pp. 357-358 in the range $\frac{1}{2} \leq \sigma \leq 2$. It remains to prove the bound in the range $-1 \leq \sigma < \frac{1}{2}$. The asymmetric form of the functional equation of the zeta function is $\zeta(s) = \chi(s)\zeta(1-s)$ where $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$. A calculation with Stirling's formula demonstrates that $|\chi(\sigma+iT)| \asymp T^{\frac{1}{2}-\sigma}$ and therefore we deduce that

$$|\zeta(s)|^{-1} = |\zeta(1-s)\chi(s)|^{-1} \ll T^{\epsilon+\sigma-\frac{1}{2}} \ll T^\epsilon$$

for $-1 \le \sigma < \frac{1}{2}$.

We now prove an explicit formula for $M(x)$. With the exception of a few minor changes, the proof follows Theorem 14.27 of [23] pp. 372-374.

Lemma 4 *Assume the Riemann hypothesis and that all zeros of $\zeta(s)$ are simple. For $x \ge 2$ and $T \in \mathcal{T}$*

$$M(x) = \sum_{|\gamma|<T} \frac{x^\rho}{\rho\zeta'(\rho)} + \tilde{E}(x,T)$$

where

$$\tilde{E}(x,T) \ll \frac{x\log x}{T} + \frac{x}{T^{1-\epsilon}\log x} + 1 \ .$$

Proof. We apply Lemma 2 with $f(s) = \zeta(s)^{-1}$, $\Phi(x) = 1$, $\alpha = 1$, and $w = 0$ to obtain

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)}ds + O\left(\frac{x^c}{T(c-1)} + \frac{x\log x}{T} + 1\right).$$

Setting $c = 1 + (\log x)^{-1}$, this becomes

$$M(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)}ds + O\left(\frac{x\log x}{T} + 1\right).$$

We introduce a large parameter U and consider a positively oriented rectangle $B_{T,U}$ with vertices at $c-iT, c+iT, -U+iT$, and $-U-iT$. Thus the integral on the right equals

$$\frac{1}{2\pi i}\int_{B_{T,U}} \frac{x^s}{s\zeta(s)}ds - \frac{1}{2\pi i}\left(\int_{c+iT}^{-U+iT} + \int_{-U+iT}^{-U-iT} + \int_{-U-iT}^{c-iT}\right)\frac{x^s}{s\zeta(s)}ds \ .$$

It is shown in Titchmarsh [23] p. 373 that the middle integral approaches 0 as $U \to \infty$. Inside the box $B_{T,U}$, $\frac{x^s}{s\zeta(s)}$ has poles at the zeros of the zeta function and $s = 0$. By Cauchy's Residue Theorem, we have

$$\begin{aligned} M(x) = &\frac{1}{2\pi i}\sum_{|\gamma|<T} \frac{x^\rho}{\rho\zeta'(\rho)} - 2 + \sum_{k\ge1} \frac{x^{-2k}}{(-2k)\zeta'(-2k)} \\ &- \frac{1}{2\pi i}\left(\int_{c+iT}^{-\infty+iT} + \int_{-\infty-iT}^{c-iT}\right)\frac{x^s}{\zeta(s)s}ds + O\left(\frac{x\log x}{T} + 1\right). \end{aligned} \tag{25}$$

The second and third terms are absorbed by the $O(1)$ term. We now bound the integrals. Break up the first integral in two pieces as

$$\int_{c+iT}^{-\infty+iT} \frac{x^s}{s\zeta(s)}\, ds = \left(\int_{c+iT}^{-1+iT} + \int_{-1+iT}^{-\infty+iT}\right) \frac{x^s}{s\zeta(s)}\, ds\ . \tag{26}$$

By Lemma 3, we have

$$\left|\int_{-1+iT}^{c+iT} \frac{x^s}{s\zeta(s)}\, ds\right| \ll \int_{-1}^{c} \frac{x^\sigma T^\epsilon}{\sqrt{\sigma^2+T^2}}\, d\sigma \le T^{\epsilon-1}\int_{-1}^{c} e^{\sigma\log x}\, d\sigma \ll \frac{x}{T^{1-\epsilon}\log x}.$$

For the second piece we apply the functional equation

$$\int_{-\infty+iT}^{-1+iT} \frac{x^s}{s\zeta(s)}\, ds = \int_{2-iT}^{\infty-iT} \frac{x^{1-s}2^{s-1}\pi^s}{(1-s)\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)}\, ds.$$

For $\sigma \ge 2$ we have the Stirling formula estimate $\frac{1}{|\Gamma(\sigma-iT)|} \ll e^{\sigma-(\sigma-\frac{1}{2})\log\sigma+\frac{1}{2}\pi T}$ and the elementary estimate $\frac{1}{|\cos(\frac{\pi(\sigma-iT)}{2})|} \ll e^{-\frac{\pi}{2}T}$ and hence the integral is

$$O\left(\int_2^\infty \frac{x}{T}\left(\frac{2\pi}{x}\right)^\sigma e^{\sigma-(\sigma-\frac{1}{2})\log\sigma}\, d\sigma\right) = O\left(\frac{x}{T}\right)\ .$$

The same argument applies to the second integral in (26) and we have shown that

$$M(x) = \sum_{|\gamma|<T} \frac{x^\rho}{\rho\zeta'(\rho)} + O\left(\frac{x\log x}{T} + \frac{x}{T^{1-\epsilon}\log x} + 1\right)\ .$$

We now remove the assumption that $T \in \mathcal{T}$ from the last lemma by applying the Gonek-Hejhal conjecture (3) for $k=-1$.

Lemma 5 *Assume the Riemann hypothesis and*

$$J_{-1}(T) \ll T\ .$$

For $x \ge 2$, $T \ge 2$

$$M(x) = \sum_{|\gamma|\le T} \frac{x^\rho}{\rho\zeta'(\rho)} + E(x,T)$$

where

$$E(x,T) \ll \frac{x\log x}{T} + \frac{x}{T^{1-\epsilon}\log x} + \left(\frac{x\log T}{T}\right)^{\frac{1}{2}} + 1\ . \tag{27}$$

Proof. Let $T \geq 2$ satisfy $n \leq T \leq n+1$. Now suppose without loss of generality that $n \leq T_n \leq T \leq n+1$. Then we have

$$M(x) = \sum_{|\gamma|\leq T} \frac{x^{\rho}}{\rho\zeta'(\rho)} - \sum_{T_n\leq|\gamma|\leq T} \frac{x^{\rho}}{\rho\zeta'(\rho)} + \widetilde{E}(x, T_n) \ .$$

By Cauchy-Schwarz the second sum is

$$\left| \sum_{T_n\leq\gamma\leq T} \frac{x^{\rho}}{\rho\zeta'(\rho)} \right| \leq x^{\frac{1}{2}} \left(\sum_{T_n\leq\gamma\leq T} \frac{1}{|\rho\zeta'(\rho)|^2} \right)^{\frac{1}{2}} \left(\sum_{T_n\leq\gamma\leq T} 1 \right)^{\frac{1}{2}} \ .$$

By Lemma 1(ii), $J_{-1}(T) \ll T$ implies $\sum_{T\leq\gamma\leq 2T} \frac{1}{|\rho\zeta'(\rho)|^2} \ll \frac{1}{T}$ and we deduce that

$$\left| \sum_{T_\nu\leq\gamma\leq T} \frac{x^{\rho}}{\rho\zeta'(\rho)} \right| \ll \left(\frac{x\log T}{T} \right)^{\frac{1}{2}}$$

which completes the proof.

Lemma 6 is the crucial step in proving the existence of the limiting distribution in the next section. The key point is that the integral in this lemma should be small in order to justify the weak convergence of a sequence of distribution functions in Theorem 2. This is also used in the proof of Theorem 1 parts (ii)-(iv).

Lemma 6 *Assume the Riemann hypothesis and $J_{-1}(T) \ll T$. Then*

$$\int_Z^{eZ} \left| \sum_{T\leq|\gamma|\leq X} \frac{x^{i\gamma}}{\rho\zeta'(\rho)} \right|^2 \frac{dx}{x} \ll \frac{(\log T)}{T^{\frac{1}{4}}} \tag{28}$$

for $Z > 0$ and $T < X$.

Proof. Making the substitution $x = e^y$ in the left hand side of (28) we obtain

$$\begin{aligned} &\int_{\log Z}^{\log Z+1} \left| \sum_{T\leq|\gamma|\leq X} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)} \right|^2 dy \leq 4 \int_{\log Z}^{\log Z+1} \left| \sum_{T\leq\gamma\leq X} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)} \right|^2 dy \\ &= 4 \sum_{T\leq\gamma\leq X} \sum_{T\leq\lambda\leq X} \frac{1}{\rho\zeta'(\rho)\overline{\rho'\zeta'(\rho')}} \int_{\log Z}^{\log Z+1} e^{i(\gamma-\lambda)y} \, dy \qquad (29) \\ &\ll \sum_{T\leq\gamma\leq X} \sum_{T\leq\lambda\leq X} \frac{1}{|\rho\zeta'(\rho)||\overline{\rho'\zeta'(\rho')}|} \min\left(1, \frac{1}{|\gamma-\lambda|}\right) \end{aligned}$$

Note that ρ and $\rho^{'}$ denote zeros of the form $\rho = \frac{1}{2} + i\gamma$ and $\rho^{'} = \frac{1}{2} + i\lambda$. We break this last sum in two sums Σ_1 and Σ_2 where Σ_1 consists of those terms for which $|\gamma - \lambda| \leq 1$ and Σ_2 consists of the complementary set. The first sum is bounded as follows

$$\Sigma_1 \ll \sum_{T \leq \gamma \leq X} \frac{1}{|\rho\zeta'(\rho)|} \sum_{\gamma-1 \leq \lambda \leq \gamma+1} \frac{1}{|\rho'\zeta'(\rho')|}.$$

It is well known that $N(t+1) - N(t-1) \ll \log t$, hence the inner sum is

$$\leq \left(\sum_{\gamma-1 \leq \lambda \leq \gamma+1} \frac{1}{|\rho'\zeta'(\rho')|^2} \right)^{\frac{1}{2}} (N(\gamma+1) - N(\gamma-1))^{\frac{1}{2}} \ll \left(\frac{\log \gamma}{\gamma} \right)^{\frac{1}{2}}$$

by an application of Lemma 1(ii). By Lemma 1(iii) we deduce that

$$\Sigma_1 \ll \sum_{T \leq \gamma} \frac{(\log \gamma)^{\frac{1}{2}}}{\gamma^{\frac{3}{2}} |\zeta'(\rho)|} \ll \frac{\log T}{T^{\frac{1}{2}}} \ . \tag{30}$$

Write the second sum as

$$\Sigma_2 = \sum_{T \leq \gamma \leq X} \frac{1}{|\rho\zeta'(\rho)|} \sum_{T \leq \lambda \leq X, |\gamma-\lambda| \geq 1} \frac{1}{|\rho'\zeta'(\rho')||\gamma - \lambda|}. \tag{31}$$

The inner sum is analyzed by splitting the sum in to ranges. The crucial range is when $|\gamma - \lambda| \approx 1$. This argument was originally employed by Cramér [4]. We eliminate the condition $\gamma \leq X$ and denote the inner sum of (31) as $S(\gamma)$ where $\gamma \geq T$. Consider the set of numbers, $\gamma^{\frac{1}{2}}, \gamma - \gamma^{\frac{1}{2}}$, and $\gamma - 1$. One of the following cases must occur $T \leq \gamma^{\frac{1}{2}}$, $\gamma^{\frac{1}{2}} < T \leq \gamma - \gamma^{\frac{1}{2}}$, $\gamma - \gamma^{\frac{1}{2}} < T \leq \gamma - 1$, or $\gamma - 1 < T \leq \gamma$. These conditions translate in to the four cases: $T^2 \leq \gamma$, $T + \sqrt{T + \frac{1}{4}} + \frac{1}{2} \leq \gamma < T^2$, $T + 1 \leq \gamma < T + \sqrt{T + \frac{1}{4}} + \frac{1}{2}$, and $T \leq \gamma < T + 1$. Suppose the first case is true, i.e. $T \leq \gamma^{\frac{1}{2}}$. Then we may write the inner sum $S(\gamma)$ as six separate sums

$$\begin{aligned} S(\gamma) = \Bigg(&\sum_{T \leq \lambda < \gamma^{\frac{1}{2}}} + \sum_{\gamma^{\frac{1}{2}} \leq \lambda < \gamma - \gamma^{\frac{1}{2}}} + \sum_{\gamma - \gamma^{\frac{1}{2}} \leq \lambda \leq \gamma - 1} \\ &+ \sum_{\gamma+1 \leq \lambda < \gamma + \gamma^{\frac{1}{2}}} + \sum_{\gamma + \gamma^{\frac{1}{2}} \leq \lambda < 2\gamma} + \sum_{2\gamma \leq \lambda} \Bigg) \frac{1}{|\rho'\zeta'(\rho')||\gamma - \lambda|} \ . \end{aligned} \tag{32}$$

Denote these sums by $\sigma_1, \dots, \sigma_6$. In the following estimates we apply Lemma 1(ii) several times. We find that

$$\sigma_1 \leq \frac{1}{\gamma - \gamma^{\frac{1}{2}}} \sum_{T \leq \lambda < \gamma^{\frac{1}{2}}} \frac{1}{|\rho' \zeta'(\rho')|} \ll \frac{1}{\gamma} \left(\sum_{T \leq \lambda < \gamma^{\frac{1}{2}}} \frac{1}{|\rho' \zeta'(\rho)|^2} \right)^{\frac{1}{2}} \left(\sum_{T \leq \lambda < \gamma^{\frac{1}{2}}} 1 \right)^{\frac{1}{2}}$$
$$\ll \frac{1}{\gamma T^{\frac{1}{2}}} (\gamma^{\frac{1}{2}} \log \gamma)^{\frac{1}{2}} = \frac{(\log \gamma)^{\frac{1}{2}}}{T^{\frac{1}{2}} \gamma^{\frac{3}{4}}}, \tag{33}$$

$$\sigma_2 \leq \frac{1}{\gamma^{\frac{1}{2}}} \sum_{\gamma^{\frac{1}{2}} \leq \lambda < \gamma - \gamma^{\frac{1}{2}}} \frac{1}{|\rho' \zeta'(\rho')|} \leq \frac{1}{\gamma^{\frac{1}{2}}} \left(\sum_{\gamma^{\frac{1}{2}} \leq \lambda < \gamma - \gamma^{\frac{1}{2}}} \frac{1}{|\rho' \zeta'(\rho')|^2} \right)^{\frac{1}{2}} \left(\sum_{\gamma^{\frac{1}{2}} \leq \lambda < \gamma - \gamma^{\frac{1}{2}}} 1 \right)^{\frac{1}{2}}$$
$$\ll \frac{1}{\gamma^{\frac{1}{2}}} \left(\frac{1}{\gamma^{\frac{1}{2}}} \right)^{\frac{1}{2}} (\gamma \log \gamma)^{\frac{1}{2}} = \frac{(\log \gamma)^{\frac{1}{2}}}{\gamma^{\frac{1}{4}}}, \tag{34}$$

and

$$\sigma_3 \leq \left(\sum_{\gamma - \gamma^{\frac{1}{2}} \leq \lambda \leq \gamma - 1} \frac{1}{|\rho' \zeta'(\rho')|^2} \right)^{\frac{1}{2}} \left(\sum_{\gamma - \gamma^{\frac{1}{2}} \leq \lambda \leq \gamma - 1} 1 \right)^{\frac{1}{2}} \ll \frac{1}{\gamma^{\frac{1}{2}}} (\gamma^{\frac{1}{2}} \log \gamma)^{\frac{1}{2}} = \frac{(\log \gamma)^{\frac{1}{2}}}{\gamma^{\frac{1}{4}}}. \tag{35}$$

The fourth sum, σ_4, gives the same error as the third sum. Similarly,

$$\sigma_5 \ll \frac{1}{\gamma^{\frac{1}{2}}} \left(\sum_{\gamma + \gamma^{\frac{1}{2}} \leq \lambda} \frac{1}{|\rho' \zeta'(\rho)|^2} \right)^{\frac{1}{2}} \left(\sum_{\gamma + \gamma^{\frac{1}{2}} \leq \lambda \leq 2\gamma} 1 \right)^{\frac{1}{2}} \ll \frac{1}{\gamma^{\frac{1}{2}}} \left(\frac{\gamma \log \gamma}{\gamma} \right)^{\frac{1}{2}} = \frac{(\log \gamma)^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}$$

and

$$\sigma_6 \le \sum_{k=1}^{\infty} \sum_{2^k\gamma \le \lambda \le 2^{k+1}\gamma} \frac{1}{|\rho'\zeta'(\rho')||\gamma-\lambda|}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{(2^k-1)\gamma} \left(\sum_{2^k\gamma \le \lambda \le 2^{k+1}\gamma} \frac{1}{|\rho'\zeta'(\rho')|^2} \right)^{\frac{1}{2}} \left(\sum_{2^k\gamma \le \lambda \le 2^{k+1}\gamma} 1 \right)^{\frac{1}{2}} \tag{36}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{(2^k-1)\gamma} \left(\frac{2^{k+1}\gamma \log(2^{k+1}\gamma)}{2^k\gamma} \right)^{\frac{1}{2}} \ll \frac{(\log\gamma)^{\frac{1}{2}}}{\gamma} \ .$$

Putting together these bounds leads to

$$S(\gamma) \ll \frac{(\log\gamma)^{\frac{1}{2}}}{\gamma^{\frac{1}{4}}}$$

as long as $T^2 \le \gamma$. In fact, the same argument applies in the other three cases. The only difference is that there would be fewer sums and we still establish $S(\gamma) \ll (\log\gamma)^{\frac{1}{2}}\gamma^{-\frac{1}{4}}$ for all $\gamma \ge T$. The assumption $J_{-1}(T) \ll T$ implies by Cauchy-Schwarz that

$$J_{-1/2}(T) \ll J_{-1}(T)^{\frac{1}{2}} N(T)^{\frac{1}{2}} \ll T^{\frac{1}{2}}(T(\log T))^{\frac{1}{2}} = T(\log T)^{\frac{1}{2}} \ . \tag{37}$$

Applying Lemma 1(*iii*) yields the bound

$$\Sigma_2 \ll \sum_{\gamma > T} \frac{(\log\gamma)^{\frac{1}{2}}}{\gamma^{\frac{5}{4}}|\zeta'(\rho)|} \ll \frac{\log T}{T^{\frac{1}{4}}}$$

and the lemma is proved.

Combining the previous lemmas we may now prove Theorem 1.

Proof of Theorem 1. (*i*) By Lemma 5,

$$M(x) \ll x^{\frac{1}{2}} \sum_{0<\gamma<T} \frac{1}{|\rho\zeta'(\rho)|} + E(x,T)$$

where $E(x,T)$ is defined by (27). By the bound (37) Lemma 1(*i*) yields

$$M(x) \ll x^{\frac{1}{2}}(\log T)^{\frac{3}{2}} + \frac{x\log x}{T} + \frac{x}{T^{1-\epsilon}\log x} + \left(\frac{x\log T}{T}\right)^{\frac{1}{2}} \ .$$

By the choice $T^{1-\epsilon} = \sqrt{x}$, we deduce $M(x) \ll \sqrt{x}(\log x)^{\frac{3}{2}}$.
(ii) The starting point is to consider the explicit formula. By Lemma 5, we have

$$M(x) = \sum_{|\gamma| \le X} \frac{x^\rho}{\rho\zeta'(\rho)} + O\left(X^\epsilon\right) \tag{38}$$

valid for $X \le x \ll X$. By Lemma 6, we have for $T^4 < X$

$$\int_X^{eX} \left| \sum_{T^4 \le |\gamma| \le X} \frac{x^\rho}{\rho\zeta'(\rho)} \right|^2 \frac{dx}{x^2} \ll \frac{(\log T)}{T}.$$

By considering the set

$$S = \left\{ x \ge 2 \mid \left| \sum_{T^4 \le |\gamma| \le X} \frac{x^\rho}{\rho\zeta'(\rho)} \right| \ge x^{\frac{1}{2}} (\log\log x)^{\frac{5}{4}} \right\}$$

it follows that

$$(\log\log X)^{\frac{5}{2}} \int_{S \cap [X, eX]} \frac{dx}{x} \le \int_X^{eX} \left| \sum_{T^4 \le |\gamma| \le X} \frac{x^\rho}{\rho\zeta'(\rho)} \right|^2 \frac{dx}{x^2} \ll \frac{(\log T)}{T}$$

and thus

$$\int_{S \cap [X, eX]} \frac{dx}{x} \ll \frac{(\log T)}{T(\log\log X)^{\frac{5}{2}}} = \frac{1}{T(\log T)^{\frac{3}{2}}}$$

for $T = \log X$. Choosing $X = e^k$ with $k = 2, 3, \ldots$ we deduce

$$\int_{S \cap [e^2, \infty]} \frac{dx}{x} \ll \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{\frac{3}{2}}} < \infty$$

and thus S has finite logarithmic measure. By the bound (37) Lemma 1(i) implies

$$\left| \sum_{0 \le |\gamma| \le T^4} \frac{x^\rho}{\rho\zeta'(\rho)} \right| \ll X^{\frac{1}{2}} \sum_{0 \le |\gamma| \le T^4} \frac{1}{|\rho\zeta'(\rho)|} \ll X^{\frac{1}{2}} (\log T)^{\frac{3}{2}} \ll X^{\frac{1}{2}} (\log\log X)^{\frac{3}{2}}$$

for $X \le x \le eX$. Hence,

$$M(x) = \sum_{T^4 \le |\gamma| \le X} \frac{x^\rho}{\rho\zeta'(\rho)} + O\left(X^{\frac{1}{2}} (\log\log X)^{\frac{3}{2}} \right)$$

for $X \leq x \leq eX$ and $T = \log X$. Define the set

$$S_\alpha = \{x \geq 2 \mid |M(x)| \geq \alpha x^{\frac{1}{2}} (\log\log x)^{\frac{3}{2}}\}.$$

Suppose $x \in S_\alpha \bigcap [X, eX]$. Then we have

$$\begin{aligned} \left| \sum_{T^4 \leq |\gamma| \leq X} \frac{x^\rho}{\rho \zeta'(\rho)} \right| &\geq |M(x)| - O\left(X^{\frac{1}{2}} (\log\log X)^{\frac{3}{2}}\right) \\ \geq \alpha x^{\frac{1}{2}} (\log\log x)^{\frac{3}{2}} &- O\left(X^{\frac{1}{2}} (\log\log X)^{\frac{3}{2}}\right) \geq x^{\frac{1}{2}} (\log\log x)^{\frac{5}{4}} \end{aligned} \tag{39}$$

for $x \in [X, eX]$ as long as X is sufficiently large and α is chosen larger than the constant that occurs in the error term of (39). Thus $S_\alpha \bigcap [X, eX] \subset S \bigcap [X, eX]$ for X sufficiently large and it follows that S_α has finite logarithmic measure. Observe that if we also assumed the conjecture $J_{-\frac{1}{2}}(t) \ll t (\log t)^{\frac{1}{4}}$ then the same arguments in (i) and (ii) would have shown that $M(x) \ll x^{\frac{1}{2}} (\log x)^{\frac{5}{4}}$ and $M(x) \ll x^{\frac{1}{2}} (\log\log x)^{\frac{5}{4}}$ except on a set of finite logarithmic measure.

(iii) Squaring equation (38), dividing by x^2, and integrating yields

$$\int_X^{eX} \left(\frac{M(x)}{x}\right)^2 dx \ll \int_X^{eX} \left| \sum_{|\gamma| \leq X} \frac{x^\rho}{\rho \zeta'(\rho)} \right|^2 \frac{dx}{x^2} + O(X^{-1+2\epsilon}) \ll 1 \tag{40}$$

by taking $Z = X$ and $T = 14$ in Lemma 6. It immediately follows that

$$\int_X^{eX} \frac{M(x)^2}{x} \, dx \ll X.$$

Substituting the values $\frac{X}{e}, \frac{X}{e^2}, \ldots$ and adding yields

$$\int_2^X \frac{M(x)^2}{x} dx \ll X \ .$$

(iv) Similarly, we obtain from (40)

$$\int_2^X \left(\frac{M(x)}{x}\right)^2 dx \ll \sum_{k=1}^{[\log(\frac{X}{2})]+1} \int_{X/e^k}^{X/e^{k-1}} \left(\frac{M(x)}{x}\right)^2 dx \ll \log X \ .$$

3 Proofs of Theorems 2 and 3

In this section we prove the existence of a limiting distribution for the function $\phi(y) = e^{-\frac{y}{2}}M(e^y)$. If we assume the Riemann hypothesis and write non-trivial zeros as $\rho = \frac{1}{2} + i\gamma$, then we obtain

$$x^{-\frac{1}{2}}M(x) = \sum_{|\gamma| \leq T} \frac{x^{i\gamma}}{\rho\zeta(\rho)} + \sum_{T < |\gamma| \leq e^Y} \frac{x^{i\gamma}}{\rho\zeta(\rho)} + x^{-\frac{1}{2}}E(x, e^Y)$$

where $T < e^Y$ and $E(x, e^Y)$ is defined in (27). Making the variable change $x = e^y$, we have

$$\phi(y) = e^{-\frac{y}{2}}M(e^y) = \phi^{(T)}(y) + \epsilon^{(T)}(y) \tag{41}$$

where

$$\phi^{(T)}(y) = \sum_{|\gamma| \leq T} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)} \quad \textit{and} \tag{42}$$

$$\epsilon^{(T)}(y) = \sum_{T \leq |\gamma| \leq e^Y} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)} + e^{-\frac{y}{2}}E(e^y, e^Y). \tag{43}$$

In order to construct a sequence of distribution functions that converge to the distribution of Theorem 2, we require the following uniform distribution result.

Lemma 7 *Let $t_1, \ldots, t_N$ be N arbitrary real numbers. Consider the curve $\psi(y) = y(t_1, \ldots, t_N) \in \mathbb{R}^N$ for $y \in \mathbb{R}$. Let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous and have period one in each of its variables. There exists an integer $1 \leq J \leq N$ and A, a J-dimensional parallelotope, such that*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f(\psi(y))\, dy = \int_A f(a)\, d\mu \tag{44}$$

where μ is normalized Haar measure on A. More precisely, A is the topological closure of $\psi(y)$ in $\mathbb{T}^N$.

Proof. This lemma is a well-known and it is a variant of the traditional Kronecker-Weyl theorem (see Hlawka [12], pp. 1-14 for the proof) . We now describe the principal idea in how the lemma is deduced from this. Let J be the maximum number of linearly independent elements over $\mathbb{Q}$ among

$t_1, \ldots, t_N$. The basic idea is to show that the topological closure of the set $\{ \left(\{y\frac{\gamma_1}{2\pi}\}, \ldots, \{y\frac{\gamma_N}{2\pi}\}\right) \mid y \in \mathbb{R} \}$ cuts out a sub-torus of $\mathbb{T}^N$ of dimension J (Note that $\{x\}$ is the fractional part of $x \in \mathbb{R}$). By a variable change, one then deduces the lemma from the Kronecker-Weyl theorem.

By an application of Lemma 7, we construct for each large T a distribution function ν_T.

Lemma 8 *Assume the Riemann hypothesis, then for each $T \geq \gamma_1$ (the imaginary ordinate of the first non-trivial zero of $\zeta(s)$) there is a probability measure ν_T on $\mathbb{R}$ such that*

$$\nu_T(f) := \int_{-\infty}^{\infty} f(x)\ d\nu_T(x) = \lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(\phi^{(T)}(y))\ dy$$

for all bounded continuous functions f on $\mathbb{R}$ where $\phi^{(T)}(y)$ is defined by (42).

Proof. This is identical to Lemma 2.3 of [22] p. 180. Let $N = N(T)$ denote the number of zeros of $\zeta(s)$ to height T. Label the imaginary ordinates of the zeros as $\{\gamma_1, \ldots, \gamma_N\}$. By pairing conjugate zeros $\rho = \frac{1}{2} + i\gamma$ and $\overline{\rho} = \frac{1}{2} - i\gamma$ we have

$$\phi^{(T)}(y) = \sum_{|\gamma|\leq T} \frac{e^{i\gamma y}}{\rho\zeta'(\rho)} = 2\text{Re}\left(\sum_{l=1}^{N} b_l e^{iy\gamma_l}\right)$$

where $b_l = \frac{1}{(\frac{1}{2}+i\gamma_l)\zeta'(\frac{1}{2}+i\gamma_l)}$. Define functions X_T and g on the N-torus $\mathbb{T}^N$ by

$$X_T(\theta_1, \ldots, \theta_N) = 2\text{Re}\left(\sum_{l=1}^{N} b_l e^{2\pi i\theta_l}\right) \text{ and } g(\theta_1, \ldots, \theta_N) = f(X_T(\theta_1, \ldots, \theta_N))\ .$$

We now apply Lemma 7 to the N numbers $\{\frac{\gamma_1}{2\pi}, \ldots, \frac{\gamma_N}{2\pi}\}$ and to the continuous function g. According to Lemma 7 there exists a torus $A \subset \mathbb{T}^N$ such that

$$\lim_{Y\to\infty} \frac{1}{Y} \int_0^Y g\left(y\left(\frac{\gamma_1}{2\pi}, \ldots, \frac{\gamma_N}{2\pi}\right)\right)\ dy = \int_A g(a)\, d\mu\ .$$

The measure $d\mu$ is normalized Haar measure on A. Note that $X_T|_A : A \to \mathbb{R}$ is a random variable and we define a probability measure ν_T on $\mathbb{R}$ by

$$\nu_T(B) = \mu\left(X|_A^{-1}(B)\right) \tag{45}$$

where B is any Borel set. By the change of variable formula, we deduce

$$\lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(\phi^{(T)}(y))\, dy = \int_{\mathbb{R}} f(x)\, d\nu_T(x)$$

and the proof is complete.

Before proceeding, we require some results from probability theory. We say that a real valued function $G(x)$ is a generalized distribution function on $\mathbb{R}$ if it is non-decreasing and right-continuous. Lemma 9(i) will enable us to construct a limiting distribution function from the set $\{\nu_T\}_{T\gg 1}$ constructed in the previous lemma.

Lemma 9 (i) *Let F_n be a sequence of distribution functions. There exists a subsequence $\{F_{n_k}\}$ and a generalized distribution function F such that*

$$\lim_{k\to\infty} F_{n_k}(x) = F(x)$$

at continuity points x of F.
(ii) *Let $\{F_n\}$ be distribution functions and F a generalized distribution function on $\mathbb{R}$ such that F_n converges to F weakly. This is equivalent to*

$$\int_{\mathbb{R}} f(x)\, dF_n(x) \to \int_{\mathbb{R}} f(x)\, dF(x)$$

for all continuous, bounded, real $f(x)$.
(iii) *Let F_n, F be distribution functions with Fourier transforms, $\hat{F}_n$, $\hat{F}$. A necessary and sufficient condition for F_n to converge weakly to F is $\hat{F}_n(t) \to \hat{F}(t)$ for each t.*

Proof. Part (i) is Helly's selection theorem and part (iii) is Levy's theorem. See [2] pp. 344-346 for proofs of (i) and (ii) and pp. 359-360 for (iii).

The next lemma shows that the error term $\epsilon^{(T)}(y)$ in (43) has small mean square. This will be crucial in deducing that a limiting distribution exists for $e^{-\frac{y}{2}}M(e^y)$.

Lemma 10 *Assume the Riemann hypothesis and $J_{-1}(T) \ll T$. For $T \geq 2$ and $Y \geq \log 2$,*

$$\int_{\log 2}^{Y} |\epsilon^{(T)}(y)|^2 dy \ll Y \frac{(\log T)}{T^{\frac{1}{4}}} + 1.$$

Proof. First we will consider the contribution from $E(x,T)$ as defined in (27). Note that

$$\int_{\log 2}^{Y} |e^{-\frac{y}{2}}E(e^y,e^Y)|^2 dy \ll \int_{\log 2}^{Y}\left(\frac{y^2e^y}{e^{2Y}}+\frac{\frac{1}{y^2}e^y}{(e^{2Y})^{1-\epsilon}}+\frac{Y}{e^Y}+\frac{1}{e^y}\right)dy \ll 1$$

and we have

$$\begin{aligned}\int_{\log 2}^{Y}|\epsilon^{(T)}(y)|^2 dy &\ll \int_{\log 2}^{Y}\left|\sum_{T\le\gamma\le e^Y}\frac{e^{i\gamma y}}{\rho\zeta'(\rho)}\right|^2 dy + O(1)\\ &\le \sum_{j=0}^{\lfloor Y\rfloor}\int_{\log 2+j}^{\log 2+j+1}\left|\sum_{T\le\gamma\le e^Y}\frac{e^{i\gamma y}}{\rho\zeta'(\rho)}\right|^2 dy + O(1) \ll Y\frac{\log T}{T^{\frac{1}{4}}}+1\end{aligned} \tag{46}$$

where Lemma 6 has been applied in the last inequality.

By applying Lemmas 7-10, we may now prove Theorem 2.

Proof of Theorem 2. Once again the proof follows Theorem 1.1 of [22] pp. 180-181. Let f be a Lipschitz bounded continuous function that satisfies $|f(x)-f(y)|\le c_f|x-y|$. By an application of the Lipschitz condition, Cauchy-Schwarz, and Lemma 10, we have

$$\begin{aligned}\frac{1}{Y}\int_{\log 2}^{Y} f(\phi(y))\,dy &= \frac{1}{Y}\int_{\log 2}^{Y} f(\phi^{(T)}(y))\,dy + O\left(\frac{c_f}{Y}\int_{\log 2}^{Y}|\epsilon^{(T)}(y)|\,dy\right)\\ &= \frac{1}{Y}\int_{\log 2}^{Y} f(\phi^{(T)}(y))\,dy + O\left(\frac{c_f}{\sqrt{Y}}\left(\int_{\log 2}^{Y}|\epsilon^{(T)}(y)|^2\,dy\right)^{\frac{1}{2}}\right)\\ &= \frac{1}{Y}\int_{\log 2}^{Y} f(\phi^{(T)}(y))\,dy + O\left(c_f\left(\frac{\log T}{T^{\frac{1}{4}}}+\frac{1}{\sqrt{Y}}\right)^{\frac{1}{2}}\right).\end{aligned} \tag{47}$$

By Lemma 8, there is a distribution function ν_T for each $T\ge\gamma_1$ such that

$$\nu_T(f)=\int_{\mathbb{R}} f(x)\,d\nu_T(x)=\lim_{Y\to\infty}\frac{1}{Y}\int_{\log 2}^{Y} f(\phi^{(T)}(y))\,dy\ .$$

Taking limits as $Y \to \infty$ we deduce that

$$\nu_T(f) - O\left(\frac{c_f(\log T)^{\frac{1}{2}}}{T^{\frac{1}{8}}}\right) \leq \liminf_{Y\to\infty} \frac{1}{Y}\int_{\log 2}^{Y} f(\phi(y))dy$$
$$\leq \limsup_{Y\to\infty} \frac{1}{Y}\int_{\log 2}^{Y} f(\phi(y))dy \leq \nu_T(f) + O\left(\frac{c_f(\log T)^{\frac{1}{2}}}{T^{\frac{1}{8}}}\right) . \tag{48}$$

By Lemma 9 (i), we may choose a subsequence ν_{T_k} of these distribution functions ν_T and a generalized distribution function ν such that $\nu_{T_k} \to \nu$ weakly. By Lemma 9 (ii)

$$\nu_{T_k}(f) = \int_{\mathbb{R}} f(x)\, d\nu_{T_k}(x) \to \int_{\mathbb{R}} f(x)\, d\nu(x) = \nu(f) .$$

Replacing T by T_k and letting $k \to \infty$ in (48) , we observe that

$$\lim_{Y\to\infty} \frac{1}{Y}\int_{\log 2}^{Y} f(\phi(y))\, dy = \int_{\mathbb{R}} f(x)\, d\nu(x) .$$

Thus (48) becomes

$$\left|\int_{\mathbb{R}} f(x)d\nu(x) - \int_{\mathbb{R}} f(x)d\nu_T(x)\right| \ll \frac{c_f(\log T)^{\frac{1}{2}}}{T^{\frac{1}{2}}} . \tag{49}$$

However, by applying equation (49) with $f(x) = 1$

$$\left|\int_{\mathbb{R}} d\nu(x) - 1\right| \ll \frac{(\log T)^{\frac{1}{2}}}{T^{\frac{1}{8}}}$$

and we conclude that ν is a distribution function by letting $T \to \infty$.

By assuming the linear independence conjecture, we may provide a concrete description of the Fourier transform of ν in terms of the zeros of $\zeta(s)$. This description will be practical in obtaining finer details regarding $M(x)$.

Proof of Corollary 1. The Fourier transform of ν is

$$\widehat{\nu}(\xi) = \int_{\mathbb{R}} e^{-i\xi t}\, d\nu(t) .$$

In the proof of Theorem 2, we demonstrated $\nu_T \to \nu$ weakly. Hence, by Levy's Theorem (Lemma 9 (iii)), $\widehat{\nu}_T \to \widehat{\nu}$. By Lemmas 7 and 8, we know that ν_T is constructed from normalized Haar measure μ on the torus $A \subset \mathbb{T}^N$ where A is the topological closure of the set $\{ \left(\{y\frac{\gamma_1}{2\pi}\}, \ldots, \{y\frac{\gamma_N}{2\pi}\}\right) \mid y \in \mathbb{R} \}$. However, the assumption of LI implies by the Kronecker-Weyl theorem that $A = \mathbb{T}^N$ and consequently normalized Haar measure $d\mu = d\theta_1 \ldots d\theta_N$ is Lesbesgue measure on $\mathbb{T}^N$. Hence, we observe by (45) and the change of variable formula for integrals that $\widehat{\nu}_T(\xi)$ equals

$$\int_{\mathbb{R}} e^{-i\xi t}\, d\nu_T(t) = \int_{\mathbb{T}^N} e^{-i\xi X_T(\theta)} d\mu = \int_{\mathbb{T}^N} e^{-i\xi \sum_{j=1}^N 2\mathrm{Re}\left(\frac{1}{\rho\zeta^{'}(\rho)} e^{2\pi i\theta_j}\right)}\, d\theta_1 \ldots d\theta_N$$

and it follows that

$$\widehat{\nu}(\xi) = \lim_{T\to\infty} \widehat{\nu}_T(\xi) = \lim_{T\to\infty} \prod_{j=1}^N \int_0^1 e^{-i\xi 2\mathrm{Re}\left(\frac{1}{\rho\zeta^{'}(\rho)} e^{2\pi i\theta}\right)}\, d\theta \ .$$

However the integral within the product equals

$$\int_0^1 e^{-i\xi 2\mathrm{Re}\left(\frac{1}{|\rho\zeta^{'}(\rho)|} e^{2\pi i(\theta-\alpha_\gamma)}\right)}\, d\theta = \int_0^1 e^{-i\xi 2\left(\frac{1}{|\rho\zeta^{'}(\rho)|}\cos 2\pi\theta\right)}\, d\theta$$

where $\alpha_\gamma = \arg(\rho\zeta^{'}(\rho))/2\pi$ and the last step follows by the periodicity of the integrand. From the well-known identity for the $\tilde{J}_0$ Bessel function

$$\int_0^1 e^{is\cos(2\pi x)} dx = \frac{1}{\pi}\int_0^\pi \cos(s\sin x) dx = \tilde{J}_0(s)$$

it follows that

$$\widehat{\nu}(\xi) = \prod_{\gamma>0} \tilde{J}_0\left(\frac{2\xi}{|(\frac{1}{2}+i\gamma)\zeta'(\frac{1}{2}+i\gamma)|}\right) \ .$$

We improve Theorem 1(iv) by following closely Cramér's argument [4].

Proof of Theorem 3. Recall that By Lemma 5, we have the decomposition

$$M(e^y)e^{-\frac{y}{2}} = \phi^{(T)}(y) + \epsilon^{(T)}(y) \tag{50}$$

where

$$\phi^{(T)}(y) = \sum_{|\gamma|\le T} \frac{e^{iy\gamma}}{\rho\zeta'(\rho)}\ ,\ \epsilon^{(T)}(y) = \sum_{T<|\gamma|\le e^Y} \frac{e^{iy\gamma}}{\rho\zeta'(\rho)} + e^{-\frac{y}{2}} E(e^y, e^Y) \tag{51}$$

and $E(x,T)$ is defined in (27). Consequently, we deduce

$$m(Y) := \frac{1}{Y}\int_0^Y \left(\frac{M(e^y)}{e^{\frac{y}{2}}}\right)^2 dy = \frac{1}{Y}\int_0^Y |\phi^{(T)}(y)|^2\, dy + \frac{1}{Y}\int_0^Y |\epsilon^{(T)}(y)|^2\, dy + O\left(\left(\frac{1}{Y}\int_0^Y |\phi^{(T)}(y)|^2\right)^{\frac{1}{2}}\left(\frac{1}{Y}\int_0^Y |\epsilon^{(T)}(y)|^2\right)^{\frac{1}{2}}\right) . \tag{52}$$

As the second integral was treated in Lemma 10, we concentrate on the first integral in (52). Squaring out the terms in $\phi^{(T)}(y)$, we deduce

$$\int_1^Y |\phi^{(T)}(y)|^2\, dy = (Y-1)\sum_{\gamma\le T}\frac{2}{|\rho\zeta'(\rho)|^2} + \sum_{\substack{0<|\gamma|,|\lambda|<T\\ \gamma\neq\lambda}}\frac{1}{(\frac{1}{2}+i\gamma)\zeta'(\rho)(\frac{1}{2}+i\lambda)\zeta'(\rho')}\int_1^Y e^{iy(\gamma+\lambda)}\, dy . \tag{53}$$

In the second sum, the contribution from pairs (γ,λ) with the same sign is

$$\sum_{0<\gamma,\lambda\le T}\frac{1}{\gamma|\zeta'(\rho)|\lambda|\zeta'(\rho')|(\gamma+\lambda)} \ll \left(\sum_{0<\gamma<T}\frac{1}{\gamma^{\frac{3}{2}}|\zeta'(1/2+i\gamma)|}\right)^2 \ll 1 .$$

Here we have applied $x+y\ge 2\sqrt{xy}$ and then evaluated the resulting sum by a partial summation similar to Lemma 1(iii). Also note that

$$\sum_{\gamma<T}\frac{1}{|\rho\zeta'(\rho)|^2} = \beta - \sum_{\gamma>T}\frac{1}{|\rho\zeta'(\rho)|^2} = \beta + O\left(\frac{1}{T}\right)$$

where β is defined by (17) and the error term is obtained by Lemma 1(ii). We have now shown that

$$\frac{1}{Y}\int_1^Y |\phi^{(T)}(y)|^2\, dy = \beta + O\left(\frac{1}{T}+\frac{1}{Y}+\frac{\Sigma(T)}{Y}\right)$$

where

$$\begin{aligned}\Sigma = \Sigma(T,Y) &= \sum_{\substack{0<\gamma,\lambda<T\\ \gamma\neq\lambda}}\frac{1}{\gamma|\zeta'(\rho)|\lambda|\zeta'(\rho')|}\min\left(Y,\frac{1}{|\gamma-\lambda|}\right) dy \\ &= \Sigma_1(T,Y)+\Sigma_2(T,Y) .\end{aligned} \tag{54}$$

The first sum is the contribution from those pairs with $|\gamma-\lambda|\le 1$ and the second sum consists of the complementary terms. We have

$$\begin{aligned}\Sigma_2(T,Y) \le \sum_{0<\gamma<T}\frac{1}{\gamma|\zeta'(\rho)|}\Bigg(\sum_{\lambda<\gamma^{\frac12}} + \sum_{\gamma^{\frac12}<\lambda<\gamma-\gamma^{\frac12}} + \sum_{\gamma-\gamma^{\frac12}<\lambda<\gamma-1} \\ + \sum_{\gamma+1<\lambda<\gamma+\gamma^{\frac12}} + \sum_{\gamma+\gamma^{\frac12}<\lambda<2\gamma} + \sum_{2\gamma<\lambda}\Bigg)\frac{1}{\lambda|\zeta'(\rho')||\gamma-\lambda|}\ . \\ = \sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6\end{aligned} \tag{55}$$

By a calculation completely analogous to the calculation in Lemma 6, we obtain

$$\sigma_1 \le \sum_{0<\gamma<T}\frac{1}{\gamma(\gamma-\gamma^{\frac12})|\zeta'(\rho)|} \ll \sum_{\gamma>0}\frac{1}{\gamma^2|\zeta'(\rho)|} \ll 1\ ,$$

$$\sigma_2 \le \sum_{0<\gamma<T}\frac{1}{\gamma^{\frac32}|\zeta'(\rho)|}\left(\sum_{\lambda\ge\gamma^{\frac12}}\frac{1}{|\lambda\zeta'(\rho')|^2}\right)^{\frac12}(\gamma\log\gamma)^{\frac12} \ll \sum_{\gamma>0}\frac{(\log\gamma)^{\frac12}}{\gamma^{\frac54}|\zeta'(\rho)|} \ll 1\ ,$$

$$\sigma_3 \le \sum_{0<\gamma<T}\frac{1}{\gamma|\zeta'(\rho)|}\left(\sum_{\gamma-\gamma^{\frac12}<\lambda}\frac{1}{|\rho'\zeta'(\rho')|^2}\right)^{\frac12}(\gamma^{\frac12}\log\gamma)^{\frac12} \ll \sum_{\gamma>0}\frac{(\log\gamma)^{\frac12}}{\gamma^{\frac54}|\zeta'(\rho)|} \ll 1$$

where we have applied Lemma 1(iii) in each of these cases. The computation of σ_4 is analogous to σ_3 and the computation of σ_5 is analogous to σ_2

$$\sigma_4 \ll \sum_{\gamma>0}\frac{(\log\gamma)^{\frac12}}{\gamma^{\frac54}|\zeta'(\rho)|} \ll 1\ ,\ \ \sigma_5 \ll \sum_{\gamma>0}\frac{(\log\gamma)^{\frac12}}{\gamma^{\frac54}|\zeta'(\rho)|} \ll 1\ .$$

For the final sum we obtain

$$\sigma_6 \le \sum_{0<\gamma<T} \frac{1}{\gamma|\zeta'(\rho)|} \sum_{k\ge 1} \frac{1}{(2^k-1)\gamma} \sum_{2^k\gamma\le\lambda\le 2^{k+1}\gamma} \frac{1}{\lambda|\zeta'(\frac{1}{2}+i\lambda)|}$$

$$\ll \sum_{0<\gamma<T} \frac{1}{\gamma^2|\zeta'(\rho)|} \sum_{k\ge 1} \frac{1}{2^k} \left(\sum_{2^k\gamma<\lambda<2^{k+1}\gamma} \frac{1}{|\lambda\zeta'(\rho')|^2} \right)^{\frac{1}{2}} ((2^k\gamma)\log(2^k\gamma))^{\frac{1}{2}} \quad (56)$$

$$\ll \sum_{0<\gamma<T} \frac{1}{\gamma^2|\zeta'(\rho)|} \sum_{k\ge 1} \frac{(\log(2^k\gamma))^{\frac{1}{2}}}{2^k} \ll \sum_{\gamma>0} \frac{\sqrt{\log\gamma}}{\gamma^2|\zeta'(\rho)|} \ll 1$$

and we deduce that

$$\Sigma_2(T,Y) \ll \sum_{\gamma>0} \frac{1}{\gamma^{\frac{5}{4}}|\zeta'(\rho)|} \ll 1\ .$$

Thus

$$\frac{1}{Y}\int_1^Y |\phi^{(T)}(y)|^2\, dy = \beta + O\left(\frac{1}{T} + \frac{1}{Y} + \frac{\Sigma_1}{Y}\right) \quad (57)$$

where

$$\Sigma_1 = \Sigma_1(T,Y) = \sum_{\substack{0<\gamma,\lambda<T \\ |\gamma-\lambda|\le 1}} \frac{1}{\gamma|\zeta'(\rho)|\lambda|\zeta'(\rho')|} \min\left(Y, \frac{1}{|\gamma-\lambda|}\right)\ .$$

In addition, we know by Lemma 10 that

$$\frac{1}{Y}\int_1^Y |\epsilon^{(T)}(y)|^2\, dy \ll \frac{\log T}{T^{\frac{1}{4}}} + \frac{1}{Y}\ . \quad (58)$$

Let $0<\eta<1$. Choose and fix $T=T_\eta$ large enough to make the $O(\frac{1}{T})$ in (57) and $O(\frac{\log T}{T^{\frac{1}{4}}})$ in (58) less than η. Choose Y_1 large enough such that if $Y \ge Y_1$ the $O(Y^{-1})$ expressions in (57) and (58) are less than η. Choose Y_η to satisfy

$$Y_\eta = \max\left(\frac{1}{\eta \min_{0<\gamma\le T_\eta} |\gamma'-\gamma|}, Y_1\right) \quad (59)$$

where if γ denotes an imaginary ordinate of a zero of $\zeta(s)$ then γ' is the next largest one (note that $\gamma' > \gamma$ since $J_{-1}(T) \ll T$ implies all zeros are simple). We will consider $Y \geq Y_\eta$ and analyze Σ_1. Decompose $\Sigma_1(T_\eta, Y) = \Sigma_{11}(T_\eta, Y) + \Sigma_{12}(T_\eta, Y)$ where the first sum contains pairs (γ, λ) with $|\gamma - \lambda|^{-1} \leq \eta Y$ and the second sum contains the complementary set. Therefore

$$\begin{aligned} \Sigma_{11}(T_\eta, Y) &\leq \eta Y \sum_{\gamma < T_\eta} \frac{1}{\gamma|\zeta'(\rho)|} \sum_{\gamma-1<\lambda<\gamma+1} \frac{1}{\lambda|\zeta'(\rho')|} \\ &\ll \eta Y \sum_{\gamma<T_\eta} \frac{1}{\gamma|\zeta'(\rho)|} \left(\sum_{\lambda>\gamma-1} \frac{1}{|\lambda\zeta'(\rho')|^2} \right)^{\frac{1}{2}} (\log\gamma)^{\frac{1}{2}} \ll \eta Y \sum_{\gamma<T_\eta} \frac{(\log\gamma)^{\frac{1}{2}}}{\gamma^{\frac{3}{2}}|\zeta'(\rho)|} \end{aligned} \tag{60}$$

and we have $\Sigma_{11}(T_\eta, Y) \leq c_3 \eta Y$ for $c_3 > 0$ by Lemma 1(iii). The second sum consists of pairs (γ, λ) such that

$$\eta Y < |\gamma - \lambda|^{-1} < \left(\min_{0<\gamma\leq T_\eta} |\gamma' - \gamma| \right)^{-1}$$

which implies

$$Y < \left(\eta \min_{0<\gamma\leq T_\eta} |\gamma' - \gamma| \right)^{-1} \leq Y_\eta$$

and thus this second sum is empty. Consequently, $\Sigma_{12}(T_\eta, Y) = 0$ and thus $\Sigma_1(T_\eta, Y)Y^{-1} \leq c_3\eta$. This demonstrates that

$$\left| \frac{1}{Y} \int_1^Y |\phi^{(T_\eta)}(y)|^2 \, dy - \beta \right| \leq (2 + c_3)\eta \quad \textit{and} \quad \frac{1}{Y} \int_1^Y |\epsilon^{(T_\eta)}(y)|^2 \, dy \leq 2\eta \tag{61}$$

if $Y \geq Y_\eta$. By (52) and (61) we deduce

$$\left| \frac{1}{Y} \int_1^Y \left(\frac{M(e^y)}{e^{\frac{y}{2}}} \right)^2 dy - \beta \right| \leq (4 + c_3)\eta + c_4\sqrt{\eta} \ll \sqrt{\eta}$$

if $Y \geq Y_\eta$ and hence the proof is finished.

4 Applications of LI

The goal of this section is to study the true order of $M(x)$. We will attempt to find the size of the tail of the probability measure ν associated to $\phi(y) = e^{-\frac{y}{2}} M(e^y)$. The tool we employ in studying tails of ν are probability results due to Montgomery [18]. We will need to assume the linear independence conjecture for our analysis. Consider a random variable X, defined on the infinite torus $\mathbb{T}^\infty$ by

$$X(\underline{\theta}) = \sum_{k=1}^{\infty} r_k \sin(2\pi\theta_k)$$

where $\underline{\theta} = (\theta_1, \theta_2, \ldots) \in \mathbb{T}^\infty$ and $r_k \in \mathbb{R}$ for $k \geq 1$. This is a map $X : \mathbb{T}^\infty \to \mathbb{R} \cup \{\infty\}$. Under the assumption $\sum_{k\geq 1} r_k^2 < \infty$, Komolgorov's theorem ensures that X converges almost everywhere. In addition, $\mathbb{T}^\infty$ possesses a canonical probability measure P. Attached to the random variable X is the distribution function ν_X defined by

$$\nu_X(x) = P(X^{-1}(-\infty, x)).$$

For these random variables, Montgomery [18] pp. 14-16 proved the following results.

Lemma 11 *Let $X(\underline{\theta}) = \sum_{k=1}^{\infty} r_k \sin 2\pi\theta_k$ where $\sum_{k=1}^{\infty} r_k^2 < \infty$. For any integer $K \geq 1$,*
(i)

$$P\left(X(\underline{\theta}) \geq 2\sum_{k=1}^{K} r_k\right) \leq \exp\left(-\frac{3}{4}\left(\sum_{k=1}^{K} r_k\right)^2 \left(\sum_{k>K} r_k^2\right)^{-1}\right)$$

(ii) and if δ is so small that $\sum_{r_k>\delta}(r_k - \delta) \geq V$, then

$$P(X(\underline{\theta}) \geq V) \geq \frac{1}{2}\exp\left(-\frac{1}{2}\sum_{r_k>\delta} \log\left(\frac{\pi^2 r_k}{2\delta}\right)\right) .$$

Observe that the linear independence assumption implies that the limiting distribution ν constructed in Theorem 2 equals ν_X where X is the random variable

$$X(\underline{\theta}) = \sum_{\gamma>0} \frac{2}{|\rho\zeta'(\rho)|} \sin(2\pi\theta_\gamma) .$$

In the above sum γ ranges over the positive imaginary ordinates of the zeros of $\zeta(s)$. We abbreviate notation by setting $r_\gamma = \frac{2}{|\rho\zeta'(\rho)|}$. By assuming the linear independence conjecture, we may now study ν via the random variable X. By applying Lemma 11, we can estimate the tails of the limiting distribution ν. Define

$$a(T) := \sum_{\gamma<T} r_\gamma = \sum_{\gamma<T} \frac{2}{|\rho\zeta'(\rho)|} \quad \textit{and} \quad b(T) := \sum_{\gamma\geq T} r_\gamma^2 = \sum_{\gamma\geq T} \frac{4}{|\rho\zeta'(\rho)|^2} .$$

By Lemma 1, the conjectured formulae are

$$a(T) \asymp (\log T)^{\frac{5}{4}} \quad and \quad b(T) \asymp \frac{1}{T} \ . \tag{62}$$

Assuming these bounds we prove upper and lower bounds for the tail of the limiting distribution ν. Let V be a large parameter. Our goal is to find upper and lower bounds for the tail of the probability distribution, namely

$$\nu([V,\infty)) := \int_V^\infty d\nu(x) = P(X(\underline{\theta}) \geq V) \ .$$

4.1 An upper bound for the tail

Choose T such that $a(T^-) < V \leq a(T)$. Note that T is the ordinate of a zero. We have the chain of inequalities

$$(\log T)^{\frac{5}{4}} \ll a(T^-) < V \leq a(T) \ll (\log T)^{\frac{5}{4}}. \tag{63}$$

This implies $\log T \asymp V^{\frac{4}{5}}$ and we have by Lemma 11(i), (62), and (63),

$$\begin{aligned} P\left(X(\underline{\theta}) \geq c_5 V\right) &\leq P\left(X(\underline{\theta}) \geq 2a(T)\right) \leq \exp\left(-\frac{3}{4}a(T)^2 b(T)^{-1}\right) \\ &\leq \exp\left(-c_6 V^2 T\right) \leq \exp\left(-c_6 V^2 e^{(c_7 V)^{\frac{4}{5}}}\right) \end{aligned} \tag{64}$$

for effective constants c_5, c_6, and c_7. By altering the constants, we obtain the upper bound

$$P(X(\underline{\theta}) \geq V) \ll \exp(-\exp(c_7 V^{\frac{4}{5}})) \ .$$

4.2 A lower bound for the tail

This is a more delicate analysis than the upper bound. We now apply Lemma 11(ii). As before, V is considered fixed and large. We would like to choose δ small enough such that

$$\sum_{r_\gamma > \delta} (r_\gamma - \delta) \geq V \ . \tag{65}$$

Introduce the notation S_δ and N_δ such that

$$S_\delta = \{\gamma \mid r_\gamma > \delta \ \} \ and \ N_\delta = \# S_\delta$$

where γ ranges over positive imaginary ordinates of zeros of $\zeta(s)$. Let ϵ be a small fixed number. Note that RH implies $|\zeta'(\rho)| \ll |\rho|^{\epsilon}$. Thus,

$$\delta < \frac{2}{c_8|\rho|^{1+\epsilon}} \Longrightarrow \delta < \frac{2}{|\rho\zeta'(\rho)|}$$

for some effective constant c_8. However, notice that

$$\delta < \frac{2}{c_8|\rho|^{1+\epsilon}} \Longleftrightarrow |\rho| \leq \left(\frac{2}{c_8\delta}\right)^{\frac{1}{1+\epsilon}}$$

and since $|\rho| \ll \gamma$, we obtain

$$\gamma \leq c_9 \left(\frac{1}{\delta}\right)^{\frac{1}{1+\epsilon}} \Longrightarrow \delta < \frac{2}{|\rho\zeta'(\rho)|}.$$

We deduce from Riemann's zero counting formula that there are at least

$$c_9 \left(\frac{1}{\delta}\right)^{\frac{1}{1+\epsilon}} \log\left(\frac{1}{\delta}\right) + O\left(\left(\frac{1}{\delta}\right)^{\frac{1}{1+\epsilon}}\right)$$

zeros in the set S_δ. We will now find an upper bound for N_δ. Gonek [8] has defined the number

$$\Theta = \text{l.u.b.}\{\ \theta \mid |\zeta'(\rho)|^{-1} \ll |\gamma|^{\theta},\ \forall\rho\ \}.$$

However $J_{-1}(T) \ll T$ implies $\Theta \leq \frac{1}{2}$. Gonek [8] has speculated that $\Theta = \frac{1}{3}$. Choose $\epsilon < \frac{1}{2}$. This implies that if $\gamma \in S_\delta$ then

$$\delta < \frac{2}{|\rho\zeta'(\rho)|} \ll \frac{|\rho|^{\frac{1}{2}+\epsilon}}{|\rho|} \leq \frac{1}{|\gamma|^{\frac{1}{2}-\epsilon}}\ .$$

We deduce that if $\gamma \in S_\delta$ then $\gamma \ll \left(\frac{1}{\delta}\right)^{2+\epsilon}$ where ϵ has been taken smaller. We conclude that $N_1(\delta) \leq N_\delta \leq N_2(\delta)$ where

$$N_1(\delta) = c_9\left(\frac{1}{\delta}\right)^{1-\epsilon} \ \textit{and}\ N_2(\delta) = c_{10}\left(\frac{1}{\delta}\right)^{2+\epsilon}\ .$$

We are trying to determine a condition on δ so that (65) will be satisfied. Note that

$$\sum_{r_\gamma > \delta}(r_\gamma - \delta) \geq \sum_{\gamma \leq N_1}(r_\gamma - \delta)\ .$$

Before evaluating the second sum, observe that

$$\delta N_1 = c_9 \delta^{\epsilon} \to 0 \ as \ \delta \to 0.$$

We will choose δ as a function of V and as $V \to \infty$ we have $\delta \to 0$. However, by (62)

$$\begin{aligned} \sum_{\gamma \le N_1} (r_\gamma - \delta) &= 2 \sum_{\gamma \le N_1} \frac{1}{|\rho \zeta'(\rho)|} - \delta \sum_{\gamma \le N_1} 1 \\ &\ge c_{11} (\log N_1)^{\frac{5}{4}} - \frac{\delta N_1}{2\pi} \log N_1 + O(\delta N_1) \ge c_{12} (\log N_1)^{\frac{5}{4}} \end{aligned} \tag{66}$$

where $0 < c_{12} < c_{11}$. The last inequality holds for N_1 sufficiently large. Hence, choosing $N_1 = \exp((V/c_{12})^{\frac{4}{5}})$ implies

$$\sum_{r_\gamma > \delta} (r_\gamma - \delta) \ge \sum_{\gamma \le N_1} (r_\gamma - \delta) \ge V \ .$$

Thus if δ satisfies

$$c_9 \left(\frac{1}{\delta} \right)^{1-\epsilon} = \exp((V/c_{12})^{\frac{4}{5}})$$

(i.e. $\delta = c_{13} \exp(-c_{14} V^{\frac{4}{5}})$) we have satisfied (65). By this choice of δ, Lemma 11(ii) implies

$$P(X(\underline{\theta}) \ge V) \ge \frac{1}{2} \exp \left(-\frac{1}{2} \sum_{r_\gamma > \delta} \log \left(\frac{\pi^2 r_\gamma}{2\delta} \right) \right) \ . \tag{67}$$

An upper bound of the sum will provide a lower bound for the tail. Note that $\frac{1}{|\rho\zeta'(\rho)|} \to 0$ under the assumption that all zeros are simple (see [23] pp. 377-380). Consequently $\frac{1}{|\rho\zeta'(\rho)|} \le c_{15}$ and we obtain

$$\sum_{r_\gamma > \delta} \log \left(\frac{\pi^2 r_\gamma}{2\delta} \right) \le \sum_{\gamma \le N_2} \log \left(\frac{\pi^2 c_{15}}{\delta} \right) \ll \log \left(\frac{\pi^2 c_{15}}{\delta} \right) N_2 \log N_2 \ .$$

By definition of $N_2(\delta)$ and our choice of δ it follows that

$$\sum_{r_\gamma > \delta} \log \left(\frac{\pi^2 r_\gamma}{2\delta} \right) \ll V^{\frac{4}{5}} \exp(c_{16} V^{\frac{4}{5}}) \ll \exp(c_{17} V^{\frac{4}{5}}) \ . \tag{68}$$

By (67) and (68) we arrive at the lower bound

$$P(X(\underline{\theta}) \geq V) \gg \exp\left(-\exp(c_{18}V^{\frac{4}{5}})\right) .$$

Putting this all together establishes the following highly conditional result.

Corollary 12 *The Riemann hypothesis, the linear independence conjecture,*

$$\sum_{0<\gamma<T} \frac{1}{|\rho\zeta'(\rho)|} \asymp (\log T)^{\frac{5}{4}} \ , \ \textit{and} \ \sum_{\gamma>T} \frac{1}{|\rho\zeta'(\rho)|^2} \asymp \frac{1}{T}$$

imply

$$\exp(-\exp(\tilde{c}_1 V^{\frac{4}{5}})) \ll \nu([V,\infty)) \ll \exp(-\exp(\tilde{c}_2 V^{\frac{4}{5}}))$$

for effective constants $\tilde{c}_i > 0$ *for* $i = 1 \ldots 2$.

4.3 Speculations on the lower order of $M(x)$

We now examine the effect that bounds for the tail of the probability measure have on the lower order of $M(x)$. Note that the following argument is only heuristic. We begin with the lower bound

$$\exp\left(-\exp(\tilde{c}_1 V^{\frac{4}{5}})\right) \ll \nu([V,\infty)) .$$

Assuming the linear independence conjecture, the Riemann hypothesis, and $J_{-1}(T) \ll T$, we have

$$\lim_{Y\to\infty} \frac{1}{Y} \text{meas}\{y \in [0,Y] \mid M(e^y) \geq e^{y/2}V \ \} = \nu([V,\infty)) . \qquad (69)$$

We will assume that the convergence of (69) is sufficiently uniform in Y. By (69) there exists a function $f(V)$, such that

$$\frac{1}{Y} \text{meas}\{y \in [0,Y] \mid M(e^y) \geq e^{y/2}V \ \} \gg \exp\left(-\exp(\tilde{c}_1 V^{\frac{4}{5}})\right)$$

if V is sufficiently large and $Y \geq f(V)$. We now choose Y as a function of V by the equation

$$V = \left(\frac{\theta}{\tilde{c}_1}\right)^{\frac{5}{4}} (\log_2 Y)^{\frac{5}{4}} \text{ or } Y = g(V) = \exp\left(\exp\left(\frac{\tilde{c}_1}{\theta} V^{\frac{4}{5}}\right)\right)$$

for $0 < \theta < 1$. If we had $g(V) \geq f(V)$ then it would follow that for large Y,

$$\exp\left(\log Y - (\log Y)^{\theta}\right) \ll \text{meas}\{y \in [0, Y] \mid M(e^y)e^{-\frac{y}{2}} \geq \alpha(\log_2 Y)^{\frac{5}{4}} \}$$

where $\alpha = \left(\frac{\theta}{\tilde{c}_1}\right)^{\frac{5}{4}}$. Since $0 < \theta < 1$ the left hand side of the equation approaches infinity as $Y \to \infty$. In turn, this implies that there exists an increasing sequence of real numbers y_m such that $y_m \to \infty$ and

$$\frac{M(e^{y_m})}{e^{\frac{y_m}{2}}} \geq \alpha \left(\log_2 y_m\right)^{\frac{5}{4}}.$$

Suppose by way of contradiction, that the above inequality is false. That is, there exists a real number u_0 such that

$$\frac{M(e^{y})}{e^{\frac{y}{2}}} < \alpha \left(\log_2 y\right)^{\frac{5}{4}}$$

for all $y \geq u_0$. Then we have that

$$\begin{aligned} &\text{meas}\{y \in [0, Y] \mid M(e^y) \geq \alpha\, e^{\frac{y}{2}}(\log_2 Y)^{\frac{5}{4}} \} \\ &= \text{meas}\{y \in [0, u_0] \mid M(e^y) \geq \alpha\, e^{\frac{y}{2}}(\log_2 Y)^{\frac{5}{4}} \} \end{aligned} \tag{70}$$

since if $u_0 \leq y \leq Y$ then

$$\frac{M(e^{y})}{e^{\frac{y}{2}}} \leq \alpha \left(\log_2 y\right)^{\frac{5}{4}} \leq \alpha \left(\log_2 Y\right)^{\frac{5}{4}}.$$

Thus we deduce that

$$\exp(\log Y - (\log Y)^{\theta}) \leq u_0 \ll 1 \tag{71}$$

which is a contradiction for large enough Y. Hence, our original assumption is false and we obtain

$$\limsup_{y\to\infty} \frac{M(e^y)}{e^{\frac{y}{2}}(\log\log y)^{\frac{5}{4}}} \geq \left(\frac{\theta}{\tilde{c}_1}\right)^{\frac{5}{4}} .$$

Letting $\theta \to 1$ we have

$$\limsup_{y\to\infty} \frac{M(e^y)}{e^{\frac{y}{2}}(\log\log y)^{\frac{5}{4}}} \geq \left(\frac{1}{\tilde{c}_1}\right)^{\frac{5}{4}} .$$

We now consider the upper bound. Arguing in the same fashion we have

$$\nu([V,\infty)) = P(\underline{\theta} \in \mathbb{T}^\infty \mid X(\underline{\theta}) \geq V) \ll \exp\left(-\exp(\tilde{c}_2 V^{\frac{4}{5}})\right) \quad . \tag{72}$$

For $n \in \mathbb{N}$ define the event

$$A_n = \left\{ \underline{\theta} \in \mathbb{T}^\infty \mid X(\underline{\theta}) \geq \left(\frac{1}{\tilde{c}_2} \log\log(n(\log n)^\theta) \right)^{\frac{5}{4}} \right\}$$

with $\theta > 1$. Therefore we have by (72)

$$\sum_{n=n_0}^{\infty} P(A_n) \ll \sum_{n=n_0}^{\infty} \frac{1}{n(\log n)^\theta} \ll 1$$

for n_0 a sufficiently large integer. By the Borel-Cantelli lemma, it follows that

$$P(A_n \ infinitely \ often) = 0 \tag{73}$$

which suggests that if the convergence of (69) is sufficiently uniform then

$$\limsup_{y\to\infty} \frac{M(e^y)}{e^{\frac{y}{2}}(\log\log y)^{\frac{5}{4}}} \leq \left(\frac{1}{\tilde{c}_2}\right)^{\frac{5}{4}} \quad .$$

Hence, our analysis shows that the bounds

$$\exp\left(-\exp(\tilde{c}_1 V^{\frac{4}{5}})\right) \ll \nu([V,\infty)) \ll \exp\left(-\exp(\tilde{c}_2 V^{\frac{4}{5}})\right)$$

suggest

$$\left(\frac{1}{\tilde{c}_1}\right)^{\frac{5}{4}} \leq \limsup_{y\to\infty} \frac{M(e^y)}{e^{\frac{y}{2}}(\log\log y)^{\frac{5}{4}}} \leq \left(\frac{1}{\tilde{c}_2}\right)^{\frac{5}{4}} \quad .$$

Thus we arrive at an argument for the conjecture (20).

By the preceding heuristic analysis and Theorems 1-3 we hope to have demonstrated that the size of $M(x)$ depends in a crucial way on the sizes of the discrete moments $J_{-\frac{1}{2}}(T)$ and $J_{-1}(T)$.

References

[1] P.T. Bateman et al., *Linear relations connecting the imaginary parts of the zeros of the zeta function*, in *Computers in Number Theory*, eds. A.O.L. Atkin and B.J. Birch, Academic Press, New York, 1971, 11-19.

[2] Billingsley, P., *Probability and measure*, John Wiley and Sons, New York, 1986.

[3] J.B. Conrey, *More than two-fifths of the zeros of the Riemann zeta function are on the critical line*, J. Reine. Angew. Math. **399** (1989), 1-26.

[4] Harald Cramér, *Ein Mittelwertsatz in der Primzahltheorie*, Math. Z. **12** (1922), 147-153.

[5] Harold Davenport, *Multiplicative number theory, Second Edition*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1980.

[6] P.X. Gallagher, *Some consequences of the Riemann hypothesis*, Acta Arith. **37** (1980), 339-343.

[7] S.M. Gonek, *On negative moments of the Riemann zeta-function*, Mathematika **36** (1989), 71-88.

[8] S.M. Gonek, *The second moment of the reciprocal of the Riemann zeta-function and its derivative*, Notes from a talk at MSRI, summer 1999.

[9] C.B. Haselgrove, *A disproof of a conjecture of Polya*, Mathematika **5** (1958), 141-145.

[10] D.R. Heath Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith., **60** (4) (1992), 389-415.

[11] D. Hejhal, *On the distribution of* $log|\zeta'(\frac{1}{2}+it)|$, in *Number Theory, Trace Formula and Discrete Groups*, eds. Aubert, Bombieri, and Goldfeld, Academic Press, San Diego, 1989, 343-370.

[12] Edmund Hlawka, *The theory of uniform distribution*, AB Academic Publishers, Berkhamsted, 1984.

[13] C.P. Hughes, J.P. Keating, and Neil O'Connell, *Random matrix theory and the derivative of the Riemann zeta function*, Proceedings of the Royal Society: A **456** (2000), 2611-2627.

[14] A.E. Ingham, *The distribution of prime numbers*, Stechert-Hafner Service Agency, New York, 1964.

[15] A.E. Ingham, *On two conjectures in the theory of numbers*, Amer. J. Math., **64** (1942), 313-319.

[16] Aleksandar Ivić, *The theory of the Riemann zeta function with applications*, John Wiley and Sons, New York, 1985.

[17] J.P. Keating and N.C. Snaith, *Random matrix theory and* $\zeta(\frac{1}{2}+it)$, Commun. Math. Phys., **214** (2000), 57-89.

[18] H.L. Montgomery, *The zeta function and prime numbers*, Proceedings of the Queen's Number Theory Conference, 1979, Queen's Univ., Kingston, Ont., 1980, 1-31.

[19] N. Ng, *Limiting distributions and zeros of Artin L-functions*, Ph.D. Thesis, University of British Columbia, fall 2000.

[20] A.M. Odlyzko and H.J.J. te Riele, *Disproof of the Mertens conjecture*, J. Reine Angew. Math., **357** (1985), 138-160.

[21] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin, 1957.

[22] Michael Rubinstein and Peter Sarnak, *Chebyshev's Bias*, Journal of Experiment. Math. **3** (1994), 173-197.

[23] E.C. Titchmarsh, *The theory of the Riemann zeta function, Second Edition*, Oxford University Press, New York, 1986.

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