Department of Mathematics and Computer Science

Probability and Statistics Challenge Problems 4 - Solution

Destiny?

Heathcliffe and Catherine maintain that destiny brought them together. One Friday they each decided to eat their lunch by the fountain in the park and, as fate would have it, they met. Suppose Heathcliffe and Catherine each arrive at the fountain in the park at some random time between 12:00pm and 1:00pm. They each remain at the fountain for 20 minutes, or until 1:00pm, whichever comes first. What is the probability that they will meet?

Consider two cases: Case I - Heathcliffe arrives first and Case II - Catherine arrives first. The probability that they meet is the same regardless of who arrives first, so we can just focus on Case I and double whatever probability we obtain in order to get our final answer.

Suppose Heathcliffe arrives between 12:00 and 12:40. This represents 40 of the total 60 minutes, so the probability that he arrives during this time is 40/60=2/3. Now Catherine must arrive during the following 20 minutes. This happens with probability 20/60=1/3. So, the probability of them meeting under these circumstances is (2/3)(1/3)=2/9. ***Because their arrival times are independent we can just multiply the individual probabilities in order to arrive at the joint probability.***

Now suppose Heathcliffe arrives between 12:40 and 1:00. This represents 20 of the total 60 minutes, so the probability that he arrives during this time is 20/60=1/3. Now Catherine must arrive before 1:00, but how many minutes is that? If Heathcliffe is equally likely to arrive anytime between 12:40 and 1:00, then on average he will arrive at 12:50. Therefore on average Catherine will have 10 minutes in which to arrive. This happens with probability 10/60=1/6. So, the probability of them meeting under these circumstances is (1/3)(1/6)=1/18.

So if Heathcliffe arrives first, the probability that they will meet is 2/9 + 1/18 = 5/18.
As stated above this means that if Catherine arrives first, the probability that they will meet is also 2/9 + 1/18 = 5/18.
So the probability that Heathcliffe and Catherine will meet is 5/18 + 5/18 = 5/9.

More formally, if X represents the arrival time for Heathcliffe and Y represents the arrival time for Catherine, where X and Y are each allowed to take values between 0 and 60 representing times between 12:00 and 1:00, then the probability density for each is $f_X(x) = f_Y(y) = 1/60$. This means that they are equally likely to arrive at any time during the hour.

Let Heathcliffe arrive between 12:00 and 12:40 and Catherine arrives within 20 minutes. Then the probability that they meet is given by the following integral

$$\int_0^{40} \int_x^{x+20} \left(\frac{1}{60}\right) \left(\frac{1}{60}\right) \, dy \, dx = \frac{2}{9}$$

Now suppose Heathcliffe arrives between 12:40 and 1:00 and Catherine arrives before 1:00pm. Then the probability that they meet is given by the following integral

$$\int_{40}^{60} \int_x^{60} \left(\frac{1}{60}\right) \left(\frac{1}{60}\right) \, dy \, dx = \frac{1}{18}$$

Problem posted at http://www.cs.uleth.ca/∼sheriff/statistics/challenge.htm.

Problems are open to all University of Lethbridge students. Solutions may be submitted to Dennis Connolly (C566) or John Sheriff (D510). If no one is in the office, just put the solution under the door. Please include your name and student number. Students with correct solutions will be acknowledged and prizes will be awarded at the end of the semester based upon performance during the semester.

If you like problem solving you should consider trying
Math Problem of the Week (see Kerri Webb in C572).

You may also be interested in the following competitions
ACM Programming Contest (see Howard Cheng in C570)
Putnam Exam (see Sean Legge in C518)