Drawing K_n in Three Dimensions
with Two Bends per Edge

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Drawing K_n in Three Dimensions with Two Bends per Edge *

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Abstract. A three-dimensional grid drawing of a graph places the vertices at integer-valued grid points and draws edges as straight line segments between adjacent vertices, with no pair of edges intersecting. A fundamental result is that the complete graph, K_n, can be drawn in $\Theta(n^3)$ volume in this model. In this paper we show that K_n can be drawn in 3D in $\Theta(n^2)$ volume if two bends per edge are permitted.

1 Introduction and Definitions

In the field of Graph Drawing there are several models of 3-dimensional (3D) drawings of graphs. Perhaps the most natural is to represent vertices as points at integer-valued grid points and represent edges as straight line segments between adjacent vertices with no pair of edges intersecting (in 3D). The **volume** of such a drawing is typically defined in terms of a smallest bounding box containing the drawing and with sides orthogonal to one of the coordinate axes. If such a box B has width w, length l and height h, then we refer to the **dimensions** of B as $(w+1) \times (l+1) \times (h+1)$ and define the volume of B, **Vol(**B**)** as $(w+1) \cdot (l+1) \cdot (h+1)$. Note that vol($B$) then exactly measures the number of grid points in (or on) the box B. A fundamental result shown by Cohen, Eades, Lin and Ruskey [3], is that it is possible to draw *any* graph in this model and indeed the complete graph K_n is drawable within a bounding box of volume $\Theta(n^3)$. Restricted classes of graphs can be drawn in this model in smaller asymptotic volume. For example, Calamonieri and Sterbini [2] showed that 2-, 3-, and 4-colorable graphs can be drawn

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in $O(n^2)$ volume, and Felsner, Liotta and Wismath [4] showed that outerplanar graphs can be drawn in $O(n)$ volume. Establishing tight volume bounds for planar graphs remains an open problem.

In 2-dimensional graph drawing, the effect of allowing bends in edges has been well studied – see for example Kaufmann and Wiese [6]. However the consequences of allowing bends in 3 dimensions has received little attention, other than the lower bound result of Bose, Czyzowicz, Morin, and Wood [1], who showed that the number of edges in a graph provides an asymptotic lower bound on the volume regardless of the number of bends permitted. In this paper we show that for K_n this lower bound of $\Omega(n^2)$ is indeed achievable with only two bends per edge.

We draw the complete graph K_n by placing vertices at integer-valued grid points and drawing each edge as a sequence of at most three connected line segments (referred to as *subedges* for convenience). The two bend points of the edge drawing are also required to be at integer grid points. No pair of drawn edges intersect.

2 Construction – Two Bends per Edge

Our drawing of K_n fits inside a bounding box of dimensions $2n \times (n+1) \times 3$. All the vertices v_i are placed on the plane $Z = 0$, at $(i, i, 0)$. Vertex v_j will be connected to v_i $(j > i)$ with 3 line segments via intermediate *bend* (grid) points $D_{j,i}$ and $E_{j,i}$. All the D bend points will lie in the plane $Z = 2$ and the E bend points in the plane $Z = 1$. More specifically,

$$D_{j,i} = (j + i, 1 + j, 2)$$
$$E_{j,i} = (i, j, 1)$$

Figure 1 shows an orthographic projection of the drawing onto the $Z = 0$ plane – only the edges from v_6 to all other vertices are displayed.

Theorem 1. *The construction produces a valid 3-dimensional drawing of K_n in volume $O(n^2)$ and with two bends per edge.*

Proof.

We first show that no pair of drawn edges intersect.

Group the subedges into three sets:

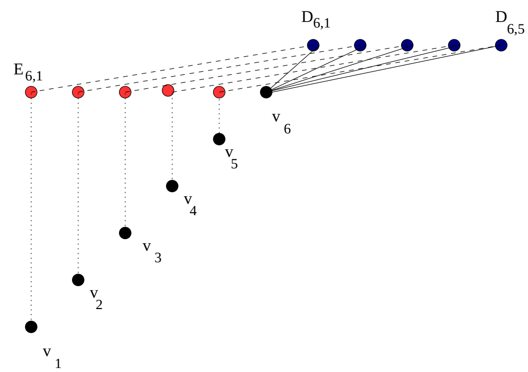

Fig. 1. Projection onto the $Z = 0$ plane (only edges from v_6 are displayed)

1. those from v_j to $D_{j,i}$.
2. those from $D_{j,i}$ to $E_{j,i}$.
3. those from $E_{j,i}$ to v_i.

(Type 1 vs. 3) Note that all the subedges of the 1st type lie entirely on one side of the plane $Y = X$ and the subedges of the 3rd type lie on the other side of that plane.

(Type 2 vs. 3) All the subedges of the 2nd type lie entirely between the planes $Z = 2$ and $Z = 1$. All the subedges of the 3rd type lie entirely between the planes $Z = 1$ and $Z = 0$.

(Type 1 vs. 2): Consider the 2 following lines for a particular value of j: l_1 passes through the $D_{j,i}$ and l_2 through the $E_{j,i}$. These 2 lines are parallel and define a plane P and thus all type 2 subedges (from j) lie in the plane P. Plane P intersects the plane $Z = 0$ forming a line at $Y = j - 1$ (i.e. all points on this line are of the form $(x, j - 1, 0)$) and thus $v_j = (j, j, 0)$ is not on plane P. Since all type 1 subedges are incident on v_j, they can intersect the plane P at only one point, $D_{j,i}$ and thus cannot intersect any of the type 2 subedges.

Thus there can be no intersection between a pair of subedges of different types.

Consider the type 1 subedges. For any particular value of j these subedges lie entirely between the planes $Y = j$ and $Y = j + 1$ and thus for different values of j cannot intersect. For a fixed value of j, the subedges have one endpoint at v_j and the other on a distinct point on a line in the plane $Y = j + 1$ and thus do not intersect (except at v_j).

Similarly the type 2 subedges lie in disjoint groups that do not intersect. Namely, these subedges lie between the planes $Y = j$ and $Y = j + 1$, and are pairwise parallel.

The type 3 subedges lie in groups in the planes $X = i$. Within each such group, the subedges all have one endpoint at $v_i = (i, i, 0)$ and the other endpoint on the plane $Z = 1$ with distinct Y values and thus no pair intersects.

By construction, the bend point and vertices are all located at distinct grid points so indeed there are no intersections. Finally, the volume bound is easily established as the drawing lies within a bounding box of size $2n \times (n + 1) \times 3$.

□

3 One Bend Per Edge

Having established that K_n can be drawn in $O(n^2)$ volume with two bends per edge and recalling that $\Omega(n^3)$ volume is required if no bends are permitted, it is natural to consider the case when one bend per edge is allowed. Unfortunately, we have no asymptotic answer. In this section we outline a drawing of K_n within a bounding box of dimensions $n(n-1)/2 \times n/2 \times 3$ (i.e. with a volume of $3(n^3 - n^2)/4$). This represents a constant factor improvement over the drawing proposed by Pach, Thiele and Tóth [8] which contains no bends and has a volume of $4n^3$.

A simple but useful insight noted by for example Calamoneri and Sterbini [2], involves skew lines[1].

Property: Let (a, b) and (c, d) be two line segments. If the two lines through $\overline{ac}$ and $\overline{bd}$ are skew, then (a, b) and (c, d) do not intersect.

Our initial construction places vertices on one line and all bend points on a skew line as follows. We place the vertices on the line parametrically defined as $\overline{v} = (0, u, 0), u \in \mathcal{R}$; specifically, v_i is located at (0,i,0). The bend points $B_{i,j}$ will join vertex v_i to v_j and are placed on the line $\overline{B} = (u, 1, 1), u \in \mathcal{R}$. We require $\binom{n}{2}$ consecutive

[1] Two lines in 3D are *skew* if they are not parallel and do not intersect.

bend points on $\overline{B}$. Specifically,

$$B_{i,j} = ((2ni - i^2 - 2n + i)/2 + (j - i), 1, 1), \forall i, j = 1..n, i < j$$

See Figure 2.

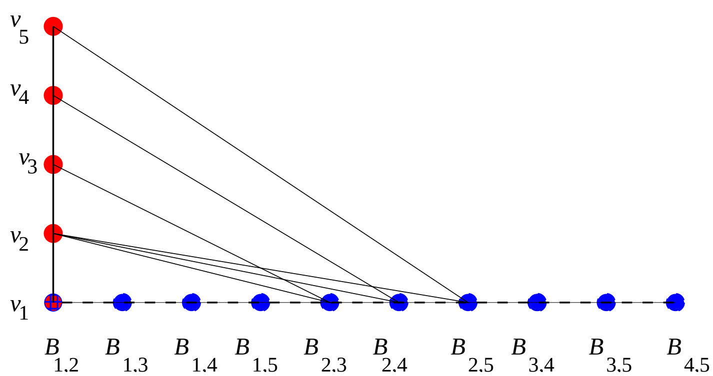

Fig. 2. Projection onto the $Z = 0$ plane (only edges from v_2 are displayed)

Since the two lines $\overline{v}$ and $\overline{B}$ are skew, no pair of subedges can intersect. Furthermore, note that *any* assignment of bend points to distinct grid points on $\overline{B}$ results in a valid drawing of K_n and furthermore, the bounding box has dimensions $\binom{n}{2} \times n \times 2$.

To achieve the claimed grid size, we modify the above construction slightly, placing the vertices on one of two parallel lines, namely, $v_i = (0, i, 0), \forall i = 1, ..., n/2$ and at $v_i = (2, i - n/2, 0), \forall i = n/2 + 1, ..., n$. The subedges incident on these two lines can easily be seen to be separated in space and hence do not intersect.

There are further minor improvements that can be applied to this basic drawing strategy, however in the absence of an asymptotic upper bound they will not be presented.

4 Conclusions and Open Problems

We have shown that allowing bends (with bends at grid points) can reduce the volume of a 3-dimensional drawing of the complete graph, K_n, to $\Theta(n^2)$. The tightness of this bound follows from a result of Bose, Czyzowicz, Morin, and Wood [1], who showed that the number of edges in a graph provides an asymptotic lower bound on the volume regardless of the number of bends.

A tight upper bound on the volume of one-bend drawings of K_n remains an open problem.

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