Tighter Bounds on the Genus of Nonorthogonal Polyhedra Built from Rectangles

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Abstract

We prove that there is a polyhedron with genus 6 whose faces are orthogonal polygons (equivalently, rectangles) and yet the angles between some faces are not multiples of 90°, so the polyhedron itself is not orthogonal. On the other hand, we prove that any such polyhedron must have genus at least 3. These results improve the bounds of Donoso and O'Rourke [4] that there are nonorthogonal polyhedra with orthogonal faces and genus 7 or larger, and any such polyhedron must have genus at least 2. We also demonstrate nonoverlapping one-piece edge-unfoldings (nets) for the genus-7 and genus-6 polyhedra.

1 Introduction

Donoso and O'Rourke [4] consider two questions, the first of which was posed by Biedl, Lubiw, and Sun [3]:

> **Question 1.** If an orthogonal polygon is creased along orthogonal chords (parallel to the edges) and folded into a polyhedron, must it be an orthogonal polyhedron?

> **Question 2.** If a polyhedron's faces are orthogonal polygons (equivalently, rectangles), must it be an orthogonal polyhedron?

The difference between these two questions is that Question 1 demands that the polyhedron has a *net*, a nonoverlapping one-piece unfolding by cutting along edges. This restriction reduces the class of candidate polyhedra; see [1, 2].

The answers to these questions turn out to depend on the allowed genus of the polyhedron. Donoso and O'Rourke [4] proved that for the originally intended case of genus-0 polyhedra, and even for genus-1 polyhedra, the answers are both YES. On the other hand, they demonstrated a nonorthogonal polyhedron with rectangular faces and genus 7, answering Question 2 with a NO for genus 7 and larger. They also modified this polyhedron to answer Question 1 with a NO for genus 7 and larger.

Our results. We extend these results in 3 ways:

1. We extend the lower bound to show that the answers to both questions are YES for genus-2 polyhedra.
2. We show that the original genus-7 polyhedron from [4] has a net, so it too answers Question 1 with a NO.
3. We give a genus-6 polyhedron that answers both questions with a NO.

2 Net for Genus 7

We begin with the most tangible result. The original genus-7 example from [4, Fig. 2], reproduced in Figure 1, is a skeletal octahedron with its edges "thickened" into thin triangular prisms. Figure 3 shows a net for this polyhedron, proving that it settles Question 1 without further modification.

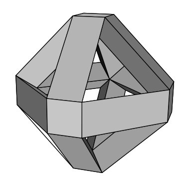

Figure 1: Genus 7 example.

3 Example with Genus 6

Figure 2 shows our polyhedron with genus 6 that answers both questions with a NO. As in Figure 1, we start with a skeletal cube whose edges are thickened into triangular prisms. This construction leaves triangular "holes" at the corners, visible from the center of the cube. To fill these holes, we add 8 triangular prisms meeting at two points in the center to form two degree-4 vertices as in Figure 2. The result is a polyhedron, shown in Figure 2 (left), with genus 11. To reduce the genus to 6, we add a thin layer around five of the faces.

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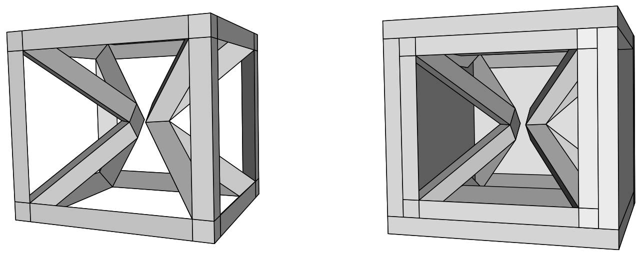


Figure 2: (Left) Base polyhedron with genus 11. (Right) Polyhedron with genus 6 after adding a thin box layer on the outside of all but one face.

Figure 4 shows a net for this polyhedron.

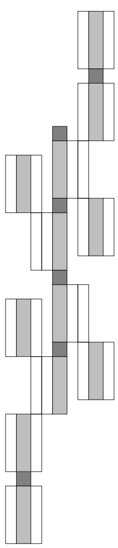


Figure 3: Net for Figure 1.

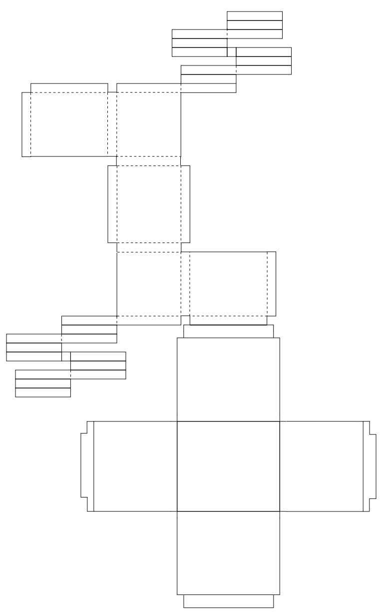


Figure 4: Net for Figure 2.

4 Genus Must Be At Least 3

Our proof that a nonorthogonal polyhedron with orthogonal faces must have genus at least 3 works as follows. First we develop a general upper bound on the number of vertices of the "nonorthogonal part" of a polyhedron in terms of its genus. In particular, for genus 2, the bound is 8. Then we prove that a nonorthogonal polyhedron with orthogonal faces must have at least 9 vertices in its nonorthogonal part, and hence must have genus more than 2.

4.1 Basic Definitions and Counts

Following [4], we think of an edge of a polyhedron as colored *green* (good) if its dihedral angle (angle between the two incident faces) is a multiple of 90°, and *red* (bad) otherwise. Define the graph G_r (the *nonorthogonal part*) by starting with all of the red edges, then removing any degree-two vertices, coalescing the incident edges, and finally focusing attention to just a single connected component. Any edges that were coalesced were already collinear [4, Lem. 7], so the graph G_r remains embedded in $\mathbb{R}^3$, defining angles and faces (combinatorial faces, which do not necessarily lie in a plane).

The following two lemmas relate G_r to the original polyhedron. For their proofs, we need the notion of an *orthogonal path* around a vertex v [4]: a path of circular arcs on the intersection of the polyhedron with a small sphere centered at v, such that every turn along the path is by $\pm 90°$. Let p' denote the projection of a point p onto this sphere.

Lemma 1 *The face angles of G_r are multiples of* 90°.

Proof: Consider a face angle at v made by two edges v, v_0 and v, v_1 in G_r. By the definition of G_r, there is an orthogonal path around v from v_0' to v_1'. By [4, Lem. 4], the great arc length between v_0' and v_1' is a multiple of 90°, and this arc length is precisely the face angle. □

Lemma 2 *Let e_0, e_1, e_2 be consecutive edges around a common vertex in G_r. Then the dihedral angle between the plane e_0, e_1 and the plane e_1, e_2 is not a multiple of* 90°.

Proof: Let v be the common endpoint of the e_i's, and let w_i be the other endpoint of e_i. As in the previous lemma, there is an orthogonal path around v from w_0' to w_1'. By [4, Lem. 5], the great arc between w_0' and w_1' meets this orthogonal path at w_0' and w_1' with two 90°-multiple angles. Similarly, there is an orthogonal path from w_1' to w_2', and the great arc between w_1' and w_2' meets this orthogonal path at two 90°-multiple angles. Thus, the two great arcs meet at w_1' with a 90°-multiple angle precisely if the two orthogonal paths meet at w_1' with a 90°-multiple angle. The former angle is the dihedral angle between planes e_0, e_1 and e_1, e_2, and the latter angle is the dihedral angle at e_1 in the original polyhedron. Because e_1 is red, the angles must not be 90° multiples. □

Let V, E, and F denote the number of vertices, edges, and faces in G_r. Let $D_{\neq 5}$ denote the number of vertices with degree not equal to 5. Because no vertex of G_r can have degree one or three [4, Lem. 6, 8], and we have eliminated all degree-2 vertices,

$$D_{\neq 5} = D_{=4} + D_{\geq 6}, \tag{1}$$

where $D_{=4}$ is the number of vertices of degree 4 and $D_{\geq 6}$ is the number of vertices of degree at least 6.

4.2 Special Angles

To get a better handle on $D_{=4}$, we introduce the notions of "flat" and "special" angles. Call a face angle *flat* if it is 180°.

Lemma 3 *Any degree-4 vertex v in G_r is incident to two faces each of which have a flat angle at v.*

Proof: Consider a degree-4 vertex v in G_r whose incident edges are e_0, e_1, e_2, e_3. By [4, Lem. 9], these edges form an orthogonal '+' in $\mathbb{R}^3$. We claim that e_0 and e_2 bound a common face, forming a flat angle at v, and symmetrically e_1 and e_3 bound a common face [Figure 5(a)]. The only other possible type of face incident to v is one that bounds e_i and e_{i+1} for some i (modulo 4). In fact, we must have all such faces [Figure 5(b)], or else there would be an edge incident to only one face, contradicting the definition of a polyhedron. But then the dihedral angles between these faces, as in Lemma 2, would each be 180°, making the edges green, not red. Thus the claim holds. □

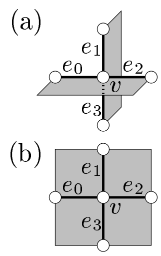


Figure 5: Possible faces around a degree-4 vertex.

Call an angle *special* if it is both flat and incident to a degree-4 vertex as in the lemma. Thus, the number s of special angles is given by

$$s = 2D_{=4}. \tag{2}$$

Lemma 4 *Every face of G_r has at least 4 nonflat angles, and hence at least 4 nonspecial angles.*

Proof: By Lemma 1, every face angle is a multiple of 90°. Such a closed polygonal chain in 3D must have at least 4 bends: there is no triangle whose angles are all ±90°. □

See also [4, Lem. 14].

Because every face of degree more than 4 has a fifth angle which is either special or nonspecial, this lemma implies that

$$F_{\geq 5} \leq F_{\geq 1s} + F_{\geq 5\neg s}, \tag{3}$$

where $F_{\geq 5}$ is the number of faces with degree at least 5, $F_{\geq 1s}$ is the number of faces with at least 1 special angle, and $F_{\geq 5\neg s}$ is the number of faces with at least 5 nonspecial angles.

4.3 Upper Bound on Vertices

We start with two relationships:

Lemma 5 *In G_r, $2E - 2D_{=4} \geq 4F + F_{\geq 5\neg s}$.*

Proof: The total number of angles is $2E$, and each degree-4 vertex contributes two special angles. So $2E - 2D_{=4}$ is the number of nonspecial angles. On the other hand, there are at least 4 nonspecial angles per face (Lemma 4), and at least one more per face with 5 or more nonspecial angles. □

Lemma 6 *In G_r, $2E \geq 5V - D_{=4} + D_{\geq 6}$.*

Proof: The sum of the vertex degrees is $2E$. As mentioned earlier, each vertex has degree at least 4. Thus, we can lower bound the sum of the vertex degrees by counting each vertex as if it had degree 5 ($5V$), then decrementing the sum for each vertex of degree 4 ($D_{=4}$), then incrementing the sum for each vertex of degree at least 6 ($D_{\geq 6}$). □

We are now ready to prove the bound:

Lemma 7 *In G_r, $V \leq 8(g-1) - \max\{D_{\neq 5}, F_{\geq 5}/2\}$.*

Proof: By Euler's Theorem,

$$F \geq 2 - 2g - V + E. \tag{4}$$

Substituting this bound on F into Lemma 5, we obtain

$$2E - 2D_{=4} \geq 8 - 8g - 4V + 4E + F_{\geq 5\neg s}.$$

Combining the E terms, negating, and rewriting, we obtain

$$2E \leq -2D_{=4} + 8(g-1) + 4V - F_{\geq 5\neg s}.$$

Combining this equation with Lemma 6, we have

$$5V - D_{=4} + D_{\geq 6} \leq -2D_{=4} + 8(g-1) + 4V - F_{\geq 5\neg s},$$

which simplifies to

$$V \leq -D_{\geq 6} - D_{=4} + 8(g-1) - F_{\geq 5\neg s}.$$

By Equation 1 and dropping the $F_{\geq 5\neg s}$ term, we obtain the first bound:

$$V \leq -D_{\neq 5} + 8(g-1).$$

On the other hand, by Equation 2, and because by definition $F_{\geq 1s} \leq s$, we can obtain

$$V \leq -D_{\geq 6} - F_{\geq 1s}/2 + 8(g-1) - F_{\geq 5\neg s}.$$

By Equation 3, and dropping the $D_{\geq 6}$ term and half of the $F_{\geq 5\neg s}$ term, we obtain the second bound:

$$V \leq -F_{\geq 5}/2 + 8(g-1). \quad \square$$

For genus 2, Lemma 7 becomes

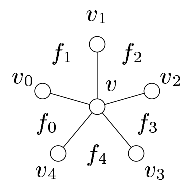
$$\max\{D_{\neq 5}, F_{\geq 5}/2\} \leq 8 - V. \tag{5}$$

4.4 Too Many Vertices for Genus 2

We now use this bound to prove the final result:

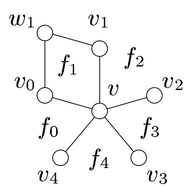
Theorem 8 *There are no nonorthogonal polyhedra with orthogonal faces and genus at most 2.*

Proof: Because every vertex in G_r has degree at least 4, $V \geq 5$. By Equation 5, $D_{\neq 5} \leq 3$. Thus, there must be a degree-5 vertex, call it v. Let v_0, v_1, v_2, v_3, v_4 denote the neighbors of v in clockwise order. Let f_i denote the face bounded by v_{i-1}, v, v_i. (Indices are modulo 5.)



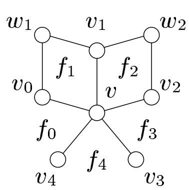
Because $v, v_0, v_1, v_2, v_3, v_4$ are 6 distinct vertices, $V \geq 6$. Substituting this bound into Equation 5, we obtain that $F_{\geq 5} \leq 4$. Thus, out of the 5 distinct faces f_0, f_1, f_2, f_3, f_4, at least one f_i has degree 4. (There are no faces with degree less than 4 by Lemma 4.) By Lemma 1, the angles of this quadrilateral face f_i must be multiples of 90°, and any such face must be planar and hence a rectangle.

Let w_i be the fourth vertex of f_i, so that v_{i-1}, v, v_i, w_i are the vertices of the face. We claim that w_i is different from each of v_0, v_1, v_2, v_3, v_4:



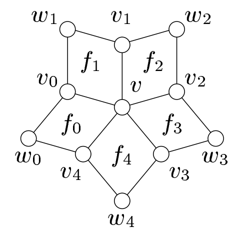
1. w_i must be different from v_{i-1} and v_i because together they make up a face.
2. w_i must be different from v_{i+1}, because the angle v_i, v, v_{i+1} is a multiple of 90° by Lemma 1, but the angle v_i, v, w_i between a side and a diagonal of the rectangle f_i must be strictly between 0 and 90°.
3. Symmetrically, w_i must be different from v_{i-2}.
4. Finally, w_i must be different from v_{i+2}. Otherwise, angles v_i, v, v_{i+1} and $v_{i+1}, v, v_{i+2} = w_i$ are both multiples of 90° by Lemma 1. Thus, the edge v, v_{i+1} is orthogonal to rectangle f_i, and hence orthogonal to both v, v_i and v, v_{i-1}. But then the dihedral angle at edge v, v_i as in Lemma 2 is a multiple of 90°, contradicting that edge v, v_i belongs to G_r.

Thus, $v, v_0, v_1, v_2, v_3, v_4, w_i$ are seven distinct vertices, so $V \geq 7$. Applying Equation 5, we find that $F_{\geq 5} \leq 2$. Thus, out of the 5 distinct faces f_0, f_1, f_2, f_3, f_4, at least two consecutive faces f_i and f_{i+1} have degree 4. Again we define w_i and w_{i+1} to be the fourth vertices of f_i and f_{i+1}, respectively.



Applying the same reasoning as above to each of w_i and w_{i+1}, w_i and w_{i+1} must be distinct from $v, v_0, v_1, v_2, v_3, v_4$. Furthermore, we claim that w_i and w_{i+1} must be distinct from each other. Otherwise, rectangles f_i and f_{i+1} would share three vertices, and hence be identical, implying that v_{i-1} and v_{i+2} are identical, contradicting that they are distinct neighbors of v.

Now we have 8 distinct vertices, $v, v_0, v_1, v_2, v_3, v_4, w_i, w_{i+1}$, so $V \geq 8$. Applying Equation 5, we obtain that $F_{\geq 5} = 0$. Thus, all of the faces in G_r must have degree 4, in particular all of the f_i's. Define w_i to be the fourth vertex of each f_i. Applying the previous arguments, w_0, w_1, w_2, w_3, w_4 are distinct from v_0, v_1, v_2, v_3, v_4, and each w_i is distinct from w_{i-1} and w_{i+1}. (Although w_i might equal w_{i+2} or w_{i+3}.) Furthermore, we cannot have all the odd-index w_i's equal, and all the even-index w_i's equal, because the number of w_i's is odd. Thus, the set $\{w_0, w_1, w_2, w_3, w_4\}$ must have at least 3 distinct members.



So there must be at least 9 distinct vertices in the set $\{v, v_0, v_1, v_2, v_3, v_4, w_0, w_1, w_2, w_3, w_4\}$. But Equation 5 tells us that $V \leq 8$, a contradiction. □

5 Conclusion

The main open problem is to settle whether there are nonorthogonal polyhedra with orthogonal faces and genus between 3 and 5.

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