Asymptotically efficient triangulations of the d-cube *

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1 Introduction

Triangulating the regular d-cube $I^d = [0,1]^d$ in a "simple" way has many applications, like solving differential equations by finite element methods or calculating fixed points. See, for example, [7]. Determining the smallest number of simplices needed has brought special attention both from a theoretical point of view and from an applied one (see [6, Section 14.5.2] for a recent survey).

Before going on, let us clarify that when we speak about *triangulations* of a polytope P of dimension d we mean decompositions of P into d-simplices (i) using as vertices only the vertices of P, and (ii) intersecting face to face (i.e., forming a geometric simplicial complex). If the second condition is not fulfilled, we call them *simplicial dissections* of P. The number of d-simplices of a triangulation or dissection T will be called its *size* and denoted $|T|$.

A general method to obtain the smallest triangulation of a polytope P of dimension d as the optimal integer solution of a linear program is described in [2]. For the d-cube, the direct application of this method is impossible in practice beyond dimension 4 or 5. With a somewhat similar method but simplifying the equations using simmetries of the cube, Anderson and Hughes [1] have calculated the smallest size of a triangulation of the 6-cube and the 7-cube, in a computational *tour-de-force* which involved a problem with 1,456,318 variables and *ad hoc* ways of decomposing the system into smaller subsystems.

In order to compare sizes of triangulations of cubes in different dimensions, Todd [7] defines the *efficiency* of a triangulation T of the d-cube to be the number $(|T|/d!)^{\frac{1}{d}}$. This is at most 1, since $|T| \leq d!$. Following [6, Section 14.5.2], let us denote by ϕ_d and ρ_d the smallest size and efficiency, respectively, among all possible triangulations of the cube of dimension d.

Their values are known up to dimension 7; see Table 1. The number ϕ_d is known as the *simplexity* of the d-cube and, obviously, $\rho_d = (\phi_d/d!)^{1/d}$.

In [4] (see also [6], pages 283-284), Haiman observes that a triangulation of I^{k+l} with $t_k t_l \binom{k+l}{k}$ simplices can be constructed from given triangulations of I^k and I^l with t_k and t_l simplices respectively. With this, one easily concludes:

Theorem 1.1 (Haiman) *For every k and l, ${\rho_{k+l}}^{k+l} \leq {\rho_k}^k {\rho_l}^l$. Thus, the sequence $(\rho_i)_{i\in\mathbb{N}}$ converges and*

$$\lim_{i\to\infty} \rho_i \leq \rho_d \qquad \forall d \in \mathbb{N}.$$

What is the limit of this sequence? In particular, is it positive or is it zero? The known values of ρ_d (up to $d = 7$) form a decreasing sequence, as shown in Table 1, but it is not even known whether this occurs in general. Putting together this result with the values in Table 1 we conclude that $\lim_{i\to\infty} \rho_i \leq \rho_7 = 0.840$.

dim.	Smallest size (ϕ_d)	Smallest efficiency (ρ_d)
1	1	1
2	2	1
3	5	.941
4	16	.904
5	67	.890
6	308	.868
7	1493	.840

Table 1: Smallest size and efficiency of triangulations of I^d.

Here we propose a method to obtain efficient triangulations of a product polytope $P \times Q$ starting from a triangulation of Q and another of $P \times \Delta^{m-1}$, where Δ^{m-1} denotes a simplex of dimension $m-1$ and m is any relatively small number. We apply this with P being a small-dimensional cube and Q a high-dimensional one, iteratively, what allows to obtain asymptotically efficient triangulations of arbitrarily high-dimensional cubes from any (efficient) triangulation of $I^l \times \Delta^{m-1}$, with l and m small.

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The asymptotic efficiency of the triangulations obtained of course depends on how good our triangulation of $I^l \times \Delta^{m-1}$ is. Finding the triangulation of $I^l \times \Delta^{m-1}$ which is optimal for our purposes reduces to an integer programming problem, similar to finding the smallest triangulation of that polytope (actually, it is the same system of equations with a different objective function). Using the linear programming software CPLEX [5], we have solved the system for some values of l and m. The best triangulation we have found is one of $I^3 \times \Delta^2$, with which we obtain

$$\lim_{i\to\infty} \rho_i \leq \sqrt[3]{\frac{44/3}{27}} \approx 0.8159$$

2 Overview of the method and results

We start describing Haiman's original idea, which is related to our method. Let T_k and T_l be triangulations of the regular cubes I^k and I^l, respectively. The product $T_k \times T_l$ of the two triangulations gives a decomposition of the cube $I^k \times I^l = I^{k+l}$ into $|T_k| \cdot |T_l|$ subpolytopes, each of them isomorphic to the product of simplices $\Delta^k \times \Delta^l$. It is well known that every triangulation of $\Delta^k \times \Delta^l$ has size $\binom{k+l}{k}$. Hence, refining $T_k \times T_l$ in an arbitrary way one gets a triangulation of I^{k+l} of size $|T_k| \cdot |T_l| \cdot \binom{k+l}{k}$.

Our method is, in a way, similar. Starting from a triangulation of I^{n-1} and another one of $I^l \times \Delta^{m-1}$, we get one of $I^{l+(n-1)}$ with the following general method to triangulate $P \times Q$ starting from a triangulation of Q and another one of $P \times \Delta^{m-1}$:

(1) We first show how to obtain triangulations of $P \times \Delta^{n-1}$ from triangulations of $P \times \Delta^{m-1}$, where $n-1 = \dim(Q)$ is supposed to be much bigger than $m-1$. We call our triangulations *multi-staircase triangulations*.

(2) A triangulation of Q induces, as in Haiman's method, a decomposition of $P \times Q$ into polytopes isomorphic to $P \times \Delta^{n-1}$. Each of them can be triangulated using the previous paragraph, although this in principle only gives a dissection of $P \times Q$; if we want a triangulation, we have to apply (1) to all the polytopes $P \times \Delta^{n-1}$ in a compatible way. For this we use a random partition of the vertices of Q into m colors.

In order to analyze the efficiency of our method we introduce the following concepts. Their interest will be apparent in Theorem 2.2, which is our main theoretical result. Compare its statement with Theorem 1.1:

Definition 2.1 Let P be a polytope of dimension l and let τ be a simplex of dimension $l+m-1$ in $P \times \Delta^{m-1}$. Let $\{v_1, \ldots, v_m\}$ be the vertices of Δ^{m-1}. Then, τ, understood as a vertex set, decomposes as $\tau = \tau_1 \cup \cdots \cup \tau_m$ with $\tau_i \subset P \times \{v_i\}$.

(i) We define the sequence $(|\tau_1|-1, \ldots, |\tau_m|-1)$ to be the *type* of the simplex τ. The *weight* of a simplex τ of type $(t_1, \ldots, t_m)$ is the number $\frac{1}{\prod_{i=1}^m t_i!}$.

(ii) The *weighted size* of a triangulation T of $P \times \Delta^{m-1}$ is $\sum_{\tau \in T} \mathrm{weight}(\tau)$, and the *weighted efficiency* is

$$\sqrt[l]{\frac{\sum_{\tau\in T} \mathrm{weight}(\tau)}{m^l}}$$

Using this method to triangulate $I^{l+(n-1)} = I^l \times I^{n-1}$ starting from a triangulation of I^{n-1} and another one of $I^l \times \Delta^{m-1}$ we conclude:

Theorem 2.2 *If there is a triangulation T_0 of $I^l \times \Delta^{m-1}$ with weighted efficiency ρ_0, then:*

$$\forall \epsilon > 0, \forall n \geq \frac{lm\rho_0}{\epsilon},\ {\rho_{l+(n-1)}}^{l+(n-1)} \leq {\rho_{n-1}}^{n-1}(\rho_0+\epsilon)^l$$

Thus,

$$\lim_{i\to\infty} \rho_i \leq \rho_0.$$

Hence, we are led to study the smallest weighted efficiency of triangulations of $I^l \times \Delta^{m-1}$. Let us denote this number $\rho_{l,m}$. To compute it we use the *universal polytope* as described in [2], which gives a bijection between the triangulations of a polytope P and integer solutions of a certain system of linear inequalities derived from the oriented matroid of P. The weighted size turns out to be a linear functional on this universal polytope. Hence, finding $\rho_{l,m}$ is restated as an integer linear programming problem.

In order to apply our method we have used the program UNIVERSAL BUILDER by Jesús A. de Loera and Samuel Peterson [3]. Given as input the vertices of P, the program generates the linear system of equations defining the universal polytope of P. The output is a file readable by the linear programming software CPLEX [5]. We have created a routine that generates our particular objective function. Table 2 shows the results obtained in the cases we have been able to solve. The computation of $\rho_{3,3}$ involved a system with 74,400 variables, whose resolution by CPLEX took 37 hours of CPU on a SUN UltraSparc.

Note that $\rho_{l,1} = \rho_l$. The fact that $\rho_{1,m} = 1$ for every m reflects that every triangulation of the prism $I \times \Delta^{m-1}$ is unimodular. In the case of $\rho_{2,m}$ we prove that the smallest weighted efficiency is always $\sqrt{\lceil 3m^2/4 \rceil / m^2}$. That is to say, $\sqrt{3/4}$ for even m and $\sqrt{3/4} + o(m)$ for odd m.

$l \setminus m$	1	2	3
1	1	1	$\sqrt[3]{\frac{5}{6}} \approx 0.941$
2	1	$\sqrt{\frac{3}{4}} \approx 0.866$	$\sqrt[3]{\frac{14}{24}} \approx 0.8355$
3	1	$\sqrt{\frac{7}{9}} \approx 0.8819$	$\sqrt[3]{\frac{44}{81}} \approx 0.8159$
≥ 3	1	$\geq \sqrt{\frac{3}{4}}$	

Table 2: Values of $\rho_{l,m}$ for $l \leq 2$ or for $l = 3$ and $m \leq 2$.

Thus, since $\lim_{i\to\infty} \rho_i \leq \rho_{3,3} \approx 0.8159$, we conclude that, asymptotically, the d-cube can be triangulated with $0.8159^d d!$ simplices instead of the $0.840^d d!$ achievable before.

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