Drawing Series-Parallel Graphs on a Box*

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Abstract

A box is a restricted portion of the three-dimensional integer grid consisting of four parallel lines of infinite length placed one grid unit apart. A box-drawing of a graph is a straight-line crossing-free drawing where vertices are located at integer grid points along the four lines. It is known that some planar graphs with tri-connected components do not admit a box-drawing. This paper shows that even structurally simpler planar graphs, namely series-parallel graphs, are not box-drawable in general. On the positive side, it is proved that every series-parallel graph whose vertices have maximum degree at most three is box-drawable. A drawing algorithm is presented that computes a box drawing of a 3-planar series-parallel graph in optimal time and with optimal volume.

1 Introduction

A straight line drawing (also known as a *Fary embedding*) is a drawing of a graph in which vertices are points and edges are straight line segments which are pairwise crossing-free. In a straight line *grid drawing* the additional constraint of having the vertices drawn as points with integer coordinates is imposed. The problem of determining the area required to draw classes of graphs in the 2D Euclidean space has been widely investigated in literature [2] and an $\Omega(n^2)$ bound is known for planar graphs [1] (here n is the number of vertices). Suprisingly, not so many results are known about compact straight line grid drawings of planar graphs in 3D space. Namely, the basic question on whether every planar graph admits a straight line drawing in the Euclidean 3D space with vertices at integer grid points and $o(n^2)$ volume is still open.

In [6] the problem is tackled from a novel perspective: Given a grid ϕ such that ϕ is a proper subset of the integer 3D space, which graphs admit a straight line crossing-free drawing with vertices located at the grid points of ϕ? If ϕ is chosen so that it has volume V, then a volume bound of V can be determined for any class of graph drawable on ϕ. In particular, [6] focuses on two restricted integer 3D grids, the *box* and the *prism*. A *box* is a grid consisting of four parallel lines of infinite length placed one grid unit apart from each other. A *prism* is a box with one fewer grid lines. In [6] it is shown that all outerplanar graphs can be drawn on a prism. This result gives the first algorithm to compute a crossing-free straight-line 3D drawing with linear volume for a non-trivial family of planar graphs. Moreover it is shown that there exist planar graphs that cannot be drawn on the prism and that even a box does not support all planar graphs. However, while the examples of non prism-drawable graphs of [6] have a very simple combinatorial structure (they are series-parallel graphs), the examples of non box-drawable graphs are more complex since they contain triconnected components.

This paper studies compact 3D straight line drawings of series-parallel graphs. Our contribution can be summarized as follows.

- The question of whether all planar graphs without triconnected components (*i.e.* series-parallel graphs) admit a box-drawing is studied. The question is answered in the negative by exhibiting a series-parallel graph that does not admit a box-drawing.

- It is shown that all series-parallel graphs with maximum vertex degree at most three are box-drawable. A drawing algorithm is presented that computes a box-drawing of a 3-planar series-parallel graph. The algorithm runs in $O(n)$ time and computes drawings of $O(n)$ volume. Series-parallel graphs are a classical subject of investigation in graph drawing [2]. A very recent work by Dujmović, Morin, and Wood shows that every series-parallel graph has a stright-line 3D drawing of $O(n \log^2 n)$ volume [4].

The remainder of the paper is organized as follows. In Section 2 some basic definitions are given. An example of a graph that can not be drawn on the box is presented in Section 3. The proof that every 3-planar

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series-parallel graph is box-drawable is given in Section 4.

2 Preliminaries

In this section we recall some basic definitions and results that are used throughout the paper. A *series-parallel digraph* (also called an *SP-digraph* in the following) is a planar digraph recursively defined as follows [2]:

- An edge joining two vertices is a SP-digraph.
- Let G' and G'' be two SP-digraphs; the digraph obtained by identifying the sink of G' with the source of G'' (*Series Composition*) is also a SP-digraph.
- Let G' and G'' be two SP-digraphs; the digraph obtained by identifying the source of G' with the source of G'' and the sink of G' with the sink of G'' (*Parallel Composition*) is also a SP-digraph.

This defines the digraphs that are sometimes called *two terminal series-parallel digraphs*. A SP-digraph has one source and one sink that are called its *poles*. The undirected underlying graph of a SP-digraph is called a *series-parallel graph* or *SP-graph* for short.

We briefly recall the definition of the *SPQR-tree* of an *st*-graph G, also called the *decomposition tree* of G (see [3] for a complete definition). A $SPQR$-tree is a rooted tree T whose nodes are one of four types: S,P,Q, or R. Each node of T has an associated planar *st*-graph called its *skeleton*. A S-node corresponds to a series composition of blocks, and it has a child for each block; the skeleton of a S-node is a chain with an edge for each block. A P-node corresponds to a parallel composition of split components with respect to a separation pair and it has a child for each split component; the skeleton of a P-node with k split components is the parallel of k edges. A Q-node corresponds to a single edge and it has no child except the Q-node associated with the edge with respect to which the decomposition is done, which is the root of the tree; the skeleton of a Q-node is the edge associated with it. The R-nodes correspond to the triconnected components of G and since they never appear in the decomposition tree of a SP-graph we do not consider them. The *pertinent graph* of a node μ of T is the subgraph of G whose decomposition tree is the tree rooted at μ.

A *restricted integer grid* is a grid ϕ, that is a proper subset of the integer plane or space [6]. We are interested in a particular grid that we call a *box*. A box is a $2 \times 2 \times \infty$ grid, *i.e.* it consists of four parallel lines of infinite length passing through the four vertices of a unit square [1]. In the following, we assume that the box is placed such that the four lines are parallel to the x-axis, so that the position of each vertex in the drawing is uniquely defined when a line and an x-coordinate are assigned to it. We will denote by l_0 the top-left line of the box, by l_1 the top-right line of the box, by l_2 the bottom-right line of the box and by l_3 the bottom-left line of the box. We define also a *strip* as a $2 \times \infty$ planar grid, *i.e.* two parallel lines of infinite length. We say that a graph G is *box-drawable* if G admits a drawing such that every vertex is drawn at integer coordinates on one of the four lines of the box and the edges are straight-line segments such that no two edges intersect except at common end-points. The drawing of G on the box is a *box-drawing* of G. A graph is *box-forbidden* if it does not admit a box-drawing. Analogously we say that a graph is *strip-drawable* if it admits a straight-line crossing-free drawing on a strip and we call such a drawing a *strip-drawing* of G. Let G be a box-drawable graph; we say that G *covers* three lines if all box drawings of G have at least three mutually adjacent vertices drawn on different lines of the box.

3 SP Box-forbidden graphs

In this section we show a series-parallel graph that cannot be drawn on a box. We start by giving two lemmas that will be useful in the following.

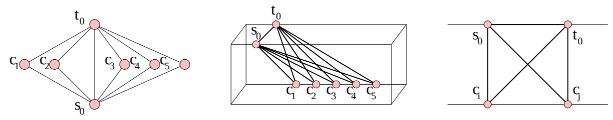
Lemma 1 *In a box-drawing, a vertex v drawn on a line l can be connected to at most two vertices drawn on the same line l.*

Lemma 2 *Let G_0 be the SP-graph of Figure 1(a) and let s_0 and t_0 be the poles of G_0. G_0 is box-drawable and in any box-drawing of G_0, s_0 and t_0 appear on different lines of the box.*

Sketch of Proof: Graph G_0 is box-drawable as shown in Figure 1(b). Suppose that there exists a box-drawing of G_0 with s_0 and t_0 on the same line l. We adopt the notation of Figure 1(a). No other vertex c_i can be drawn on l since a 3-cycle cannot be drawn on a line. Therefore, since there are three lines different from l and five vertices different from s_0 and t_0, at least two vertices, say c_i and c_j, are on a same line l'. However, since both c_i and c_j are adjacent to s_0 and t_0, the box-drawing would have a crossing (see also Figure 1(c)) – a contradiction. □

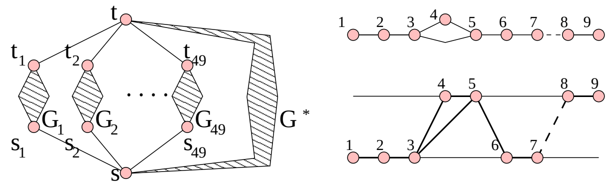
Lemma 3 *Let G_0 be the SP-graph of Figure 1(a). Graph G_0 covers three lines.*

[1] We adopt the convention of measuring the size of the grid by counting the number of grid lines used in each dimension rather than the distance between the maximal pair.



(a) (b) (c)

Figure 1: (a) Graph G_0. (b) A box-drawing of G_0. (c) Vertices s_0 and t_0 of G_0 cannot be drawn on the same line.



(a) (b)

Figure 2: (a) Graph G of Theorem 1 (b) An example of drawing of a virtual pertinent graph (the dashed edge is a virtual edge).

Sketch of Proof: By Lemma 2, vertices s_0 and t_0 are on different lines (call them l_s and l_t respectively). To prove the statement it suffices to prove that one vertex c_i different from s_0 and t_0 is on a third line. Suppose that all vertices c_i $(i = 1, \ldots, 5)$ were either on line l_s or on line l_t. At least three of them would then be on the same line (suppose l_s), but by Lemma 1, this is not possible. □

We are now ready to prove the main result of the section.

Theorem 1 *There exist box-forbidden series-parallel graphs.*

Sketch of Proof: Let G be the graph of Figure 2(a), consisting of 49 copies G_i $(i = 1, \ldots, 49)$ of the graph G_0 depicted in Figure 1(a) and by another copy G^* of G_0 such that poles s and t of G^* are connected to poles s_i of G_i $(i = 1, \ldots, 49)$ and to poles t_i of G_i $(i = 1, \ldots, 49)$ respectively. We show that G is not box-drawable.

By Lemma 3, each G_i covers three lines of the box. Consider the maximum number N of graphs G_i that can be arranged on the four lines of the box so that there are not two graphs G_h and G_k $(h \neq k, 1 \leq h, k \leq 49)$ covering the same three lines and having s_h and s_k on the same line and t_h and t_k on the same line. Since the number of permutations of length p from n elements is $\frac{n!}{(n-p)!}$, then $N = \frac{4!}{(4-3)!} = 24$. The number of copies of G_i is $2 * N + 1 = 49$, so if a box-drawing Γ of G existed, then there would be at least three components G_x, G_y, G_z $(x \neq y \neq z, 1 \leq x, y, z \leq 49)$ such that:

- the vertices of G_x, G_y, G_z are drawn on the same three lines of the box;
- the sources s_x, s_y, and s_z are drawn on the same line of the box; and
- the sinks t_x, t_y, and t_z are drawn on the same line of the box different from the one of the sources.

Let l_1, l_2 and l_3 be the three lines covered by G_x, G_y, G_z in Γ; also let l_1 be the line of the sources s_x, s_y, and s_z and let l_2 be the line of the sinks t_x, t_y, and t_z. Also let l_s be the line of the source s of G. Observe that l_s is not one of the lines l_1, l_2 and l_3. Namely, if $l_s = l_1$ then by Lemma 1 s cannot be connected to the three vertices s_x, s_y and s_z. If either $l_s = l_2$ or $l_s = l_3$, then edges (s, s_x), (s, s_y), and (s, s_z) lie on the strip σ_a defined by l_s and l_1. Since G_x, G_y, G_z cover l_1, l_2 and l_3, there are three edges on σ_a incident on s_x, s_y and s_z. It follows that there is no room for drawing (s, s_x), (s, s_y), and (s, s_z) without creating a crossing. Therefore s must be drawn on a line different from l_1, l_2, and l_3. By a similar argument, t can not be drawn on any of lines l_1, l_2, and l_3. However, s and t are the poles of G^* and by Lemma 2 they cannot be on the same line, thus establishing that no box-drawing of G exists. □

4 3-planar SP-Graphs on a Box

Theorem 1 induces the question of characterizing those series-parallel graphs that admit a box-drawing. In this section we show that all 3-planar series-parallel graphs, *i.e.* series-parallel graphs with maximum vertex degree at most three, are box-drawable. We start by providing some more necessary definitions.

Throughout this section G is a 3-planar series-parallel graph with n vertices. Since each planar SP-graph has a planar embedding with a given edge (s, t) on the external face and since each SP-graph can be st-oriented in $O(n)$ time [5], we can assume that G is st-oriented and that edge (s, t) exists in G on the external face.

Let T be the SPQR tree of G with respect to edge (s, t). Let μ be an S-node and let $(u_1, u_2), (u_2, u_3), \ldots, (u_k, u_{k+1})$ be the edges of the skeleton of μ where $u_1 \rightarrow u_{k+1}$ in G, and let $\nu_j, (j = 1, \ldots, k)$ be the child of μ_i in T corresponding to edge (u_j, u_{j+1}). We assume that the left-to-right order of ν_j $(j = 1, \ldots, k)$ is such that, given two children ν_i and ν_l of μ, ν_i precedes ν_l if $i < l$.

Since G is 3-planar, the decomposition tree T is such that each P-node μ has exactly two children, one of which is a S-node. Also, for each S-node μ of T:

- μ has at least two children,
- the leftmost and rightmost child of μ are Q-nodes,
- each child of μ which is a P-node is preceded and followed by a Q-node.

We say that a P-node of T is *simple* if its pertinent graph is the parallel composition of two simple paths; a subtree of T is *simple* if all its P-nodes are simple; a subgraph of G is *simple* if its decomposition tree is simple. Given a S-node μ of T we call the *pertinent core graph* of μ the graph G^-_μ obtained from the pertinent graph G_μ of μ by removing the edges associated with the leftmost child of μ and with the rightmost child of μ. We call the *source core pole* of μ the vertex which is adjacent to the source pole of μ in the pertinent graph. Analogously we call the *sink core pole* of μ the vertex which is adjacent to the sink pole of μ in the pertinent graph. Let $G_{\mu,i}$ $i = 1, \ldots, n_\mu$ be the pertinent graphs associated with non-simple children of μ (if there are any). The graph obtained from G^-_μ by replacing each $G_{\mu,i}$ with a virtual edge e_i, is simple. We call such a graph a *virtual pertinent graph* of μ and denote it by G^*_μ. If all children of μ are simple then its virtual pertinent graph coincides with its pertinent core graph. We call *jumping edges* the edges removed in the pertinent core graph.

A *level-numbering* of T is a numbering of each node μ of T defined as follows:

- if μ is a P-node then:
  - if $(father(\mu) = root(T))$
    then $level(\mu) := 0$
    else $level(\mu) := level(father(father(\mu))) + 1$
- if μ is an S-node then:
  - if ($father(\mu)$ is simple)
    then $level(\mu) := level(father(\mu))$
    else $level(\mu) := level(father(\mu)) + 1$
- if μ is a Q-node then:
  - if $(\mu = root(T))$
    then $level(\mu) := 0$
    else $level(\mu) := level(father(\mu))$

Using the level-numbering defined above we assign a level to each vertex v of G. If v is a pole of some P-node μ then $level(v) = level(\mu)$; otherwise v is shared by two edges whose Q-nodes have the same level number and $level(v)$ is the level of these Q-nodes. Observe that a jumping edge connects either a source pole of a P-node of level k with a source core pole of level $k+1$ or a sink pole of a P-node of level k with a sink core pole of level $k+1$.

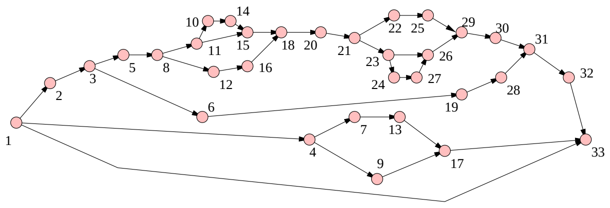


Figure 3: A 3-planar Series-Parallel digraph G.

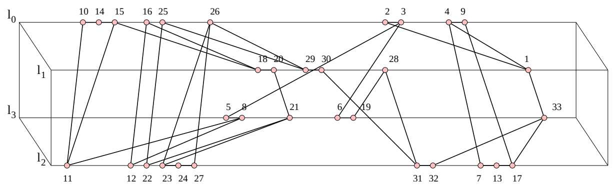


Figure 4: A box-drawing of graph G of Figure 3.

Lemma 4 *Let G be a 3-planar SP-digraph, let T be its decomposition tree, let μ be a S-node of T and let G^*_μ be its virtual pertinent graph. G^*_μ has a strip drawing such that:*

- *the source core pole s_μ and the sink core pole t_μ are on different lines;*
- *for each child ν of μ such that ν is a P-node, $source(\nu)$ is on the same line as s_μ and $sink(\nu)$ is on the same line as t_μ;*
- *s_μ has the smallest x-coordinate and t_μ has the largest x-coordinate among the vertices of G_μ.*

Sketch of Proof: Since the virtual pertinent graph G^*_μ is simple and 3-planar, it is composed of an alternating sequence of parallel components (simple cycles or virtual edges) and simple paths. The last vertex of each path (cycle) is the first vertex of the following cycle (path). Let l_1 and l_2 be the two lines defining the strip. If the sequence starts with a path, this is drawn along line l_1. Each parallel component is drawn with the source pole on l_1, the sink pole on l_2 and if it is a cycle with one of the two simple paths from source to sink on line l_1 following the source and with the other on line l_2 preceding the sink. Each path between cycles is drawn on line l_2 (except the last vertex, which is the source pole of the following cycle and is drawn on line l_1). If the sequence ends with a path, this is drawn along line l_2. If G^*_μ is composed of a single simple path it is drawn along line l_1 except the last vertex (the sink core pole) which is drawn on line l_2. An example drawing is shown in Figure 2(b). □

Theorem 2 *Let G be a 3-planar series-parallel graph with n vertices. There exists a $O(n)$-time algorithm that computes a box-drawing of G with $O(n)$ volume.*

Sketch of Proof: We first give a drawing algorithm, then prove its correctness and discuss its performance in terms of computation time and resulting volume.

Let T be the decomposition tree of G. Compute a level-numbering of T and the corresponding numbering of the vertices of G and let l_{max} be the maximum level number assigned. For $i = l_{max}, \ldots, 0$ consider all the S-nodes of level i having a father whose level is $i-1$, in the left to right order that they have in T. Draw the virtual pertinent graph of these nodes as described in Lemma 4 with sources on line l_s with $s = i \ mod \ 4$ and sinks on line l_t with $t = (i+2) \ mod \ 4$, and by using increasing values of x-coordinates. At the end of the for loop, the jumping edges are added.

We now prove that in the drawing obtained there is no crossing. The drawing of the virtual pertinent graph of each S-node is crossing-free by construction. The drawings of the pertinent graphs of two different S-nodes can not cross each other since each pertinent graph is drawn with x-coordinates greater than any preceding drawing. Now we show that the jumping edges can be added without creating crossings. Recall that the jumping edges connect source (sink) poles of level k to source (sink) core poles of level $k+1$. Consider the drawing Γ_k of the vertices of level k and assume without loss of generality that the sources are on line l_0 and sinks are on line l_2. The drawing algorithm is such that sources and sinks of level $k-1$ are on line l_3 and l_1, respectively; also sources and sinks of level $k+1$ are on line l_1 and l_3, respectively. Thus jumping edges between level $k-1$ and k are on the strip defined by l_0 and l_3 (*i.e.* the left side of the box) and on the strip defined by l_2 and l_1 (*i.e.* the right side of the box), while the jumping edges between level k and $k+1$ are on the strip defined by l_0 and l_1 (*i.e.* the top side of the box) and on the strip defined by l_2 and l_3 (*i.e.* the bottom side of the box). This implies that jumping edges on different sides of the box do not cross each other since the four sides of the box do not intersect except at the four lines. Also, jumping edges drawn on a same side of the box do not cross each other, since the order according to which the S-nodes of the same level are considered (*i.e.* the left to right order in the $SPQR$-tree) is such that if a source (sink) pole u of level k precedes another source (sink) pole v of level k then all the source (sink) core poles of level $k+1$ connected to u precede all the source (sink) core poles of level $k+1$ connected to v. It follows that the computed drawing is a valid box-drawing.

Since a decomposition tree T of G can be constructed in $O(n)$ time [3, 7] and both the level numbering and the drawing strategy can be easily accomplished by an $O(n)$-time visit of T, the statement about the time complexity holds. Also, the volume of the computed drawing is $4 \cdot n$ since the n vertices are drawn at consecutive x-coordinates and the four lines of the box are one grid unit apart. □

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References

[1] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[2] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.

[3] G. Di Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15(4):302–318, 1996.

[4] V. Dujmović, P.Morin, and D. Wood. Pathwidth and three-dimensional straight line grid drawings of graphs. In *Graph Drawing (Proc. GD '02)*, to appear.

[5] S. Even and R. E. Tarjan. Computing an st-numbering. *Theoret. Comput. Sci.*, 2:339–344, 1976.

[6] S. Felsner, G.Liotta, and S. Wismath. Straight line drawings on restricted integer grids in two and three dimensions. In P. Mutzel, M. Junger, and S. Leipert, editors, *Graph Drawing (Proc. GD '01)*, volume 2265 of *Lecture Notes Comput. Sci.* Springer-Verlag, 2001.

[7] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. *SIAM J. Computing*, 2:135–158, 1973.