Straight-Line Drawings of Graphs in Three Dimensions *

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Abstract. This is an overview of results and open problems pertaining to straight-line drawings of graphs in integer grids, in particular when drawn in three dimensions. As presented at: Journées de Géométrie Algorithmique 2002.

1 Introduction

Graph theory has a long and rich history including Euler's work on the Königsburg bridge problem in 1736. Drawings or representations of abstract graphs have also been investigated and the notion of *planarity* is motivated entirely from the *drawing* perspective – Can a graph be drawn in the plane with no pair of edges crossing?

Every school child knows Fáry's result [14]: Any planar graph can be drawn with edges drawn as crossing-free *straight-line* segments. (See also Stein [25] and Wagner [27]).

Graph Drawing is a recent field that deals with some of the more modern implications of these results. Since 1990 there has been an annual conference by this name and there are now two textbooks describing the field in detail [10][20].

Early *algorithmic* results for constructing straight-line drawings of planar graphs include the work by Tutte [26] and Read [22]. These results rely on real arithmetic and generate drawings with vertices arbitrarily close together. A natural constraint is to insist that the vertices be placed at *integer-valued coordinates.* We refer to such a drawing as an **integer grid straight-line drawing**.

Partially motivated by VLSI design issues, de Fraysseix, Pach and Pollack [8, 9] phrased the following problem:

Problem 1. Can all n-vertex planar graphs be drawn in $\Re^2$ on an integer grid with an area[1] that is polynomially bounded in n?

They showed that indeed all planar graphs can be drawn on a grid of size $(n-2) \times (2n-4)$ and the area is $\Omega(n^2)$ for some planar graphs. Similar results were independently shown by Schnyder [23].

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[1] of a bounding box

1.1 Outline of algorithm

Determine a *canonical ordering* of the vertices: For a maximal planar graph G with external face u, v, w, there is an ordering $u = v_1, v = v_2, v_3, ...v_n = w$ such that:

1. Subgraph G_{k-1} induced by $v_1...v_{k-1}$ is 2-connected and the boundary of its exterior face is a cycle C_{k-1} containing (u, v).
2. vertex v_k is in the exterior face of G_{k-1} and its neighbours in G_{k-1} are consecutive on C_{k-1}.

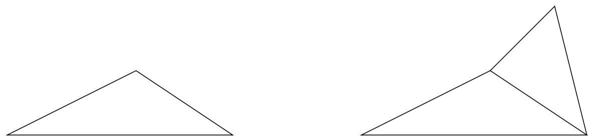

Fig. 1. First steps

The algorithm initially draws as in Figure 1. In general, v_k has y-coordinate $k-2$ and is joined to C_{k-1}. The drawing is then *shifted* to maintain an invariant on the slopes of C_k – see Figure 2.

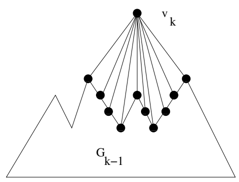

Fig. 2. General case

1.2 Lower bound argument

Note that a maximum planar graph consisting of *concentric triangles* as in Figure 3 needs a grid of size $w \times h$, where $w, h > n/3$. A single triangle requires that $w, h > 1$ and each subsequent triangle requires at least $+1$ in w and h.

A sequence of papers reduced the grid size to $(n-1) \times (n-1)$. A partial list of results includes: Kant [18, 19], Chrobak and Kant [4], Schnyder and Trotter [24], Felsner [16] and Chrobak, Goodrich, and Tamassia [5].

Theorem 1. *Any planar graph can be drawn with straight-line edges on an integer grid in $\Theta(n^2)$ area.*

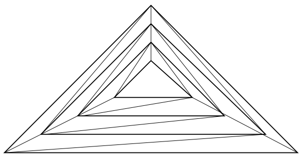

Fig. 3. Lower bound on area

1.3 Can quadratic area be beaten for classes of planar graphs?

The obvious candidate to consider is the class of trees.

There are many papers proving that linear or almost-linear area bounds can be achieved for particular classes of trees, including the result by Garg, Goodrich and Tamassia [17] and the result by Chan [3]. Summarizing tables and more references can be found in the book by Di Battista, Eades, Tamassia, and Tollis [10]. $O(n\log \log n)$ area is the best result for trees of unbounded degree.

Problem 2. (Open) Are there other classes of planar graphs that can be drawn in $o(n^2)$ area?

For example the situation for outerplanar graphs is unknown but conjectured to be $O(n \log n)$.

2 Three Dimensions

The extension to 3D is easy to imagine. Given a graph G is it possible to draw it in $\Re^3$, with vertices located at integer-valued points, and edges represented as straight-line segments and with no pair of edges intersecting (in $\Re^3$)?

What sort of volume bounds[2] pertain?

The first two critical results are due to Cohen, Eades, Lin and Ruskey [7]:

Theorem 2. *1. Every graph admits a straight-line crossing-free 3D drawing on an integer grid of $O(n^3)$ volume, and*
2. K_n has a volume bound $\in \Omega(n^3)$.

Proof. (2) Note that if there exist any 5 points that are co-planar then there must be a crossing. Now the argument involves a sweep plane: Consider the planes $X = k, k \in Z$. Since none of these planes may contain 5 points, at least $n/5$ planes are required. Similarly, $n/5$ planes are required in each dimension and thus the Volume $> n/5 \cdot n/5 \cdot n/5 \in \Omega(n^3)$.

Proof. (1) As a warm up, draw K_n in a $n \times n^2 \times n^3$ grid: Place each vertex at (i, i^2, i^3). The chords joining these vertices do not intersect. (Exercise: prove it).

An improved drawing can be obtained:

[2] of a bounding box

– choose a prime p, $n < p \leq 2n$.
– For i=1 to n do
  draw v_i at $(i, i^2 (\text{mod } p), i^3 (\text{ mod } p))$.

Again it can be shown that no 4 points are coplanar and hence there are no intersections. Since $p \leq 2n$, the grid size is $n \times n \times 2n$.

Thus all graphs can be drawn in $\Theta(n^3)$ volume.

Are there classes of graphs that can be drawn in $o(n^3)$ volume? (We know that planar graphs can be drawn in $O(n^2)$ volume in 2-D.)

Calamoneri and Sterbini [2] showed that:

Theorem 3. *All 2-, 3-, and 4-colourable graphs can be drawn in a 3D grid of $O(n^2)$ volume and there is a lower bound of $\Omega(n^{1.5})$ on the volume of such graphs.*

Proof. The critical insight is that if 2 line segments cross, their 4 endpoints must be coplanar.

– 2-Colourable (*i.e.* Bipartite): Suppose the vertices are coloured R and B. Draw the R vertices consecutively on the plane $Z = 0$ on the line $Y = 0$. Draw the B vertices consecutively on the plane $Z = 1$ on the line $X = 0$. Since these 2 lines are skew, no plane can go through two R and two B vertices. The grid is $O(n) \times O(n)$.

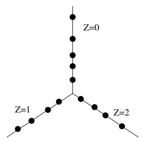

Fig. 4. 3 skew lines

– 3-Colourable: Similarly, 3 skew lines can be used. Start with the configuration in Figure 4 (each line is on a separate Z plane). And with some care, convert to Figure 5. Note that two edges between any pair of lines do not intersect but the argument for two edges with endpoints involving 3 lines is more involved.
– 4-Colourable: Similarly, 4 skew lines in *convex position* can be used – details omitted.

Pach, Thiele and Tóth [21] strengthened this result to:

Theorem 4. *For a fixed value $r \geq 2$, any r-colourable graph with n vertices, can be drawn in $\theta(n^2)$ volume.*

Proof. Let p be the smallest prime with $p \geq 2r - 1$. Let $N = p \cdot n/r$.
For i=0 to $r - 1$
set $V_i = \{(i, t, i \cdot t) \mid 0 \leq t < N, t \equiv i^2 \pmod p\}$
There are n/r elements in each set and the grid is $r \times 4n \times 4rn$.

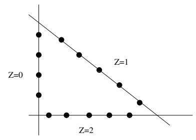

Fig. 5. 3 more skew lines

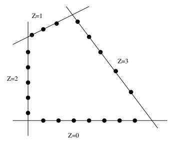

Fig. 6. 4 skew lines

3 Planar graphs in 3D

Are there other classes of graphs that are drawable in 3D in $o(n^3)$ volume?
In particular,

Conjecture 1. Can all planar graphs be drawn in $O(n)$ volume in 3D?

There are many properties of planar graphs that might be useful: 4-colourability, the planar separator theorem, Kirkpatrick's subdivision hierarchy, relation to polytopes, etc.

Very recently Wood, Dujmovič and Morin [12] have shown a strong connection to the notion of path-width, tree-width and queue number.

3.1 Restricted grids

One approach to solving the 3D case for planar graphs is to choose a fixed shape grid (of linear volume) and ask what class of graphs can be drawn on that grid.

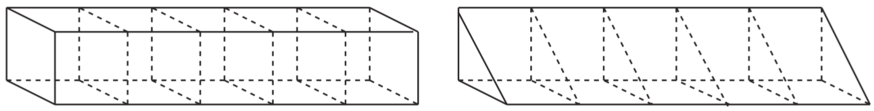

Fig. 7. A box and a prism

Felsner, Liotta and Wismath [15] studied two different types of $n \times 2 \times 2$ grids[3]. A *box* is a $n \times 2 \times 2$ grid where each side of the bounding box is also a

[3] typically 3D grids are measured by the number of lines in each dimension

grid line. Therefore, a box has four tracks which lie on two parallel planes and are one grid unit apart from each other. A *prism* is a $n \times 2 \times 2$ grid obtained by removing a track from a box. Figure 7 shows an example of a box of size $6 \times 2 \times 2$ and an example of a prism. The prism supports a large subclass of

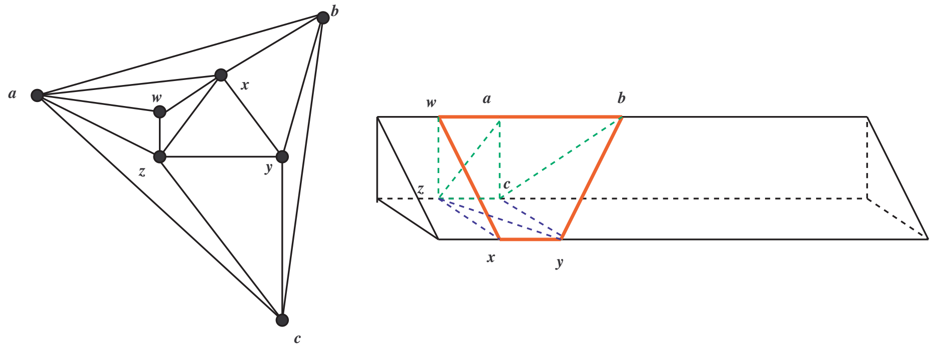

Fig. 8. A prism-drawable graph G and its drawing

planar graphs. For example, Figure 8 shows a maximal planar graph, and its prism drawing.

The first non-trivial class of graphs drawable in linear volume are the outerplanar graphs[4].

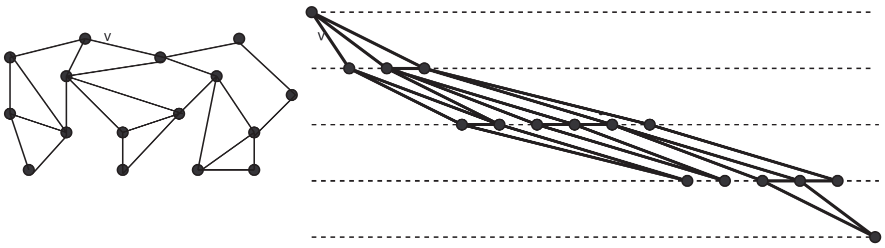

Fig. 9. An outerplanar graph drawn by `Step 1` .

Theorem 5. *[15] All outerplanar graphs can be drawn on a prism.*

[4] outerplanar means $\exists$ a plane embedding with all vertices on the exterior face

Proof. Let G be an outerplanar graph with a specified *outerplanar embedding*. First a 2D drawing of G is computed on a grid that consists of $O(n)$ horizontal tracks and such that adjacent vertices are at grid points whose y-coordinates differ by at most one; this is done by visiting G in a breadth-first fashion. Secondly, the drawing is "wrapped" onto the faces of a prism by folding it along the tracks. Figure 9 shows an example of the output of Step 1 of the algorithm. An animation of the procedure is availble at: `www.cs.uleth.ca/~wilsdoj9/graphanims`.

Unfortunately, it turns out that not all planar graphs are prism-drawable.

Theorem 6. *There exist planar graphs not drawable on a prism.*

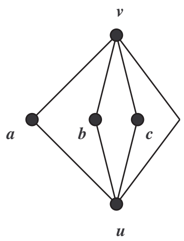

Fig. 10. A strictly prism-drawable graph.

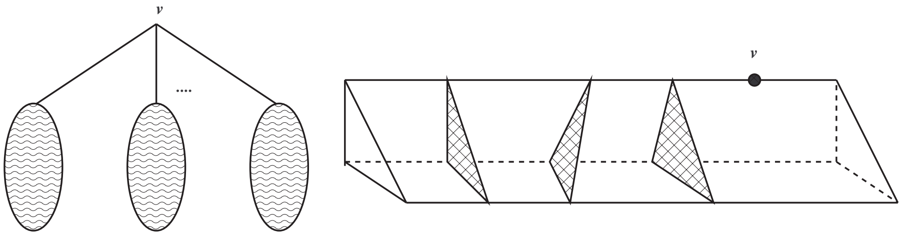

Fig. 11. The "slicing" argument.

Proof. A graph G is *strictly prism-drawable* if it is prism-drawable and all prism drawings of G have at least three edges (x, y), (y, z) and (z, x) such that x, y and z are on different tracks as in Figure 10. Now 3 copies of such a graph can be used to create a graph that is not prism-drawable (Figure 11). The 3 copies slice the prism and there is no way to join vertex v.

3.2 Box-Drawable Graphs

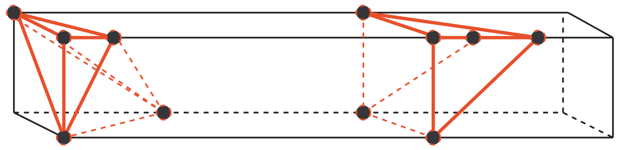

Fig. 12. K_5 and $K_{3,3}$ drawn on a box

Although a *box* supports a larger class of graphs (including some non-planar ones – Figure 12), it is not sufficiently powerful:

Theorem 7. *There exist planar graphs not drawable on a box.*

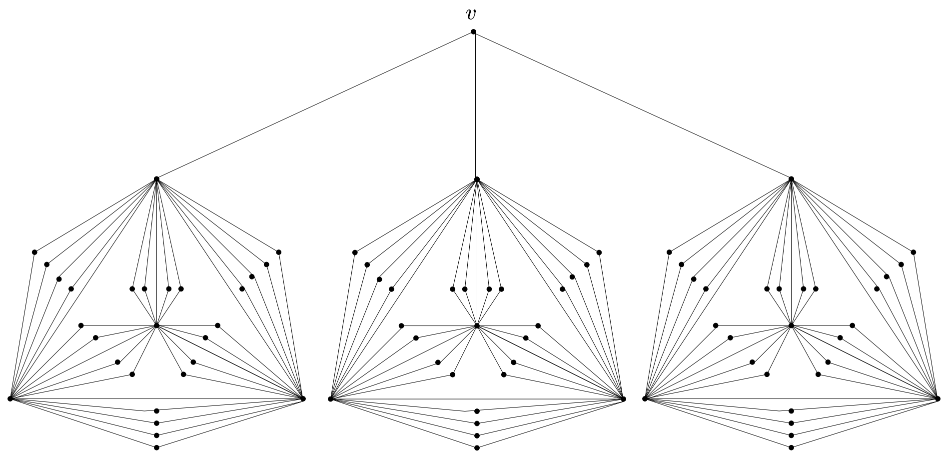

Fig. 13. A planar graph that is not box-drawable.

The proof involves a similar technique – a graph is created that uses all 4 tracks and the 6 line segments between them, thus blocking any edges from passing through it. Then 3 copies of it are arranged as in Figure 13.

It can even be shown that there are series-parallel graphs that are not box-drawable, but *almost all* series-parallel graphs are drawable on 8 lines [11].

3.3 Other Directions

There are several directions being considered to solve the 3D case for planar graphs.

Can all planar graphs be drawn on these grids:

- $2 \times k\sqrt{n} \times k\sqrt{n}$
- $n^{1/3} \times n^{1/3} \times n^{1/3}$ (this has a constant aspect ratio)

Complexity Issues: Given a graph G and a grid Φ, can G be drawn on Φ?

Problem 3. Can all planar graphs of maximum degree 3 be drawn in linear volume?

Other approaches:

- Bose, Czyzowicz, Morin, Wood [1] – bounds on the total number of edges possible in a given box.
- canonical ordering technique applied to 3D – terrains.
- Another related problem due to H. de Fraysseix is to prove:

  Conjecture 2. All planar graphs can be drawn with vertices lying on k parallel lines arranged on a cylinder.

  Here *real* coordinates are permitted but there is some hope that the integer version would hold true and imply linear volume for planar graphs. It has been shown that $k > 5$.

3.4 Variants

Orthogonal 3D graph drawing: Vertices are at integer coordinates but each edge consists of a *sequence* of line segments parallel to an axis. Note that each vertex has maximum degree 6.

There are several bend/volume tradeoff results. See for example, [13] and [6].

The big unresolved problem here is:

Conjecture 3. All graphs of maximum degree 6 can be drawn orthogonally in 3D *with at most 2 bends per edge.*

Exercise: Draw K_7 with 2 bends per edge.

4 Summary of Open Problems and Conjectures

1. Can all outerplanar graphs be drawn in $o(n^2)$ area on a 2D integer grid? Does there exist a 2D universal grid set of small area that supports all outerplanar graphs?
2. Can all planar graphs be drawn in linear volume on a three-dimensional integer grid? Does there exist a 3D universal grid set of linear volume that supports all planar graphs?

3. Can all planar graphs of maximum degree 3 be drawn in linear volume?
4. Given a graph G and a grid Φ, what is the complexity of determining whether G be drawn on Φ?
5. *The k-lines drawability problem*: (posed by H. de Fraysseix) asks if all planar graphs can be drawn on k parallel lines that lie on the surface of a cylinder, for a fixed value of k.
6. Can all graphs of maximum degree 6 be drawn orthogonally in 3D *with at most 2 bends per edge*?

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