Journal of Graph Algorithms and Applications
http://www.cs.brown.edu/publications/jgaa/
vol. ?, no. ?, pp. ?–? (????)

Straight-Line Drawings on Restricted Integer Grids in Two and Three Dimensions

Stefan Felsner[1], *Giuseppe Liotta*[2], *Stephen Wismath*[3]

[1] Freie Universität Berlin, Fachbereich Mathematik und Informatik, Takustr. 9, 14195 Berlin, Germany.
felsner@inf.fu-berlin.de

[2] Dipartimento di Ingegneria Elettronica e dell'Informazione, Università di Perugia, via G. Duranti 93, 06125 Perugia, Italy.
liotta@diei.unipg.it

[3] Dept. of Mathematics and Computer Science, U. of Lethbridge, Alberta, T1K-3M4 Canada.
wismath@cs.uleth.ca

Abstract

This paper investigates the following question: Given a grid ϕ, where ϕ is a proper subset of the integer 2D or 3D grid, which graphs admit straight-line crossing-free drawings with vertices located at (integral) grid points of ϕ? We characterize the trees that can be drawn on a strip, *i.e.*, on a two-dimensional $n \times 2$ grid. For arbitrary graphs we prove lower bounds for the height k of an $n \times k$ grid required for a drawing of the graph. Motivated by the results on the plane we investigate restrictions of the integer grid in 3D and show that every outerplanar graph with n vertices can be drawn crossing-free with straight lines in linear volume on a grid called a prism. This prism consists of $3n$ integer grid points and is *universal* – it supports all outerplanar graphs of n vertices. We also show that there exist planar graphs that cannot be drawn on the prism and that extension to an $n \times 2 \times 2$ integer grid, called a box, does not admit the entire class of planar graphs.

Communicated by: Submitted on: ; Revised on:

Research supported in part by: the CNR Project "Geometria Computazionale Robusta con Applicazioni alla Grafica ed al CAD", the project "Progetto ALINWEB: Algoritmica per Internet e per il Web", of the Italian Ministry of University and Scientific and Technological Research; and by the Natural Sciences and Engineering Research Council of Canada.

1 Introduction

This paper deals with crossing-free straight-line drawings of planar graphs in two and three dimensions. Given a graph G, we constrain the vertices in a drawing of G to be located at integer grid points and aim at computing drawings whose area/volume is small. The interest in these two requirements is motivated in part by the fact that the screen of a computer is an integer grid of limited size and by the fact that a drawing algorithm should not be affected by round-off errors when representing the output coordinates. Also, the increasing demand of visualization algorithms to draw and browse very large networks makes it natural to study what families of graphs can be entirely visualized on a two dimensional screen and to investigate how much benefit can be obtained from the third dimension to represent the overall structure of a huge graph in a small portion of a virtual 3D environment. A rich body of literature has been published on computing straight-line drawings of graphs, such that the vertices are the intersection points of an integer 2D grid and the overall area of the drawing is kept small. Typically, papers that deal with this subject focus on lower bounds on the area required by straight-line drawings of specific classes of graphs and on the design of algorithms that possibly match these lower bounds. A very limited list of mile-stone papers in this field includes the works by de Fraysseix, Pach, and Pollack [11, 12] and by Schnyder [38] who independently showed that every n-vertex triangulated planar graph has a crossing-free straight-line drawing such that the vertices are at grid points, the size of the grid is $O(n) \times O(n)$, and that this is worst case optimal; the work by Kant [26, 27], Chrobak and Kant [6], Schnyder and Trotter [39], Felsner [21] and Chrobak, Goodrich, and Tamassia [7] who studied convex grid drawings of triconnected planar graphs in an integer grid of quadratic area; and the many papers proving that linear or almost-linear area bounds can be achieved for classes of trees, including the result by Garg, Goodrich and Tamassia [23] and the result by Chan [5]. Summarizing tables and more references can be found in the book by Di Battista, Eades, Tamassia, and Tollis [14].

While the problem of computing small-sized crossing-free straight-line drawings in the plane has a long tradition, its 3D counterpart has become the subject of much attention only in recent years. Chrobak, Goodrich, and Tamassia [7] gave an algorithm for constructing 3D convex drawings of triconnected planar graphs with $O(n)$ volume and non-integer coordinates. Cohen, Eades, Lin and Ruskey [9] showed that every graph admits a straight-line crossing-free 3D drawing on an integer grid of $O(n^3)$ volume, and proved that this is asymptotically optimum. Calamoneri and Sterbini [3] showed that all 2-, 3-, and 4-colourable graphs can be drawn in a 3D grid of $O(n^2)$ volume with $O(n)$ aspect ratio and proved a lower bound of $\Omega(n^{1.5})$ on the volume of such graphs. For r-colourable graphs, Pach, Thiele and Tóth [31] showed a bound of $\theta(n^2)$ on the volume. Garg, Tamassia, and Vocca [24] showed that all 4-colorable graphs (and hence all planar graphs) can be drawn in $O(n^{1.5})$ volume and with $O(1)$ aspect ratio but using a grid model where the coordinates of the vertices may not be integer. In this paper we study the problem of computing drawings of graphs on integer

2D or 3D grids that have small area/volume. The area/volume of a drawing Γ is measured as the number of grid points contained in or on a *bounding box* of Γ, *i.e.* the smallest axis-aligned box enclosing Γ. Note that along each side of the bounding box the number of grid points is one more than the actual length of the side. We approach the drawing problem with the following point of view: Instead of "squeezing" a drawing onto a small portion of a grid of unbounded dimensions, we assume that a grid of *specified* dimensions (involving a function of n) is given and we consider what the graphs are whose drawings fit that restricted grid. For example, it is well-known that there are families of graphs that require $\Omega(n^2)$ area to be drawn in the plane, the canonical example being a sequence of $n/3$ nested triangles (see [12, 8, 38]). Such graphs can be drawn on the surface of a three dimensional triangular prism of linear volume and using integer coordinates. Thus a natural question is whether there exist specific restrictions of the 3D integer grid of linear volume that can support straight-line crossing-free drawings of meaningful families of graphs. For planar graphs the best known results for three dimensional crossing-free straight-line drawings on an integer grid are by Calamoneri and Sterbini [3] who show $O(n^2)$ volume for general planar graphs and by Eades, Lin and Ruskey [9] who show $O(n \log n)$ volume for trees.

The main contributions of the present paper are investigations concerning the drawability of graphs on 2D and 3D restricted integer grids and new drawing algorithms for some classes of graphs. An overview of the results is as follows.

- We characterize those trees that can be drawn on a strip, *i.e.*, an integer 2D grid restricted to two consecutive horizontal grid lines. From the characterization we derive a linear time algorithm to generate such drawings, if possible. This result was independently obtained by Schank [36] in his Master's thesis.
- Generalizing the result for strips, we present a lower bound "the strictness of a tree" for the number k of horizontal grid lines required for grid drawings of trees. A consequence of this bound is that for any given k there always exist some trees that are not drawable on the $n \times k$ grid.
- We show that the strictness of a tree is closely related to the well-known parameter *path-width*. For general graphs the path-width is shown to be a lower bound for the height of grid drawings.
- Motivated by the results on restricted integer 2D grids we explore the capability of restricted 3D integer grids for supporting linear volume drawings of graphs. In particular, we focus on two types of 3D integer grids to be defined subsequently, both having linear volume, called the *prism* and the *box*. We show that all outerplanar graphs can be drawn in linear volume on a prism. Note that this is the first result on 3D straight-line drawings of a significant class of planar graphs that achieves linear volume with integer coordinates.

- We further explore the class of graphs that can be drawn on a prism by asking whether the prism is a *universal* 3D integer grid for all planar graphs. We answer this question in the negative by exhibiting examples of planar graphs that cannot be drawn on a prism. We also investigate the relationship between prism-drawable and hamiltonian graphs.

- We extend our study to box-drawability and present a characterization of the box-drawable graphs. While the box would appear to be a much more powerful grid than the prism, we prove that not all planar graphs are box-drawable.

Several recent related results about 3D straight line drawings of limited volume have been published after the conference version of this paper was presented at the Symposium on Graph Drawing GD 2001 [22]. Dujmovic, Morin, and Wood [20] present $O(n \log^2 n)$ volume drawings of graphs with bounded tree-width and $O(n)$ volume for graphs with bounded path-width. Wood [42] shows that also graphs with bounded queue number have 3D straight-line grid drawings of $O(n)$ volume. A very recent result by Dujmovic and Wood [17] shows that linear volume can also be achieved for graphs with bounded tree-width; they show 3D straight-line grid drawings of volume $c \times n$ for these graphs, where c is a constant whose value exponentially depends on the tree-width. Di Giacomo, Liotta, and Wismath [16] show $4 \times n$ volume for a subclass of series-parallel graphs. The problem of computing straight-line 3D drawings of planar graphs on an integer grid of $o(n^2)$ volume is still open. A recent lower bound on the volume of 3D straight line drawings as a function of the number of edges is obtained by Bose, Czyzowicz, Morin, and Wood [2].

The remainder of the paper is organized as follows. Preliminaries and basic definitions are in Section 2. The study of trees drawable on restricted integer 2D grids is the topic of Section 3. In this section we also investigate the connection of grid drawings and path-width. Section 4 presents the linear volume algorithm for outerplanar graphs. Combinatorial properties of the graphs that can be drawn on the surface of a prism and on a box are studied in Sections 5 and 6. Final remarks, directions for further research and open problems can be found in Section 7.

2 Preliminaries

We assume familiarity with basic graph drawing, and computational geometry terminology; see for example [33, 14]. Since in the remainder of the paper we shall only study crossing-free straight-line drawings of planar graphs, from now on we shall simply talk about "graphs" to mean "planar graphs" and about "drawings" to mean "crossing-free straight-line drawings". We use the terms "vertex" and "edge" for both the graph and its drawing. We will draw graphs such that vertices are located at integer grid points. The dimensions of a grid are specified as the number of different grid points along each side of a bounding box of the grid. In two dimensions, a $p \times q$ grid consists of all points (i, j) with

$1 \le i \le p$ and $1 \le j \le q$ that have integer coordinates. In three dimensions, a $p \times q \times r$ grid consists of all points (i, j, k) with $1 \le i \le p$, $1 \le j \le q$ and $1 \le k \le r$ that have integer coordinates; p, q and r are referred to as the x-, y-, and z-*dimension* of the grid, respectively.

We shall deal with the following grids and drawings.

- A 2D 1-*track* (or simply a *track*) is an $\infty \times 1$ grid; a 1-*track* drawing of a graph G is a drawing of G where the vertices are at distinct grid points of the track.

- A 2D *strip* is an $\infty \times 2$ grid; note that a strip contains two tracks. A *strip drawing* of a graph G is a drawing of G with the vertices located at distinct grid points of the strip and the edges either connect vertices on the same track or connect vertices on different tracks.

- Next we extend the notion of a strip to multiple overlapping strips. Let k be a given positive integer value. A 2D k-*track* grid is an $\infty \times k$ grid consisting of k consecutive parallel tracks. A k-*track drawing* of a graph G is a drawing of G where the vertices are at distinct grid points of the k-track and edges are only permitted between vertices that are either on the same track or that are one unit apart in their y-coordinates. Note that the previous two grids are the specific cases of $k = 1$ and $k = 2$.

- Let k be a given positive integer value. In an $n \times k$-*grid drawing* of a graph G, the vertices are located at distinct grid points and the edges may connect any pair of vertices on that grid. To avoid confusion with track drawings, we refer to the value k as the number of *grid lines* in the grid drawing.

- We will also study two different types of $n \times 2 \times 2$ grids. A *box* is an $n \times 2 \times 2$ grid where each side of the bounding box is also a grid line. Therefore, a box has four tracks which lie on two parallel planes and are one grid unit apart from each other. A *prism* is a subset of an $n \times 2 \times 2$ grid obtained by removing a track from a box. Figure 1 shows an example of a box of size $6 \times 2 \times 2$ and an example of a prism.

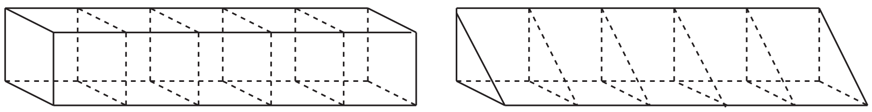

Figure 1: A box and a prism

Note that k-track drawings differ from the so-called k-level drawings (see, e.g. [25]) as in a k-track drawing (consecutive) vertices on the same track are permitted to be joined by an edge and the given graph is undirected. Let ϕ be one of the grids defined above. We say that a graph G is ϕ *drawable* if G admits a ϕ drawing Γ where each vertex is mapped to a distinct grid point of ϕ.

Property 1 *A graph is 1-track drawable if and only if it is a forest whose vertices have degree at most two.*

While in a k-track drawing no edge can connect vertices that are on non-consecutive tracks, in an $n \times k$-grid this is allowed. As the following property shows, this difference has immediate consequences on the families of k-track drawable and $n \times k$-grid drawable graphs.

Property 2 *Let $k \geq 3$ be a fixed positive integer. There exist graphs with n vertices that are $n \times k$-grid drawable but are not k-track drawable.*

Proof. The graph K_4 has an $n \times 3$-grid drawing but it does not have a 3-track drawing – indeed K_4 is not drawable on tracks. Furthermore any graph containing K_4 as a subgraph is not track-drawable. Given a drawing of a graph on an $n \times k$-grid (for $k > 3$), we can attach a copy of K_4 which makes the resulting graph not track-drawable. □

In the extended abstract for the graph drawing conference GD'01[22] we incorrectly argued that for trees $n \times k$-grid drawable is equivalent to k-track drawable. It was first observed by Matthew Suderman that this was false. In a recent manuscript Suderman [40] describes a family S^k of trees, such that S^k can be drawn on the $n \times (k+1)$ grid but requires $2k-1$ tracks. Suderman's results are actually stated in terms of the pathwidth pw of a tree (a notion introduced in the next section). He shows that $\mathsf{pw}(S^k) = k$ and every tree T with $\mathsf{pw}(T) \leq k$ admits a drawing on $2k-1$ tracks.

3 Grids, Path-width and Trees

In this section we investigate the connections between drawability of a graph G on grids and the path-width of G. The notion of the *path-width* of a graph G was introduced by Robertson and Seymour[34] in the first paper of their series on graph minors. A *path decomposition* of a graph $G = (V, E)$ is a sequence $W_1, W_2, \ldots, W_t$ of subsets of V such that

- $\forall (u, v) \in E, \quad \exists i$ such that $u, v \in W_i$
- $\forall v \in V$ the set $I(v) = \{i : v \in W_i\}$ is an interval of $\{1, \ldots, t\}$, *i.e.* if $a < b < c$ and $a, c \in I(v)$ then $b \in I(v)$.

The width of the path decomposition is $\max(|W_i| - 1 : i = 1, \ldots, t)$ and the path-width of G, denoted $\mathsf{pw}(G)$ is the minimum width of a path decomposition of G. The path-width of an independent set is defined as zero.

Several graph parameters have been shown to be equivalent to path-width. The *interval thickness* $\Theta(G)$ is the smallest max-clique over all interval supergraphs of G. Since interval graphs are perfect this can also be stated in terms of the chromatic number: $\Theta(G) = \min\{\chi(H) : H$ is an interval graph with $E(G) \subseteq E(H)\}$. Möhring [30] has shown that $\Theta(G) = \mathsf{pw}(G) + 1$. In *node searching*, an undirected graph is considered as a system of tunnels in which a fugitive is hidden. The *node search number* $\mathsf{ns}(G)$ is the least number of searchers required to capture the fugitive when the search is governed by the following rules: A search move consists of placing a searcher at a node or removing a searcher from a node. The fugitive is captured if both ends of the edge where he hides are simultaneously occupied by a searcher. The fugitive is allowed to move (at any speed) along edges subject to the condition that he never passes a node occupied by a searcher. Kiriousis and Papadimitriou [28] have shown that $\Theta(G) = \mathsf{ns}(G)$. Moreover there is an optimal search, *i.e.*, a search requiring only $\mathsf{ns}(G)$ searchers, such that after an edge has been cleared by two searchers simultaneously guarding its end-nodes it never recontaminates, *i.e.*, there never appears a path that carries no searcher connecting the cleared edge with a contaminated (uncleared) one.

We now show that the path-width is a lower bound on the number k of grid lines needed for an $n \times k$ grid drawing for general planar graphs. We note that a similar result was obtained in [19] however in the context of h-layer graph drawings.

Theorem 1 *Let G be a planar graph. Then*
$\mathsf{pw}(G) \leq \min_k$ *(G is drawable on an $n \times k$ grid).*

Proof. We prove the inequality in the node searching context; recall $\mathsf{ns}(G) = \mathsf{pw}(G) + 1$. Given a planar graph G which is drawn on an $n \times k$ grid, we show that the layout can be used to design a node search strategy using $k+1$ searchers for G. At the beginning a grid-line-searcher is placed on the leftmost node of each of the k grid lines of the drawing. The invariant is that at any intermediate step of the search there is a searcher on each grid line and the k nodes occupied by these searchers form a node separator. Left of the searchers there are cleared edges and nodes; edges and nodes to the right are not yet cleared and there is no edge connecting a cleared with an uncleared node. A *move* consists of identifying a searcher s sitting on a node $v(s)$ on grid line $t(s)$, such that either there is no edge connecting $v(s)$ to an uncleared node or there is exactly one such edge and this edge connects $v(s)$ to the next node right of $v(s)$ on the same grid line. In both cases the $(k+1)$st searcher s^* is placed on the next node v' on the same grid line $t(s)$. Then s moves to v' thus setting s^* free again. We claim that a move as described is possible as long as not all the k grid-line-searchers are sitting on the rightmost node of their line. Since a move clearly keeps the invariant valid the claim implies that the graph can be decontaminated from left to right in a sequence of moves. It remains to prove the existence of a move. Label the grid lines $t_1, t_2, \ldots, t_k$ from lowermost to topmost. Denote the node occupied by the grid-line-searcher on t_i by v_i. With v_i associate the number h_i of the highest grid line such that there is an uncleared edge from v_i to a node on grid line h_i; if there is no uncleared edge leaving v_i we set $h_i = 0$. Since $h_k \leq k$ there is a least index i such that $h_i \leq i$. We claim that we can choose

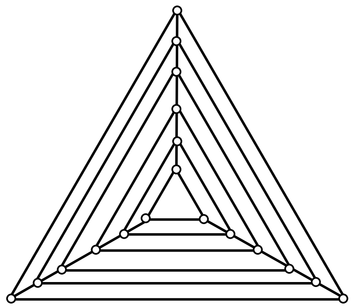

Figure 2: Six nested triangles.

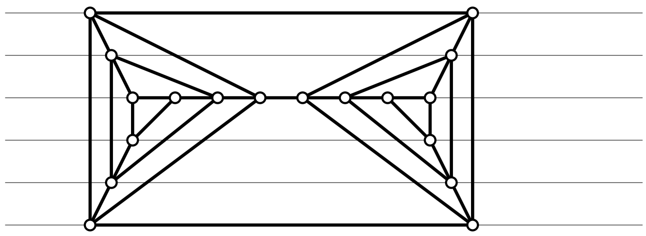

Figure 3: A drawing of G_3 on an $n \times 6$ grid.

the searcher of grid line t_i for the move. If $i = 1$ condition $h_1 \leq 1$ implies that we can advance the searcher on the first grid line. If $i > 1$ consider the edge emanating from node v_{i-1} to a node on grid line $h_{i-1} \geq i$. By planarity and since $h_{i-1} \geq i$ this edge shields v_i from all uncleared nodes on grid lines t_j for $j < i$. Therefore, there can be at most one edge leaving v_i to an uncleared node to the right and this node must sit on the same grid line. □

For the case of trees Suderman [40] has given tight bounds for the gap between path-width and the required height of the grid. Depending on the drawing model chosen, this height is bounded by the β fold of the path-width with $\beta \in \{\frac{3}{2}, 2, 3\}$.

In contrast to the situation with trees, the gap in the inequality of Theorem 1 can be arbitrarily large for other classes of planar graphs. Let G_k be the graph consisting of $2k$ nested triangles (see Figure 2 for the case $k = 3$). It is not difficult to see that a grid drawing of G_k requires at least $2k$ grid lines (*i.e.* an $n \times 2k$ grid). Namely, since G_k is three-connected it is enough to study the drawings given by all choices of outer faces. If the outer face is a three cycle, by induction we have that a drawing of G_k requires a number of grid lines that is at least twice the number of nested triangles, *i.e.* it requires at least $4k$ grid lines. If the outer face is a four cycle, we still have to draw a three cycle with at least k nested triangles inside, and therefore at least $2k$ grid lines are required. However, the node-search number of G_k is 4, independent of k. In an optimal search the searchers sweep the triangles starting at the innermost and moving

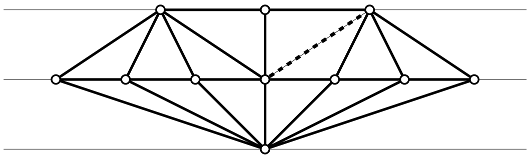

Figure 4: The graph G drawn on a grid of height 3.

out. An optimal drawing for G_3 is given in Figure 3.

3.1 Minor monotone issues

The contraction of an edge $e = (x, y)$ in a graph means replacing x and y by a single vertex z which is made adjacent to all the remaining neighbours of x and y.

Given a graph G we can generate smaller graphs by repeatedly deleting and contracting edges and deleting isolated vertices. These smaller graphs are called *minors* of G. Define $G' \prec G$ if G' is a minor of G. The relation $\prec$ is an order relation on the set of all graphs. A set $\mathcal{P}$ of graphs is *minor monotone* if $G \in \mathcal{P}$ and $G' \prec G$ implies $G' \in \mathcal{P}$.

Important examples of minor monotone sets of graphs are: forests, outer-planar graphs, planar graphs and for any fixed k, the set of graphs G with $\mathsf{pw}(G) \leq k$. A fundamental result of Robertson and Seymour [35] asserts that every minor monotone set of graphs M is characterized by a finite set of obstructions, *i.e.* there is an integer t and a list of graphs $O_1, O_2, \ldots, O_t$ such that $G \in M$ iff $O_i \not\prec G$ for $i = 1, \ldots, t$. A classical instance is the theorem of Wagner [41]: G is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$.

From Theorem 1 it is conceivable to view the minimal height k such that a planar graph admits a drawing on an $n \times k$ grid as a more discriminating, *i.e.* refined, version of path-width. The following theorem shows that this parameter 'grid-height' lacks one very important property.

Theorem 2 *Being drawable on an $n \times k$ grid is not a minor monotone graph property for $k \geq 3$.*

Proof. The graph G shown in Figure 4 is drawn on three grid lines. By contracting the dotted edge we obtain the graph G' shown to the left in Figure 5. We show that G' requires a grid of height at least four.

In Figure 5 the vertex resulting from the contraction is emphasized and labeled c, another vertex of degree two is labeled t. Let G'_t be the graph obtained by deleting t.

The first step of the proof is to show that in every drawing of G'_t the bold vertices are drawn on the middle line. This is based on an observation which is interesting in its own right.

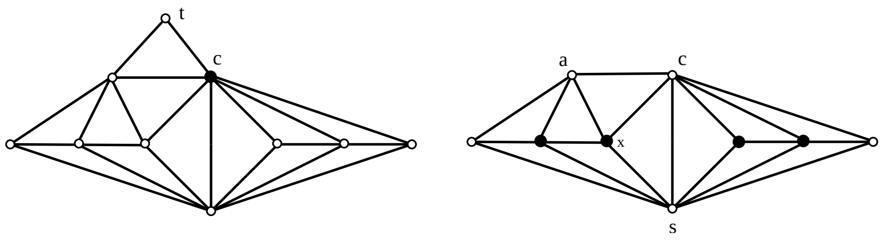

Figure 5: The contraction graph G' and its subgraph G'_t

> Let G be embedded on a grid of height k and let C be the outer cycle of G. The interior vertices of G, *i.e.*, those not on C, are embedded on the grid lines from 2 to $k-1$.

For the case of three grid lines, $k = 3$, this implies that all vertices not on the outer cycle must be drawn on the middle grid line. As a consequence, the graph induced by the inner vertices must be a subgraph of a path.

In the case of G'_t there are two candidates for the outer cycle namely, the one shown in the figure and the cycle of length four including the edge (s, c). In both cases the four black vertices are interior and must be drawn on the middle grid line. Assume that s is placed on grid line 1; then both vertices a and c have to go on grid line 3. A drawing of G' contains a drawing of G'_t, hence, for a drawing of G' on a grid of height three we also have: If s is on line 1 then a and c are on line 3. However, since x and t are common neighbours of a and c this placement of a and c makes a crossing of edges unavoidable. In conclusion the minor G' of G has no drawing on a grid of height 3. $\square$

An interesting open problem suggested by one of the referees is whether for trees and forests the required height for grid drawings might be a minor monotone parameter.

3.2 Drawings of Trees

Next we consider drawings of trees. As a first step we characterize the family of strip-drawable trees and give linear time recognition and drawing algorithms for such trees.

The approach taken applies to strip-drawable trees and has a natural generalization which leads to the notion of a k-strict tree. We show that a $(k+1)$-strict tree cannot be drawn on an $n \times k$ grid. We then show that the strictness of a tree is closely related to the path-width. More precisely, there is a difference of at most one between the two parameters. In his recent paper Matthew Suderman [40] has obtained tight bounds on the height of a grid required for drawings of trees of path-width k in several drawing models.

3.2.1 Strip-Drawable Trees

Property 1 establishes that all paths are strip-drawable, since they are in fact 1-track drawable. We define a tree as *2-strict* if it contains a vertex of degree greater than or equal to three. An immediate consequence of Property 1 is the following.

Property 3 *A 2-strict tree is not* 1*-track drawable.*

An edge is defined as a *core edge* if its removal results in two 2-strict components. For an edge $e = (u, v)$, we refer to the two subtrees resulting from its removal (but including vertices u, v) as T_u and T_v.

Lemma 1 *Core edges are connected.*

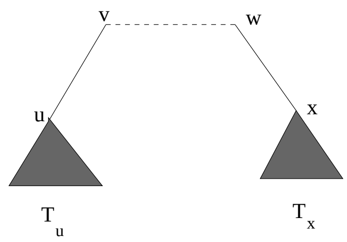

Figure 6: Core edges are connected

Proof. Let $e_1 = (u, v)$ and $e_2 = (w, x)$ be two core edges and consider any edge e on the path connecting e_1 to e_2. Refer to Figure 6. Edge e receives one 2-strict component from T_u and one from T_x and thus must be core. □

Lemma 2 *A tree is strip drawable if and only if its core edges form a path.*

Proof. ($\Rightarrow$) (by contradiction) By the previous lemma, if the core edges do not form a path, then there is a vertex v with at least three incident core edges $(v, a), (v, b), (v, c)$ – see Figure 7. If the subtrees T_a, T_b, T_c are drawable then by Property 3 their associated drawings $\Gamma_a, \Gamma_b, \Gamma_c$ each require two tracks. There is no location for v that permits a crossing-free connection to all three subdrawings. ($\Leftarrow$) Refer to Figure 8. If the core edges form a non-degenerate path (*i.e.* a non-zero length path), then draw them consecutively on track t_1. Consider an arbitrary non-core edge $e = (u, v)$ with u on track t_1. Since e is non-core, T_v must *not* be 2-strict and is thus 1-track drawable. Therefore v can be placed on track t_2 with the drawing of T_v also on the same track, as in Figure 8. There is one degenerate case to consider. If there are no core edges (*i.e.* a path of length 0), then either the tree has no vertex of degree three and is in fact drawable on a single track, or there exists at most one vertex v with neighbours $w_1, w_2, ...w_k$ and each T_{w_i} is *not* 2-strict. Each of the subtrees can

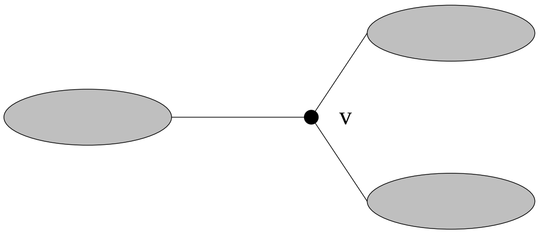

Figure 7: T is not strip-drawable if the core edges are not a path

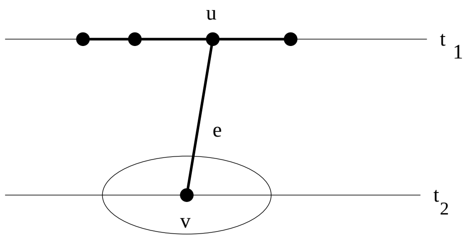

Figure 8: Drawing a tree on a strip

thus be drawn on track t_2 and v on track t_1 as in Figure 9. □

Based on this characterization, we now consider the complexity of recognizing and drawing the trees that are strip-drawable.

Lemma 3 *Let T be a tree with n vertices. There exists an $O(n)$-time algorithm that recognizes whether T is strip-drawable and, if so, computes a strip drawing of T.*

Proof. Note that a tree is 2-strict if and only if it has more than two leaves; thus counting leaves is the crucial operation. First the core edges must be

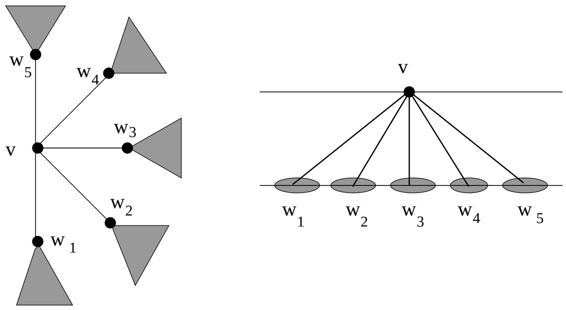

Figure 9: A degenerate core path

established and then the path condition on the core edges checked. With each edge $e = (u, v)$ we associate two counters: l_u will be the number of leaves in T_u, and l_v will be the number of leaves in T_v. Let l be the number of leaves in the entire tree T. Since $l_u + l_v = l$ it follows that e is a core edge if and only if both l_u and l_v are larger than 2. Choose an arbitrary non-leaf vertex r as a root. Each vertex v reports the number of leaves in the subtree below it to its parent u – thus establishing l_v for the edge (u, v) and hence l_u. If v has no children then it is a leaf and reports 1. A simple recursive function can be used to implement this counting step in linear time. Finally, checking that the core edges form a path is easily accomplished in linear time and the proof of the previous Lemma described the construction of the strip drawing. □

We can summarize Lemmas 2 and 3 as follows.

Theorem 3 *A tree T with n vertices is strip drawable if and only if its core edges form a path. Furthermore, there exists an $O(n)$-time algorithm that determines whether T is strip drawable and, if so computes a strip drawing of T.*

3.2.2 k-Strict Trees

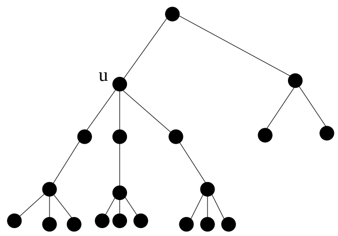

Figure 10: A 3-strict tree

The results of Theorem 3 can be extended to give a necessary condition for trees to be drawable on an $n \times k$ grid by generalizing the concepts of the previous section. A tree is *k-strict* if it contains a vertex adjacent to at least three vertices whose subtrees are $(k-1)$-strict. For example, the tree of Figure 10 is 3-strict since the vertex u is adjacent to three 2-strict subtrees.

Lemma 4 *A $(k+1)$-strict tree is not drawable on an $n \times k$ grid.*

Proof. The proof is by induction on k and Property 3 provides the base case. If a tree T is $(k+1)$-strict then it contains a vertex v adjacent to at least three vertices whose subtrees are k-strict and by the inductive hypothesis each subtree requires at least k tracks to be drawn. In this case, there is no location for v on the k tracks that allows it to connect to the three subtrees without creating an intersection. □

Corollary 1 *The complete ternary tree of height*[1] $k+1$ *is not drawable on an* $n \times k$ *grid.*

Proof. Such a tree is $(k+1)$-strict and hence not drawable on the $n \times k$ grid. □

3.2.3 k-strict trees and Path-width

In this subsection we show that the strictness of a tree is tightly related to the well known parameter path-width.

Theorem 4 *Let T be a tree. Then*

$$\mathsf{pw}(T) \leq \max_k \; (T \textit{ is } k\textit{-strict}) \leq \mathsf{pw}(T) + 1.$$

Proof. For the proof we compare the strictness of a tree T to its node search number $\mathsf{ns}(T) = \mathsf{pw}(T) + 1$ (refer to the beginning of this section for the definitions). In the proof we make use of a lemma attributed to Parsons [32] in [29]

> [Parsons' Lemma] *For any tree T and integer $k \geq 1$, $\mathsf{ns}(T) \geq k+1$ if and only if T has a vertex v at which there are three or more branches that have search number k or more.*

First we show that for a tree T k-strictness implies $\mathsf{ns}(T) \geq k$, by induction on k. Assume T is 2-strict. Then T contains a vertex of degree 3; in particular therefore at least one edge and $\mathsf{ns}(T) \geq 2$. Now assume T is k-strict, $k > 2$. Then T contains a vertex v at which three branches T_1, T_2, T_3 are $(k-1)$-strict. By induction, each T_i satisfies $\mathsf{ns}(T_i) \geq k-1$, and by Parsons' lemma $\mathsf{ns}(T) \geq k$.

Next we show that $\mathsf{ns}(T) \geq k$ implies that T is $(k-1)$-strict, again by induction on k. Assume that $\mathsf{ns}(T) = 2$. Then T contains an edge, and therefore it is 1-strict. Now $\mathsf{ns}(T) = k > 2$. By Parson's lemma T contains a vertex v with three branches T_1, T_2, T_3 such that $\mathsf{ns}(T_i) \geq k-1$. By induction, T_i is $(k-2)$-strict, and therefore T is $(k-1)$-strict. □

4 Three-Dimensional Drawings of Outerplanar Graphs

In Section 3, Corollary 1 showed that, for a fixed k, there is no $n \times k$ grid that supports all trees of n vertices. This motivates us to investigate the existence of three-dimensional restricted grids that support all trees. As it turns out, the situation in three dimensions is distinctly different. Namely, we show that all outerplanar graphs are prism-drawable by providing a linear time algorithm

[1] The height is measured as the number of vertices on the path from the root to the deepest leaf.

that computes this drawing. This is the first known three-dimensional straight-line drawing algorithm for the class of outerplanar graphs that achieves $O(n)$ volume on an integer grid. A high level description of our drawing algorithm, called `Algorithm Prism Draw`, is as follows. Let G be an outerplanar graph with a specified *outerplanar embedding*, *i.e.* a circular ordering of the edges incident around each vertex such that all vertices of G belong to the external face. (Such an embedding can be computed in linear time). `Algorithm Prism Draw` computes a prism drawing of G by executing two main steps. Firstly a 2D drawing of G is computed on a grid that consists of $O(n)$ horizontal tracks and such that adjacent vertices are at grid points whose y-coordinates differ by at most one. This is done by visiting G in a breadth-first fashion and setting the x-coordinate to be the breadth-first search (BFS) number and the y-coordinate to be the depth in the BFS tree. Secondly, the drawing is "wrapped" onto the faces of a prism by folding it along the tracks.

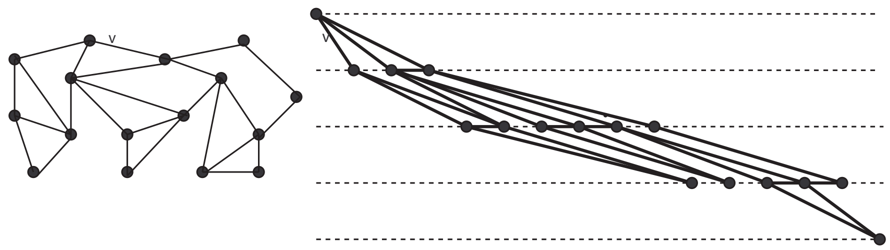

Figure 11: An outerplanar graph drawn by `Step 1` of `Algorithm Prism Draw`.

`Algorithm Prism Draw`

input: An outerplanar graph G with a given outerplanar embedding.
output: A prism drawing of G.

`Step 1. The 2D Drawing Phase:` A 2D track drawing Γ of G where vertices are assigned to different x-coordinates is computed as follows.

- Add a dummy vertex d on the external face and an edge connecting d to an arbitrary vertex v.
- mark d
- i:=0
- $currx$:=0
- draw v on track t_0 by setting $X(v)$:= $currx$; $Y(v)$:=i
- $currx$:= $currx$ +1
- mark v

- while there are unmarked vertices of G do
  - visit the vertices on track t_i from left to right and for each encountered vertex u do
    - let w be a marked neighbour of u in G
    - visit the neighbours of u in counterclockwise order starting from w, and for each encountered vertex r such that r is unmarked do
      - draw r on track t_{i+1} by setting $X(r) := currx$; $Y(r) := i{+}1$
      - $currx := currx + 1$
      - mark r
  - $i := i{+}1$

`Step 2: The 3D Wrapping Phase:` A prism drawing Γ' is obtained by folding Γ along its tracks as follows.

- for each vertex v of Γ define its coordinates $X'(v)$, $Y'(v)$ and $Z'(v)$ in Γ' by setting:
  - $X'(v) := X(v)$
  - if $Y(v) = 0, 1 \; mod \; 3$ then $Y'(v) := 0$, else $Y'(v) := 1$
  - if $Y(v) = 0, 2 \; mod \; 3$ then $Z'(v) := 0$, else $Z'(v) := 1$

`End of Algorithm Prism Draw`

Figure 11 shows an example of the output of Step 1 of the algorithm; for consistency with track layout terminology, the Y axis points downwards. The following results establish that `Algorithm Prism Draw` computes a prism drawing of any outerplanar graph G. First observe that *currx* is incremented each time a vertex is drawn, and therefore we have the following proposition.

Proposition 1 *No two vertices of* Γ *are assigned the same X-coordinate.*

Also, since the unmarked neighbours of a vertex u are all drawn on the track consecutive to that of u during Step 1, we have the following.

Proposition 2 *A vertex is assigned to track* t_{i+1} *if and only if it has not yet been marked/assigned and has a neighbour on track* t_i*, for* $i \geq 0$*.*

The following lemmas establish that the drawing between any two consecutive tracks forms a strip drawing, and therefore Step 1 of `Algorithm Prism Draw` computes a k-track drawing of the input graph.

Lemma 5 *Let* G *be an outerplanar graph with a given embedding and let* Γ *be the drawing computed by Step 1 of* `Algorithm Prism Draw`*. Then* Γ *is a* k*-track drawing of* G *for some* $k \leq n$*.*

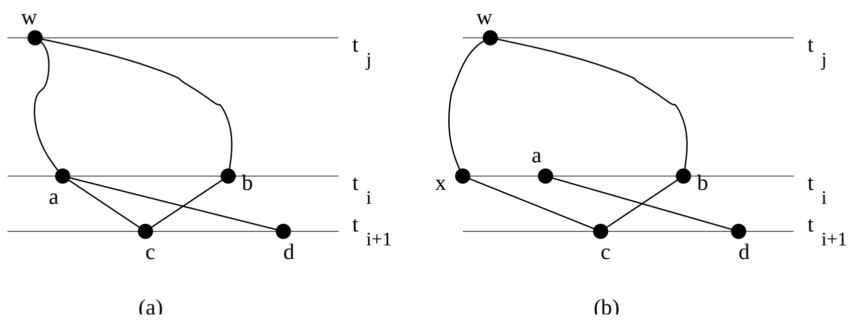

Figure 12: Two cases for the proof of Lemma 5.

Proof. Step 1 of `Algorithm Prism Draw` draws G on a 2D k-track, where $k \leq n$. Also, by Proposition 2 we have that an edge of Γ can connect only vertices that are drawn either on the same track or on two consecutive tracks. In order to complete the proof we must show that Γ satisfies the following properties.

1. No two edges connecting vertices on consecutive tracks intersect.

2. Let u and v be two vertices of Γ which are drawn on the same track t. If u and v are adjacent in G, then they appear consecutively on t.

We start by proving that Γ has the first property. Suppose there exist four vertices a, b, c, d such that

- a and b are on track t_i and a is to the left of b.
- c and d are on track t_{i+1}.
- There is an edge (a, d) intersecting an edge (b, c).

Note that `Algorithm Prism Draw` draws all the unmarked neighbours of a on track t_{i+1} by following the counterclockwise order of the edges around a given by the outerplanar embedding of G. Also note that all the neighbours of a on track t_{i+1} are assigned an X-coordinate that is strictly smaller than the X-coordinates assigned to the neighbours of any vertex drawn to the right of a on track t_i. Therefore we may conclude that if (a, d) and (b, c) cross, then one of the two cases must hold: Case 1: Vertex c is a neighbour of both b and a (see Figure 12a). Case 2: There is a vertex x, drawn on track t_i to the left of a and such that c is a neighbour of both x and b (see Figure 12b). Consider Case 1. By Proposition 2 and by the fact that `Algorithm Prism Draw` places only vertex v on track t_0, it follows that there exists a lowest common ancestor of both a and b, say w, drawn on some track t_j with $0 \leq j < i$. Let Π_{wa} and Π_{wb} be two disjoint paths connecting w to a and w to b respectively. Observe that Step 1 of `Algorithm Prism Draw` computes a drawing that preserves the given outerplanar embedding of G since it draws the vertices on consecutive tracks with increasing x values and by following the circular ordering of the edges around the vertices. Therefore, if edge (a, d) follows edge (a, c) in the counterclockwise ordering of the edges around a, then in the outerplanar embedding of G there must be a cycle (namely that formed by the path Π_{wa}, edge (a, c), edge (c, b) and path Π_{wb}) with vertex d in its interior. But this is a contradiction. Now consider Case 2. By the same argument used for the previous case, there is a lowest common ancestor w of x and b on some track t_j with $0 \leq j < i$. Let Π_{wx} and Π_{wb} be two disjoint paths connecting w to x and w to b respectively. Now observe that vertex d would necessarily lie in the interior of the cycle defined by Π_{wx}, edge (x, c), edge (c, b) and path Π_{wb} thus contradicting the fact that Γ preserves the outerplanar embedding of G. Finally, we prove that Γ satisfies the

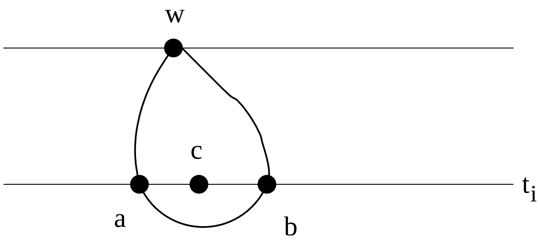

Figure 13: Edges on a given track only join consecutive vertices.

second property and again the proof is by contradiction. Suppose there exist two vertices a and b on track t_i $(0 < i \leq k)$ such that:

- a and b are adjacent in G.
- There exists a vertex c in Γ such that c is drawn on track t_i between a and b. See also Figure 13.

By the same reasoning as that used in the previous cases, let w be the lowest common ancestor of a and b on some track t_j with $0 \leq j < i$ and let Π_{wa} and Π_{wb} be two disjoint paths connecting w to a and w to b respectively. Consider the cycle formed by Π_{wa}, Π_{wb} and edge (a, b). This cycle has vertex c in its interior contradicting the fact that Γ preserves the outerplanar embedding of G. □

Theorem 5 *Every outerplanar graph G with n vertices admits a crossing-free straight-line grid drawing in three dimensions in optimal $O(n)$ volume. Furthermore,* `Algorithm Prism Draw` *computes such a drawing of G in $O(n)$ time and with the vertices of G drawn on the grid points of a prism.*

Proof. Lemma 5 ensures there are no crossings in the k-track drawing. Concerning `Step 2 (3D Wrapping Phase)`, observe that the coordinate assignment is such that the vertices of Γ' are grid points of a $n \times 2 \times 2$ grid and they all belong to just three of the four possible tracks of the $n \times 2 \times 2$ grid. Therefore, the vertices of Γ' are drawn on grid points of a prism. Also, no two edges of Γ' intersect because, as shown above, each subdrawing of Γ' induced by vertices on two different tracks is a strip drawing and because the vertices on each track have distinct X-coordinates. Finally, note that `Algorithm Prism Draw` runs in linear time since it is essentially a breadth-first traversal of the graph. □

Remark: Note that Step 2 of the algorithm is applicable given any track-drawing of a graph after suitable shifting to ensure increasing x-coordinates. Thus, graph G is track-drawable implies that G is prism-drawable. The converse however does not hold; K_4 is an example of a graph that is prism-drawable but not track-drawable.

5 Prism-Drawable Graphs

Motivated by Theorem 5, we study in this section whether the prism is a universal grid for planar graphs. For example, Figure 14 shows a maximal planar graph, and its prism drawing. As another example, note that the family of maximal planar graphs consisting of a sequence of nested triangles (as in Fig-

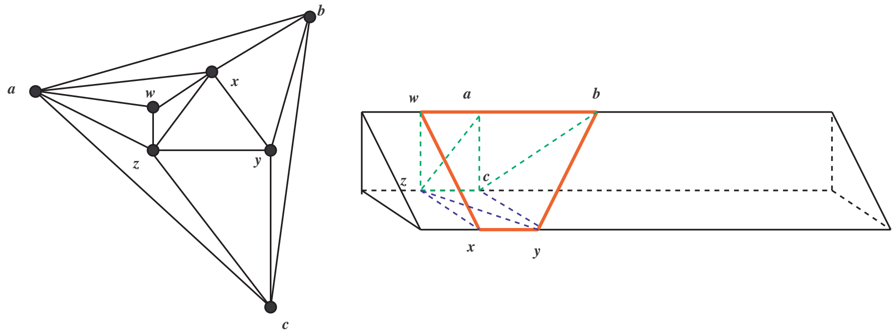

Figure 14: A prism-drawable graph G and its drawing

ure 2) and that are known to require $\Omega(n^2)$ area in the plane [11], can easily be drawn on the prism in $O(n)$ volume. Unfortunately, it turns out that not all planar graphs are prism-drawable. In Section 5.1 we give a characterization of prism-drawable graphs and in Sections 5.2 and 5.3 we illustrate two different approaches for constructing planar graphs that violate the characterization. The first approach is based on the concept of *strictly-prism drawable* graphs and the second exploits the relationship between hamiltonicity and prism-drawability.

5.1 Characterization of Prism-Drawable Graphs

An essential prerequisite of our characterization of prism-drawable graphs, is the study of the strip-drawable graphs since a prism effectively consists of three strips. Independently, Cornelsen, Schank and Wagner [10] developed a linear time algorithm for determining if a graph is strip-drawable. Clearly such graphs must be (a subset of the) outerplanar graphs since the two tracks, which contain all vertices, form the exterior face. Our characterization differs significantly from that contained in [10].

We define a *spine* in a graph G as a sequence $v_0, v_1, \ldots v_m$ of vertices such that the subgraph induced by $v_0, v_1, \ldots v_m$ is a path. The definition of spine precludes any edge between non-consecutive vertices; we refer to such an edge as a *chordal* edge. The characterization of prism-drawable graphs is based on the observation that in a strip drawing, there must exist two sub-spines (each defined by the vertices on one of the two tracks of the strip) and that edges connecting vertices on these two sub-spines must not intersect. Since every graph with less than four vertices is clearly strip-drawable (a three cycle is strip-drawable and therefore every subgraph of a three cycle is strip drawable), the next theorem considers graphs with at least four vertices. Note that although this characterization is not efficiently implementable, it is a basis for characterizing the prism-drawable graphs, the box-drawable graphs and provides a means for showing that planar graphs are not necessarily prism-drawable.

Theorem 6 *A graph G with at least four vertices is strip-drawable if and only if it is possible to augment G with edges to produce a graph G' such that:*

- *G' contains two edges (r_0, b_0) and (r_z, b_t).*
- *There are two vertex-disjoint spines $r_0, r_1, \ldots r_z$ and $b_0, b_1, \ldots b_t$ in G' such that all vertices of G are on the two spines.*
- *If there exists an edge (r_i, b_j) with $0 \le i \le z$ and $0 \le j \le t$ then there are no edges of the form (r_k, b_l) with $(0 \le k < i$ and $j < l \le t)$ or $(i < k \le z$ and $0 \le l < j)$.*

Proof. ($\Rightarrow$)We show how to construct a graph G' that satisfies the statement. Let Γ be a strip drawing of G with t_1 and t_2 as the two tracks of the strip. Let r_0, b_0 be the leftmost pair of vertices and let r_z, b_t be the rightmost pair of vertices of Γ such that b_0 and b_t are on track t_1. If r_0, b_0 are not adjacent in Γ, then edge (r_0, b_0) is added; similarly, if r_t, b_t are not adjacent in Γ, then edge (r_t, b_t) is added. Also, for each pair of consecutive non-adjacent vertices encountered when walking along each track an edge is added so to form two paths Π_1 and Π_2. Let Γ' be the new drawing and let G' be the graph represented by Γ'. Note that G' has two edges (r_0, b_0) and (r_t, b_t). Since there are no chordal edges between any two non consecutive vertices of Γ that are on the same track, it follows from the construction that Π_1 and Π_2 are spines for G'. Each vertex of G is drawn either on track t_1 or on track t_2 and by construction of Γ' it belongs either to Π_1 or to Π_2; it follows that all vertices of G are on the two spines of G'. Also, since in Γ there are no crossings between any two edges connecting vertices on different tracks and since in Γ' edges (r_0, b_0) and (r_t, b_t) do not cross any other edge, it follows that if there exists an edge (r_i, b_j) in G' with $0 \le i \le z$

and $0 \leq j \leq t$ then there cannot be edges of the form (r_k, b_l) with $(0 \leq k < i$ and $j < l \leq t)$ or $(i < k \leq z$ and $0 \leq l < j)$.
$(\Leftarrow)$Given an augmented graph G', a strip drawing Γ' of G' is obtained as follows. Spine $r_0, r_1, \dots r_z$ is drawn on one track such that r_j is drawn to the right of r_i for $0 \leq i < j \leq z$. Spine $b_0, b_1, \dots b_t$ is drawn on the second track such that b_j is drawn to the right of b_i for $0 \leq i < j \leq t$. Edges connecting vertices along the same track and between the two tracks are drawn as straight-line segments. Since the subgraph induced by each spine is a path and since for each edge (r_i, b_j) with $0 \leq i \leq z$ and $0 \leq j \leq t$ there are no edges of the form (r_k, b_l) with $(0 \leq k < i$ and $j < l \leq t)$ or $(i < k \leq z$ and $0 \leq l < j)$, it follows that the drawing of G' does not have crossings. Finally, a strip drawing of G is obtained by deleting edges from Γ'. □

The characterization of prism drawable graphs generalizes Theorem 6 to three dimensions. Intuitively, it must be possible to augment a given graph to obtain three spines with two "lids" (three cycles) and between each pair of spines the strip drawability condition must hold. Since every strip-drawable graph is also prism-drawable, the next theorem assumes that G has at least four vertices.

Theorem 7 *A graph G with at least four vertices is prism-drawable if and only if it is possible to augment G with edges to produce a graph G' such that:*

- *G' contains two three-cycles r_0, b_0, g_0 and r_z, b_t, g_s, where $z, t, s \geq 0$.*
- *There are three vertex-disjoint spines $r_0, r_1, \dots r_z$, $b_0, b_1, \dots b_t$, and $g_0, g_1, \dots g_s$ in G' such that all vertices of G are on the three spines.*
- *For each pair of spines $x_0, x_1, \dots x_m$ and $y_0, y_1, \dots y_p$ $(x, y \in \{r, b, g\}$, $x \neq y$, $m, p \in \{z, t, s\})$, if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with $(0 \leq k < i$ and $j < l \leq p)$ or $(i < k \leq m$ and $0 \leq l < j)$.*

Proof. $(\Rightarrow)$ We show how to construct a graph G' that satisfies the statement. Let Γ be a prism drawing of G and let t_1, t_2, and t_3 be the three tracks of the prism. Consider the subgraph G_{ij} induced by the vertices that are on two different tracks t_i and t_j $(i, j = 1, 2, 3, i \neq j)$; G_{ij} is strip-drawable and therefore there exists an augmented graph G'_{ij} with the properties stated by Theorem 6. A graph G' that satisfies the statement is then defined as $G' = G'_{12} \cup G'_{13} \cup G'_{23}$.
$(\Leftarrow)$ Given an augmented graph G', a prism drawing Γ' of G' is obtained as follows. Spine $r_0, r_1, \dots r_z$ is drawn on track t_1 such that r_j is drawn to the right of r_i for $0 \leq i < j \leq z$. Spine $b_0, b_1, \dots b_t$ is drawn on track t_2 such that b_j is drawn to the right of b_i for $0 \leq i < j \leq t$. Spine $g_0, g_1, \dots g_s$ is drawn on track t_3 such that g_j is drawn to the right of g_i for $0 \leq i < j \leq s$. Edges connecting vertices along the same track and between two different tracks are drawn as straight-line segments. Since the subgraph induced by each spine is a path and since for each edge (x_i, y_j) $(x, y \in \{r, b, g\}$; $x \neq y$; $m, p \in \{z, t, s\})$ there are no edges of the form (x_k, y_l) with $0 \leq k < i$ and $j < l \leq p$ or $i < k \leq m$ and

$0 \leq l < j$, it follows that Γ' does not have crossings. Finally, a prism drawing of G is obtained by deleting edges from Γ'. □

In the rest of the paper it will be convenient to imagine the three spines $r_0, r_1, \ldots r_z$, $b_0, b_1, \ldots b_t$, and $g_0, g_1, \ldots g_s$ of Theorem 7 as coloured red, blue and green, respectively. Also, we shall refer to a graph G' described in Theorem 7 as an *augmented graph* of G.

5.2 Prism-Drawability and Planarity

In this section we show that prism drawable graphs are a proper subset of planar graphs. In the next section we shall further restrict the set of prism-drawable graphs.

Theorem 8 *Let G be a prism-drawable graph. Then G is planar.*

Proof. Any prism-drawing of G can be augmented by edges to form a convex polytope and therefore by the theorem of Steinitz [43] only planar graphs are prism-drawable. □

Corollary 2 *If G is a maximal planar graph and is prism drawable, then the augmented graph G' coincides with G.*

One approach for constructing planar *prism-forbidden* graphs, *i.e.* planar graphs which do not admit a prism drawing, is based on the following definition and lemma. A graph G is *strictly prism-drawable* if it is prism-drawable and all prism drawings of G have at least three edges (x, y), (y, z) and (z, x) such that x, y and z are on different tracks.

Lemma 6 *Let G be a 1-connected planar graph that has a cut vertex v whose removal separates the graph into h strictly prism-drawable components G_1, ... G_h $(h \geq 3)$. Then G is prism-forbidden.*

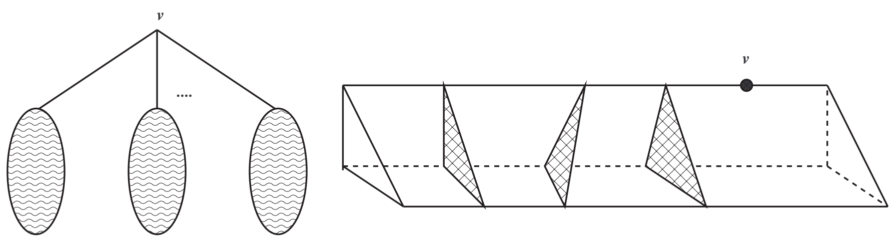

Figure 15: The "slicing" argument in the proof of Lemma 6.

Proof. Consider any prism drawing Γ_i of G_i $(0 \leq i \leq h)$. Γ_i has a three-cycle which defines a plane that intersects all three facets of the prism, because G_i

is strictly prism-drawable. Thus, $\Gamma_1, ..., \Gamma_h$ slice the prism into $h+1$ slices (see Figure 15). Now there is no location for v that permits it to be connected to all Γ_i $(0 \leq i \leq h)$ without crossing at least one three-cycle. □

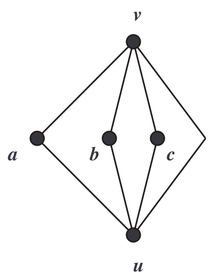

Figure 16: A strictly prism-drawable graph.

Lemma 6, provides the key to creating a prism-forbidden graph. Although K_4 can easily be shown to be strictly prism-drawable we choose to show that the graph in figure 16 is strictly prism-drawable for three reasons: the extension to the box-forbidden case is more natural, the series-parallel case follows as a consequence, and the proof portrays the importance of the spine characterization of Theorem 7.

Lemma 7 *The graph in Figure 16 is strictly prism-drawable.*

Proof. Let G be the graph of Figure 16. We first show that G satisfies Theorem 7. The augmented graph G' is defined by adding to G edges (a, b) and (b, c). We choose the two three-cycles of G' as follows: One three-cycle consists of edges (u, v), (v, a), (a, u); the second three-cycle consists of edges (u, v), (u, c), (c, v). The three spines of G' are as follows: the red spine is the path of vertices a, b, c, the blue spine consists of vertex u and the green spine consists of vertex v. Since G' is planar and two of the three spines consist of a single vertex, the non-crossing condition stated by Theorem 7 among edges connecting vertices on different spines is trivially verified. It follows that G is prism-drawable. It remains to show that G is *strictly* prism-drawable. This is done by proving the following two claims.

1. In any prism drawing of G vertices u and v cannot be on the same track.
2. In any prism drawing of G at least one of vertices a, b, c is drawn on a track different from that of u and different from that of v.

To prove Claim 1, suppose that there existed a prism drawing Γ of G with u and v on the same track t_1 (see Figure 17(a)). Since u and v are adjacent in G, there exists in Γ a straight-line segment connecting u to v. As a consequence neither a, nor b, nor c can be drawn on t_1 or else there would be an edge overlapping edge (u, v). It follows that vertices a, b, and c are drawn as points of the other two tracks, and therefore at least two of them are on the same track. Suppose without loss of generality that a and b are both drawn on track t_2. Observe that there is no way of drawing edges (a, u), (a, v), (b, u), (b, v) avoiding a crossing. It follows that in any prism drawing of G u and v must appear on different tracks.

To prove Claim 2 we assume that u is drawn on track t_1 and that v is drawn on track t_2. Let t_3 be the third track of the prism. Assume there existed a prism drawing with vertices a, b, and c all on tracks t_1 and t_2. Assume without loss of generality that both a and b are on track t_1. In order to avoid crossings it must be that one vertex, say a, is on the right-hand side and the other is on the left-hand side of u (see Figure 17(b)). Note however that c cannot be drawn on track t_1 or else edge (u, c) would intersect one of the edges (a, u), (b, u). But if c were drawn on track t_2, then edge (c, u) would intersect either (a, v) or (c, v). It follows that c is drawn on track t_3 and therefore G is strictly prism-drawable.
□

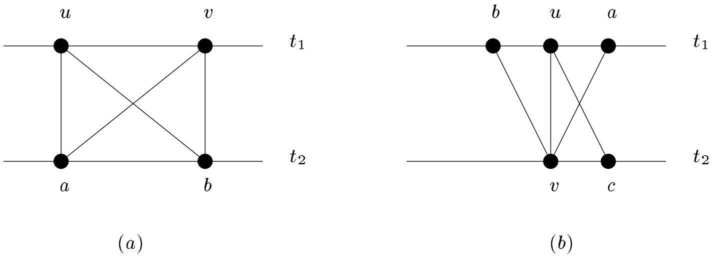

Figure 17: Illustration for the proof of Lemma 7.

Theorem 9 *There exist prism-forbidden planar graphs.*

Proof. Let G be a planar graph with a vertex v adjacent to three copies of the graph of Figure 15. Let G_0 be the subgraph of G induced by v and by these three copies. By Lemma 6 G_0 is prism-forbidden. It follows that also G is prism-forbidden. □

A consequence of the previous lemmas is the following.

Corollary 3 *There exist prism-forbidden series-parallel graphs.*

Proof. Let G be the graph in Figure 18. By using the same argument in the proof of Theorem 9 the corollary follows. □

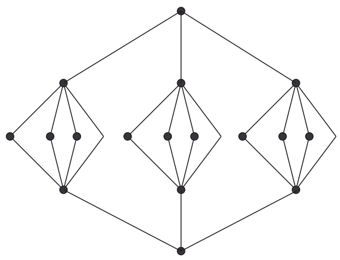

Figure 18: A prism-forbidden series-parallel graph.

5.3 Prism-Drawability and Hamiltonicity

Theorem 9 shows that not all planar graphs admit a prism drawing. In this section we further restrict the family of drawable graphs by exploiting the relation between prism-drawability and hamiltonicity.

A graph is *subhamiltonian* if it can be augmented with edges to produce a planar graph having a hamiltonian cycle.

Lemma 8 *Let G be a prism-drawable graph. Then G is subhamiltonian.*

Proof. Let Γ be a prism drawing of G and let G' be a maximally augmented graph obtained from G by adding edges to satisfy the conditions of Theorem 7. Let the three tracks of the prism be labeled t_1, t_2 and t_3 (see for example Figure 19(a)). Label the vertices on track t_2, as $r_0, \ldots r_z$ and the vertices on track t_3 as $b_0, \ldots b_y$. The hamiltonian cycle begins at the start vertex of t_1, visits the start vertex of t_2, (r_0), the start vertex of t_3 (b_0) and then alternates between the t_2 and t_3 track visiting all vertices on those two tracks and ending at the end vertex of either tracks t_2 or t_3 (see for example Figure 19(b)). In either case, the cycle then visits the end vertex of t_1 and then the entire t_1 spine in reverse order.

We now give a more formal description of the cycle's traversal of the strip between tracks t_2 and t_3. Note that r_0 is adjacent to b_0 (and 0 or more consecutive vertices on track t_3). Since G' is maximally augmented, each vertex b_i ($0 < i < y$) is adjacent to b_{i-1} and b_{i+1} and to a non-empty set of consecutive vertices on the r-track $r_j, \ldots r_{j+k}$. Furthermore, r_j is adjacent to b_{i-1}, and r_{j+k} is adjacent to b_{i+1}. The hamiltonian cycle goes from r_0 to b_0 and then applies a greedy-like approach. In general, from b_i, the cycle goes to the first (*i.e.* lowest-indexed) neighbour r_j that has not previously been visited. Only if all the neighbours of b_i on the r-track have been visited, does the cycle go to b_{i+1}. The rule on the r track is symmetric. Since at each vertex, there is always at least one neighbour that has not been visited, namely the next vertex on the same track, it is clear that the cycle can always proceed. To establish that all vertices are visited, a straightforward proof by contradiction can be applied.

The (partial) hamiltonian cycle thus ends at the endpoint of either track t_2 or t_3 and can be completed as described above. □

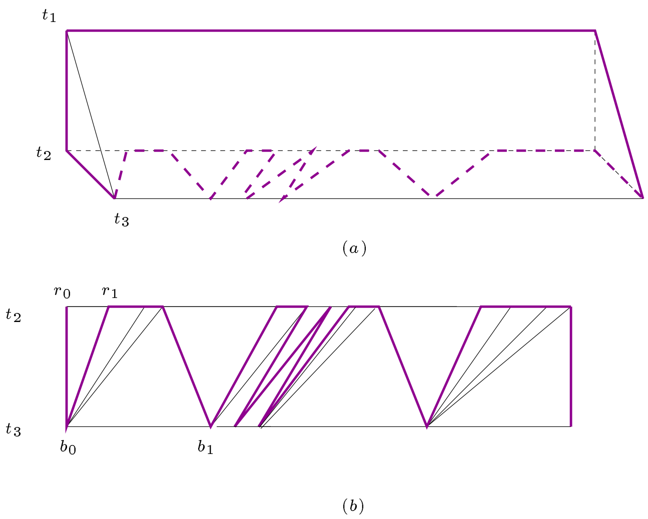

Figure 19: Illustration for the proof of Lemma 8

A natural question arising from Lemma 8 is whether all subhamiltonian planar graphs are prism drawable. This is not the case, for example the graph of Figure 18 is subhamiltonian but not prism drawable. The following lemma shows that even subhamiltonian graphs with only 9 vertices may not be prism drawable. In this instance it is the proof technique that is of primary interest.

Lemma 9 *The hamiltonian maximal planar graph G of Figure 20 is prism-forbidden.*

Proof. Suppose for a contradiction that G were prism-drawable and let G' be the augmented graph of G described in Theorem 5.

Consider the vertex a of G displayed in Figure 20. We start by showing that vertex a must be an endvertex in one of the three spines of G'.

Let Γ be a prism drawing of G. Since G is maximal planar, by Corollary 2 it follows that G coincides with its augmented graph G' and that Γ is also a prism drawing of G'. If a were not an endvertex of a spine, then the point representing a in Γ would be adjacent to two other points on the same track. But all neighbours of a are mutually adjacent in G and this would imply a chordal edge in Γ, which is impossible.

By the same argument, also vertices b and c of G (see Figure 20) are endvertices of some spine in G'.

By Theorem 5, the endvertices of the spines in G' belong to two three-cycles. Therefore, at least two vertices among a, b, and c must be connected in G'. But

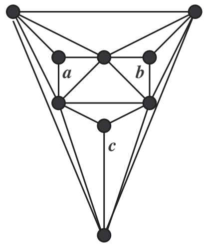

Figure 20: A hamiltonian planar graph G that is not prism-drawable.

G is identical to G' by Corollary 2 and no pair of these vertices are adjacent in Figure 20. This provides the required contradiction and proves that G is indeed prism-forbidden. □

Based on Lemmas 8 and 9 we can summarize the discussion of this section as follows.

Theorem 10 *The family of prism-drawable graph is a proper subset of the family of subhamiltonian planar graphs.*

6 Box-Drawable Graphs

Motivated by Theorems 9 and 10, we consider a restricted integer 3D grid consisting of four tracks, namely the *box*, and ask whether this grid supports all planar graphs. Clearly, the class of box-drawable graphs is larger than the class of prism-graphs: Every prism-drawable graph can be drawn on the box and it is easy to draw some non-planar graphs on a box. For example, Figure 21 shows a box drawing of K_5 and a box drawing of $K_{3,3}$. However we will show that even the box is not a universal grid for planar graphs.

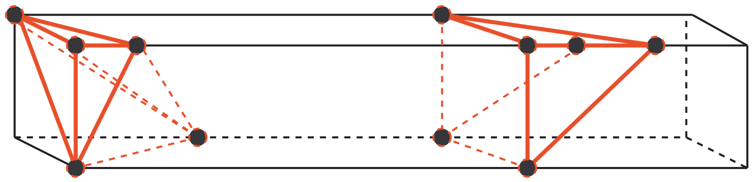

Figure 21: K_5 and $K_{3,3}$ drawn on a box

6.1 Characterization of Box-Drawable Graphs

We start our investigation by characterizing the family of box-drawable graphs with more than five vertices. Note that any graph with at most five vertices is box drawable and its box drawing can be obtained by deleting edges and vertices from the drawing of K_5 depicted in Figure 21. The next theorem follows the same pattern as Theorem 7.

Theorem 11 *A graph G with at least six vertices is box-drawable if and only if it is possible to augment G with edges to produce a graph G' such that:*

- *G' contains two four-cycles r_0, b_0, g_0, w_0 and r_z, b_t, g_s, w_q, where $z, t, s, q \geq 0$.*
- *The graph G' contains four vertex-disjoint spines $r_0, r_1, \ldots r_z$, $b_0, b_1, \ldots b_t$, $g_0, g_1, \ldots g_s$, and $w_0, w_1, \ldots w_q$ such that all vertices of G are on the four spines.*
- *For each pair of spines $x_0, x_1, \ldots x_m$ and $y_0, y_1, \ldots y_p$ ($x, y \in \{r, b, g, w\}$, $x \neq y$, $m, p \in \{z, t, s, q\}$) , if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with ($0 \leq k < i$ and $j < l \leq p$) or ($i < k \leq m$ and $0 \leq l < j$).*

Proof. ($\Rightarrow$) We show how to construct a graph G' that satisfies the statement. Let Γ be a box drawing of G and let t_1, t_2, t_3, and t_4 be the four tracks of the box. Consider the subgraph G_{ij} induced by the vertices that are on two different tracks t_i and t_j ($i, j = 1, 2, 3, 4,\ i \neq j$); G_{ij} is strip-drawable and therefore there exists an augmented graph G'_{ij} with the properties stated by Theorem 6. A graph G' that satisfies the statement is then defined as $G' = G'_{12} \cup G'_{13} \cup G'_{14} \cup G'_{23} \cup G'_{24} \cup G'_{34}$.
($\Leftarrow$) Among the four tracks t_1, t_2, t_3, and t_4 of the box, we assume that the pairs t_1, t_3 and t_2, t_4 are diagonally opposite and that the four tracks are horizontal lines. Given an augmented graph G', a box drawing Γ' of G' is constructed by spiralling the vertices on the four tracks as follows.

- Spine $r_0, r_1, \ldots r_z$ is drawn on track t_1 such that r_j is drawn to the right of r_i for $0 \leq i < j \leq z$.
- Spine $b_0, b_1, \ldots b_t$ is drawn on the diagonally opposite track t_3 such that b_0 is given an x-coordinate larger than the x-coordinate of r_z and b_j is drawn to the right of b_i for $0 \leq i < j \leq t$.
- Spine $g_0, g_1, \ldots g_s$ is drawn on track t_2 such that g_0 is given an x-coordinate larger than the x-coordinate of b_z and g_j is drawn to the right of g_i for $0 \leq i < j \leq s$.
- Spine $w_0, w_1, \ldots w_q$ is drawn on track t_4 such that w_0 is given an x-coordinate larger than the x-coordinate of g_s and w_j is drawn to the right of w_i for $0 \leq i < j \leq q$.

Then, edges connecting vertices along the same track and between two different tracks are drawn as straight-line segments. In order to see that the computed drawing does not have edge crossings observe the following.

- There are no edges between non consecutive vertices on the same track: For each spine, the subgraph of G' induced by the spine is a path which is drawn sequentially on a track.
- For each pair of tracks t_h, t_f ($h, f \in \{1, 2, 3, 4\}$, $h \neq f$) the edges connecting vertices on t_h with vertices on t_f do not cross with each other. Namely, for an edge (x_i, y_j) connecting a vertex on t_h with a vertex on t_l, there are no edges of the form (x_k, y_l) with ($k < i$ and $j < l$) or ($i < k$ and $l < j$).
- For each pair of diagonally opposite edges e_1, e_2 such that e_1 connects a vertex on track t_1 with a vertex on track t_3 and e_2 connects a vertex on track t_2 with a vertex on track t_4 there are no crossings. By construction, the coordinates of the endpoints of e_2 are both strictly larger than those of the endpoints of e_1.

Since Γ' is a box drawing and G' is a supergraph of G, a box drawing of G can be obtained by deleting edges from Γ'. □

6.2 Box-Drawability and Planarity

We extend the approach of Section 5.2 to construct planar graphs that cannot be drawn on a box. We call these graphs *box-forbidden*. In order to construct a box-forbidden graph, we need a preliminary lemma.

Lemma 10 *In any box drawing of the graph of Figure 22 vertices u and v are on different tracks.*

Proof. Let G be the graph of Figure 22. It is trivial to see that G is prism-drawable and hence it is also box-drawable. Let Γ be a box drawing of G. Suppose for a contradiction that vertices u and v in Γ were represented as points of the same track t_1. Observe that no other vertex of Γ can be a point of t_1 or else there would be a crossing. Since G has six vertices, a box consists of four tracks, and track t_1 cannot contain more than two vertices, it follows that at least two other vertices of Γ must be on another track, say t_2 of the box. But each vertex on t_2 is adjacent to both u and v, which forces a crossing in Γ; contradiction. □

A graph G is *strictly box-drawable* if it is box-drawable and there are four mutually adjacent vertices a, b, c and d and in all box drawings of G, a, b, c and d appear on separate tracks.

Lemma 11 *The graph of Figure 23(a) is strictly box-drawable.*

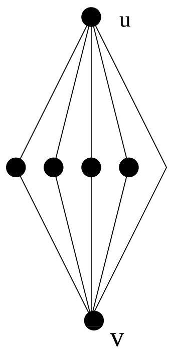

Figure 22: Graph for Lemma 10.

Proof. Let G be the graph of Figure 23(a). We first prove that G is box-drawable and then that it is strictly box-drawable. We adopt the notation of Figure 23(a).

We apply Theorem 11 to G. The four spines of G' are $r_0, .., r_8$, $b_0, .., b_4$, $g_0, .., g_8$, and $w_0, .., w_4$. The two four-cycles have the vertices r_0, b_0, g_0, w_0 and r_8, b_4, g_8, w_4. Also, as Figures 23(b), (c), (d), (e), (f), and (g) show, the subgraphs of G' induced by vertices on two different spines are strip-drawable. It follows that G is box-drawable.

We now prove that G is strictly box-drawable. Graph G consists of six copies of the graph in Figure 22. By Lemma 10 in any box drawing of G vertices r_0, g_4, b_4, w_4 must be on four different tracks. Furthermore, those four vertices are mutually adjacent. It follows that G is strictly box-drawable. □

Theorem 12 *There exist box-forbidden planar graphs.*

Proof. Let G be a planar graph with a vertex v adjacent to three copies of the graph of Figure 23. Let G_0 be the subgraph of G induced by v and by these three copies; see also Figure 24. Removing v and all its incident edges from the graph G_0 splits it into three components that we name G_1, G_2, and G_3. In any box drawing Γ_i of G_i $(0 \leq i \leq 3)$ there are four mutually adjacent vertices on four different tracks because by Lemma 11 G_i is strictly box-drawable. Thus, Γ_1, Γ_2, and Γ_3 slice the box into 4 slices and there is no location for v that permits it to be connected crossing-free to all Γ_i $(0 \leq i \leq 3)$. □

7 Conclusions and Open Problems

In this paper we showed that all outerplanar graphs can be drawn in linear volume on a prism – a restriction of the three dimensional integer grid. Proofs that certain classes of planar graphs are not prism-drawable nor box-drawable

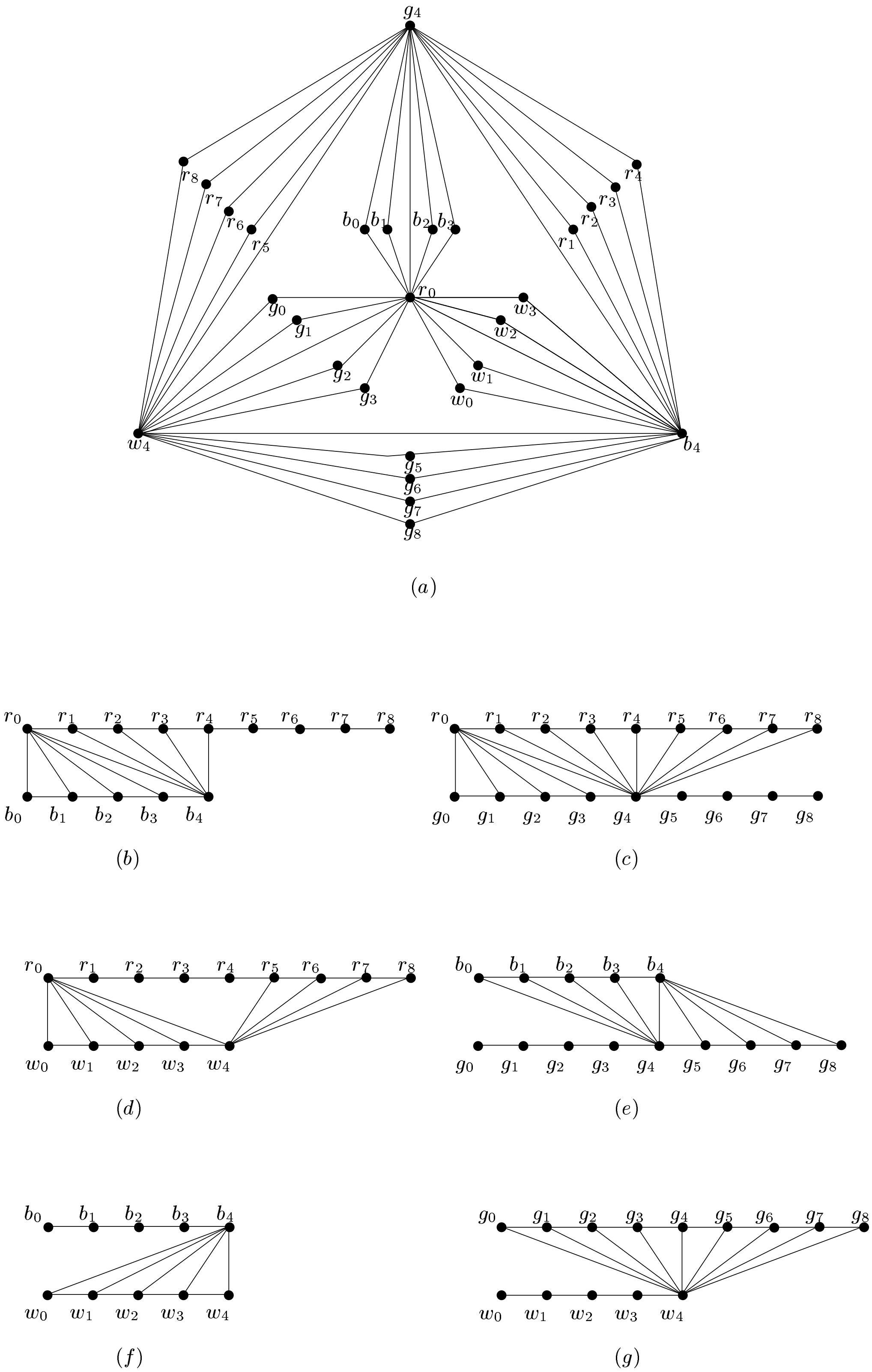

Figure 23: (a) Graph G for Lemma 11. (b)–(e) The subgraphs of G induced by vertices on two different spines are strip-drawable.

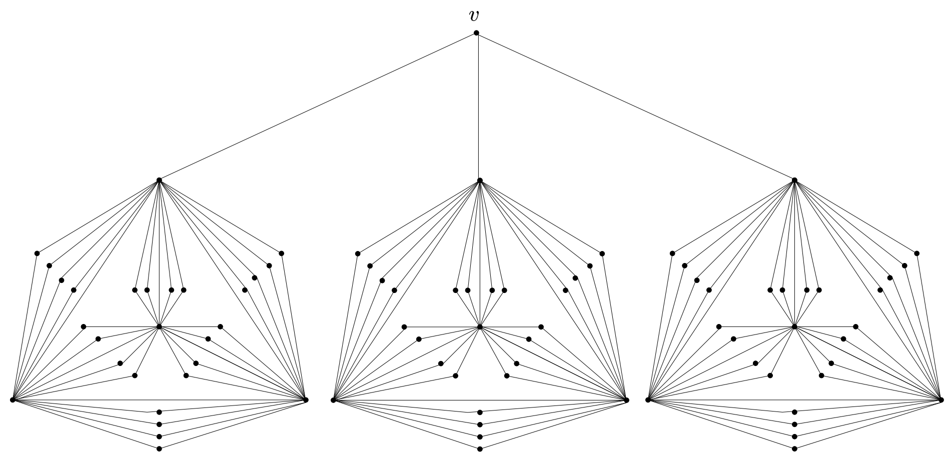

Figure 24: A planar graph that is not box-drawable.

were also provided. Although the problem of finding a universal integer 3D grid of linear volume that supports crossing-free straight line drawings of all planar graphs is still far from being solved, the drawing techniques and characterization results of this paper may provide a critical starting point for attacking such an ambitious research programme. We believe that the results on the restricted 2D grid are not only useful preliminary results for the study in three dimensions, but they may also shed some new light on the problem of drawing trees in linear area on the plane. There remain several interesting problems and directions for further research. We conclude the paper by describing some of the more intriguing open problems.

1. Can all outerplanar graphs be drawn in linear area on a 2D integer grid? Does there exist a 2D universal grid set of linear area that supports all outerplanar graphs?

2. Characterize the graphs drawable on an $n \times k$ grid.

3. Can the strong algorithms for recognizing graphs of bounded pathwidth be applied to devise polynomial dynamic programming algorithms to decide k-track drawability for fixed k? Such an approach has been applied in [18] for the recognition of proper k-track drawability for fixed k. Also, Schank [36] gave a direct linear time algorithm for the task of recognizing 2-track drawable graphs.

4. Can all planar graphs be drawn in linear volume on a three-dimensional integer grid? Does there exist a 3D universal grid set of linear volume that supports all planar graphs?

5. *The k-lines drawability problem*: A related problem posed by H. de Fraysseix [13] asks if all planar graphs can be drawn on k parallel lines that lie on the surface of a cylinder, for a fixed value of k. Our results on box-drawability imply that k would have to be strictly greater than 4.

6. *The aspect ratio problem*: Our results about linear volume come at the expense of aspect ratio. Is it possible to achieve both linear volume and $o(n)$ aspect ratio for outerplanar graphs? We conjecture that it is in fact not possible in 2D to simultaneously attain linear area and $O(1)$ aspect ratio for some classes of planar graphs.

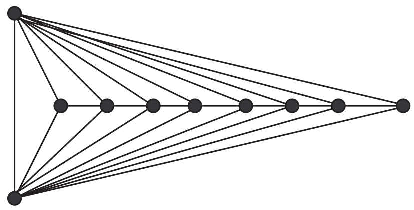

Figure 25: A graph S_n with poor aspect ratio

Conjecture 1 *There is no fixed constant k for which the family of graphs S_n (in Figure 25) can be drawn in a 2D integer grid of size $k\sqrt{n} \times \sqrt{n}$.*

Note that the graph S_n can be drawn on an $n \times 3$ grid (and hence in linear area but with *linear* aspect ratio). Recently this conjecture was verified and the result is reported in [1].

Acknowledgments
We thank Helmut Alt, Pino di Battista, Hazel Everett, Ashim Garg, Mark Keil, Sylvain Lazard and David Wood for useful discussions related to the results contained in this paper. We are grateful to Matthew Suderman for pointing out an error in earlier versions of this paper and to the referees for their many helpful suggestions.

References

[1] T. Biedl, T.M. Chan, A. López-Ortiz. Drawing $K_{2,n}$: A Lower Bound. In *Proc. 14th Can. Conf. on Comput. Geom.*, 146–148, 2002

[2] P. Bose, J. Czyzowicz, P. Morin, D. Wood. The Maximum Number of Edges in a Three-Dimensional Grid-Drawing. *Technical Report TR-02-04*, School of Computer Science, Carleton University, 1–4, Oct. 2002.

[3] T. Calamoneri and A. Sterbini. Drawing 2-, 3-, and 4-colorable graphs in $o(n^2)$ volume. In S. North, editor, *Graph Drawing (Proc. GD '96)*, volume 1190 of *Lecture Notes Comput. Sci.*, pages 53–62. Springer-Verlag, 1997.

[4] M. Capobianco and J. Molluzzo. *Examples and Counterexamples in Graph Theory.* North-Holland p.169 1978.

[5] T.M. Chan. A near-linear area bound for drawing binary trees. In *Proc. 10th Annu. ACM-SIAM Sympos. on Discrete Algorithms.*, pages 161–168, 1999.

[6] M. Chrobak and G. Kant. Convex grid drawings of 3-connected planar graphs. *Internat. J. Comput. Geom. Appl.*, 7(3):211–223, 1997.

[7] M. Chrobak, M.T. Goodrich, and R. Tamassia. Convex drawings of graphs in two and three dimensions. In *Proc. 12th Annu. ACM Sympos. Comput. Geom.*, pages 319–328, 1996.

[8] M. Chrobak, and S. Nakano. Minimum-width grid drawings of plane graphs. *Comput. Geom. Theory Appl.*, 11:29–54, 1998.

[9] R. F. Cohen, P. Eades, T. Lin, and F. Ruskey. Three-dimensional graph drawing. *Algorithmica*, 17:199–208, 1997.

[10] S. Cornelsen, T. Schank, and D. Wagner. Drawing Graphs on Two and Three Lines. Graph Drawing 2002, *Lecture Notes in Comput. Sci.* 2528, pages 31-41, Springer, 2002.

[11] H. de Fraysseix, J. Pach, and R. Pollack. Small sets supporting Fary embeddings of planar graphs. In *Proc. 20th ACM Sympos. Theory Comput.*, pages 426–433, 1988.

[12] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[13] H. de Fraysseix, personal communication.

[14] G. Di Battista, P. Eades, R. Tamassia, and I.G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.

[15] R. Diestel. *Graph theory.* Graduate Texts in Mathematics. 173. Springer, 2000. Transl. from the German. 2nd ed.

[16] E. Di Giacomo, G. Liotta and S.K. Wismath. Drawing Series-Parallel Graphs on a Box. *14th Canadian Conference On Computational Geometry (CCCG '02)*, pages 149–153, 2002.

[17] V. Dujmovic and D.R. Wood. Tree-Partitions of k-Trees with Applications in Graph Layout. *29th Workshop on Graph Theoretic Concepts in Computer Science (Proc. WG '03)*, *Lecture Notes in Comput. Sci.*, Springer-Verlag, to appear.

[18] V. Dujmovic, M. Fellows, M. Hallett, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. Rosamond, M. Suderman, S. Whitesides. A Fixed-Parameter Approach to Two-Layer Planarization. Graph Drawing 2001, *Lecture Notes in Comput. Sci.* 2265, pages 1-15, Springer, 2002.

[19] V. Dujmovic, M. Fellows, M. Hallett, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. Rosamond, M. Suderman, S. Whitesides, D.R. Wood. On the Parameterized Complexity of Layered Graph Drawing. *ESA*, 1–12, 2001.

[20] V. Dujmovic, P. Morin, and D.R. Wood. Path-Width and Three-Dimensional Straight-Line Grid Drawing of Graphs. Proc. of 10th International Symposium on Graph Drawing (GD '02), *Lecture Notes in Comput. Sci.* 2528, pages 42-53, Springer, 2002.

[21] S. Felsner. Convex drawings of planar graphs and the order dimension of 3-polytopes. *Order*,18, 19–37, 2001.

[22] S. Felsner, G. Liotta and S. Wismath. Straight-Line Drawings on Restricted Integer Grids in Two and Three Dimensions. In M. Jünger and P. Mutzel, editor, *Graph Drawing (Proc. GD '01)*, volume 2265 of *Lecture Notes Comput. Sci.*, pages 328–343. Springer-Verlag, 2002.

[23] A. Garg, M. T. Goodrich, and R. Tamassia. Planar upward tree drawings with optimal area. *Internat. J. Comput. Geom. Appl.*, 6:333–356, 1996.

[24] A. Garg, R. Tamassia, and P. Vocca. Drawing with colors. In *Proc. 4th Annu. European Sympos. Algorithms*, volume 1136 of *Lecture Notes Comput. Sci.*, pages 12–26. Springer-Verlag, 1996.

[25] M. Juenger and S. Leipert. Level planar embedding in linear time. In J. Kratochvil, editor, *Graph Drawing (Proc. GD '99)*, volume 1731 of *Lecture Notes Comput. Sci.*, pages 72–81. Springer-Verlag, 1999.

[26] G. Kant. A new method for planar graph drawings on a grid. In *Proc. 33rd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 101–110, 1992.

[27] G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16:4–32, 1996.

[28] L.M. Kirousis and C.H. Papadimitriou. Searching and pebbling. Theor. Comput. Sci. 47, 205–218 (1986).

[29] N. Meggido, S. Hakimi, M. Garey, D. Johnson, and C. Papadimitriou. The complexity of searching a graph. *JACM* 35, 18–44, 1988.

[30] R.H. Möhring, Graph problems related to gate matrix layout and PLA folding. (English) Computational graph theory, Comput. Suppl. 7, 17–51 (1990).

[31] J. Pach, T. Thiele, and G. Tóth. Three-dimensional grid drawings of graphs. In G. Di Battista, editor, *Graph Drawing (Proc. GD '97)*, volume 1353 of *Lecture Notes Comput. Sci.*, pages 47–51. Springer-Verlag, 1997.

[32] T. Parsons. Pursuit-evasion in a graph. In Alavi and Lick editors, *Theory and Applications of Graphs*, Springer-Verlag, 426–441, 1976.

[33] F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, 3rd edition, October 1990.

[34] N. Robertson and P.D. Seymour. Graph Minors I. Excluding a Forest. J. of Comb. Th (B) 35, 39–61 (1983).

[35] N. Robertson and P.D. Seymour. Graph minors - a survey. in Surveys in combinatorics 1985, (Pap. 10th Br. Combin. Conf. 1985), Lond. Math. Soc. Lect. Note Ser. 103, 153-171 (1985)

[36] T. Schank. Algorithmen zur Visualisierung planarer, partitionierter Graphen. Master Thesis, Universität Konstanz, 2001. http://scampi.physik.uni-konstanz.de/ thomas/fmi/download/zula.ps

[37] P. Scheffler. Optimal embedding of a tree into an interval graph in linear time. Combinatorics, graphs and complexity, Proc. 4th Czech. Symp., Prachatice/ Czech. 1990, Ann. Discrete Math. 51, 287–291 (1992).

[38] W. Schnyder. Embedding planar graphs on the grid. In *Proc. 1st ACM-SIAM Sympos. Discrete Algorithms*, pages 138–148, 1990.

[39] W. Schnyder and W.T. Trotter. Convex embeddings of 3-connected plane graphs. *Abstracts of the AMS*, 13(5):502, 1992.

[40] M. Suderman. Pathwidth and Layered Drawings of Trees. *Technical Report, School of Comp. Sci., McGill Univ.*, SOCS-02.8, 2002.

[41] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. Math. Ann. 114, 570-590 (1937).

[42] D.R. Wood. Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing. 22nd Foundations of Software Technology and Theoretical Computer Science (FSTTCS '02), *Lecture Notes in Computer Science* 2556, Springer-Verlag, pages 348–359, 2002.

[43] G. M. Ziegler. Lectures on Poytopes. *Springer-Verlag*, 1995.