Homework 8

(1) Show that there is no positive integer \( n \) such that \( \phi(n) = 14 \).

(2) Find all positive integers \( n \) such that \( \phi(n)|n \).

(3) Show that if \( m \) and \( n \) are positive integers with \( m|n \), then \( \phi(m)|\phi(n) \).

(4) An integer \( n \) is perfect if \( \sigma(n) = n \). For example

\[
\sigma(6) = 1 + 2 + 3 + 6 = 2 \cdot 6.
\]

Let \( p \) be a prime number such that \( 2^p - 1 \) is also prime. Prove that

\[
\sigma(2^p(2^p - 1)) = 2^{p+1}(2^p - 1).
\]

(5) The multiplicative function \( g \) is said to be the inverse of the multiplicative function \( f \) if \( f \cdot g = g \cdot f = i \) (recall that \( i(1) = 1 \) and \( i(n) = 0 \) for \( n > 1 \)). Show that if \( f \) is a non-zero multiplicative function, then \( f \) has an inverse.

Solutions:

(1) Let \( n = \prod_{i=1}^{r} p_i^{a_i} \). Assume that \( \phi(n) = 14 \). Note that \( \phi(n) = \prod_{i=1}^{r} p_i^{a_i-1}(p_i-1) \). Since we have that 7|\( \phi(n) \) we get that for some prime \( p_i|n \) we have \( 7|p_i \) or \( 7|\( p_i - 1 \). If \( 7|p_i \), then \( p_i = 7 \), so \( 6 = p_i - 1 |\phi(n) \). However, 3 \( | n \), so that’s not possible.

Therefore, we get \( 7|p_i - 1 \), that is \( p_i = 7k + 1 \). Then \( 7k|\phi(n) \) = 14, which implies \( k \leq 2 \). However, 7 \( \cdot 1 + 1 = 8 \) and 7 \( \cdot 2 + 1 = 15 \), and neither of them is a prime. Therefore, for no positive integer \( n \) can we have \( \phi(n) = 14 \).

(2) Assume that for some positive integer \( n \) we have \( \phi(n)|n \). First we prove that there is at most one odd prime dividing \( n \). Let \( n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \), where \( p_1, p_2, \ldots, p_k \) are distinct odd prime numbers. Then we get \( \phi(n) = 2^{a-1} \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) \). Note that \( p_i - 1 \) is even, so \( p_i - 1 = 2m_i \) for some integer \( m_i \). Therefore

\[
\phi(n) = 2^{a-1+k} \prod_{i=1}^{k} p_i^{a_i-1} m_i,
\]

and in particular, \( 2^{a-1+k}|\phi(n) \). However, \( 2^{a+1} \not| n \). Therefore, if \( k > 1 \) then we can not have \( n|\phi(n) \). That is, there is at most one odd prime divisor of \( n \).

So, there are four cases to consider: \( n = 1 \), \( n = 2^a \), \( n = p^b \), and \( n = 2^a p^b \), where \( p \) is an odd prime number, and \( a, b \) are positive integers. When \( n = 1 \), we get \( \phi(n) = 1 \), so \( \phi(n)|n \). When \( n = 2^a \) we get \( \phi(n) = 2^{a-1} \), so \( \phi(n)|n \). When \( n = p^b \) we get \( \phi(n) \) is even, so we can’t have \( \phi(n)|n \). So the only case that is left is when \( n = 2^a p^b \). In this case, if \( \phi(n)|n \) we get

\[
(p - 1)2^{a-1}p^{b-1}|2^a p^b,
\]

which mean \( p - 1 \) is coprime to \( p \), this can only happen if \( p - 1 = 1 \) or \( 2 \). However, since we are assuming that \( p \) is odd, we get that \( p - 1 = 2 \), which means \( p = 3 \). Therefore, this can only happen when \( n = 2^a 3^b \) with \( a, b > 0 \).

So, \( \phi(n)|n \) when \( n = 1, 2^a \) or \( 2^a 3^b \).

(3) Let \( m, n \) be positive integers such that \( m|n \). Write \( n \) as product of two integers \( n_1 \) and \( n_2 \) so that \( n_1 \) is divisible only by primes that divide \( m \), and \( n_2 \) is divisible by all the primes that don’t divide \( m \). So \( \gcd(n_1, n_2) = \gcd(m, n_2) = 1 \) and \( n = n_1 n_2 \), and \( m|n_1 \). (Alternatively, we can write
\( n_1 = \gcd(m^*, n) \) for a large value of \( * \). Then we get \( \phi(n) = \phi(n_1)\phi(n_2) \) since \( \phi \) is multiplicative. If we show that \( \phi(m)|\phi(n_1) \) then, since \( \phi(n_2) \) is an integer, we get that \( \phi(m)|\phi(n) \).

Let

\[
\begin{align*}
n_1 &= p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}, \\
m &= p_1^{b_1}p_2^{b_2} \cdots p_k^{b_k},
\end{align*}
\]

where \( 0 < b_i \leq a_i \) (since \( m|n_1 \)) for \( 1 \leq i \leq k \). Therefore

\[
\frac{\phi(n_1)}{\phi(m)} = \frac{\prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1)}{\prod_{i=1}^{k} p_i^{b_i-1}(p_i - 1)} = \prod_{i=1}^{k} p_i^{a_i-b_i},
\]

and since \( a_i \geq b_i \) we get \( a_i - b_i \geq 0 \), which means

\[
\prod_{i=1}^{k} p_i^{a_i-b_i}
\]

is an integer. Therefore \( \phi(m)|\phi(n_1) \), which is the desired result.

4) Assume that \( 2^p - 1 \) is prime. Note that \( \sigma(2^{p-1}) = 1 + 2 + 2^2 + \cdots + 2^{p-1} = 2^p - 1 \) and if \( 2^p - 1 \) is prime then \( \sigma(2^p - 1) = 2^p \). Therefore

\[
\sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)2^p,
\]

which means \( 2^{p-1}(2^p - 1) \) is a perfect number.

5) Let \( f \) be a non-zero multiplicative function. We will construct the inverse of \( f \) recursively. That is we will construct \( g \) such that \( f* g(n) = e(n) \). First note that \( f(1) = 1 \) since \( f \) is assumed to be multiplicative. Define \( g(1) = 1 \) and

\[
g(n) = - \sum_{d|n, d < n} f(n/d)g(d),
\]

for \( n > 1 \). Then \( f* g(n) = e(n) \). To see this note that \( f* g(1) = f(1)g(1) = 1 = e(1) \), and for \( n > 1 \) we get

\[
f* g(n) = \sum_{d|n} f(n/d)g(d)
\]

\[
= f(1)g(n) + \sum_{d|n, d < n} f(n/d)g(d)
\]

\[
= - \sum_{d|n, d < n} f(n/d)g(d) + \sum_{d|n, d < n} f(n/d)g(d)
\]

\[
= 0.
\]