2-DISTANCE-TRANSITIVE DIGRAPHS PRESERVING A CARTESIAN DECOMPOSITION

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ABSTRACT. In this paper, we study 2-distance-transitive digraphs admitting a cartesian decomposition of their vertex set.

1. INTRODUCTION

One of the most interesting families of highly symmetric graphs is the family of *distance-transitive* graphs. We refer the reader to the beautiful survey article [13] for the current status of the project of classifying these graphs. A major step towards this classification is a theorem of Praeger, Saxl and Yokoyama [10], which investigates the structure of distance-primitive graphs (that is, distance-transitive graphs admitting a group of automorphisms acting primitively on the vertex set). The main tool in [10] is the O'Nan-Scott reduction theorem for finite primitive permutation groups. In [10], the Classification of Finite Simple Groups is used first for dealing with primitive groups of product action type and then only via the Schreier conjecture (when dealing with primitive groups of twisted wreath type).

In this paper, we remove the dependency of [10] on the Classification of the Finite Simple Groups, in the case of product action. Our proof can be used to replace the proofs in Section 2 of [10], and does not use the CFSG. It should be mentioned that in [10] the authors say they have found a Classification-free proof of the results in Section 2 in the case where "Γ is self-paired" - i.e., the graph is not a digraph (which is the only case they are actually considering in the paper) - but they do not provide even a sketch of this proof. In this paper, we provide such a proof and, in addition, are able to deal with the case of digraphs without resorting to the CFSG. The full result of [10] also uses the CFSG (specifically, the Schreier conjecture) to deal with primitive groups of twisted wreath type, which we do not consider in this paper. However, ours is a major step towards removing [10]'s dependence on the Classification.

The paper [10] and the other work surveyed in [13] is about distance-transitive graphs: that is, graphs that have the property that for any distance d and any pairs (v_1, v_2) and (w_1, w_2) of vertices at distance d, there is an automorphism of the graph that takes v_1 to w_1 and v_2 to w_2. In this paper, we relax these conditions in three ways: first, we consider digraphs rather than only graphs. Second, our (di)graphs are only 2-distance-transitive: that is, transitive on pairs of vertices that are either adjacent or at distance 2, but not necessarily transitive on vertices that are at higher distances. Third, our (di)graphs do not necessarily admit a group of automorphisms acting primitively on their vertex set. We are able to show that even with these relaxed conditions, graphs that admit a group of automorphisms that preserve a cartesian decomposition (see Section 2 and Definition 2.1 for a precise statement of our hypothesis) must have one of a few very specific structures. One of these structures is a family of digraphs ((iii) from Theorem 1.1 below) that (to the best of our knowledge) has not been previously studied. These digraphs have a strong algebraic structure and might well provide nice examples or counter-examples to other interesting problems.

In most of the arguments of this paper, we assume that our graphs are not Cayley graphs. Additional examples do arise if we allow Cayley graphs; Lemma 4.1, for example, shows that Payley tournaments are one such class; cycles (directed or undirected) are another.

Our main result is the following. The notation $\mathrm{H}(m, n)$ is used for the Hamming graph that is isomorphic to the direct product of the complete graph K_m with itself n times.

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Theorem 1.1. *Let G be a product action type group acting 2-distance transitively on the digraph Γ, where Γ is not a Cayley digraph. Then Γ is isomorphic to one of:*

(i) $\mathrm{H}(m,n)$;
(ii) *the complement of* $\mathrm{H}(m,2)$; *or*
(iii) *one of the graphs $X_q(n)$ in Example 3.3. In this case, Γ is a digraph.*

In the light of Theorem 1.1 and the proof of the main theorem of [10] we see (without having to appeal to the Classification of Finite Simple Groups) that if Γ is a distance-primitive graph, then either Γ is as in Theorem 1.1 (i) and (ii), or Γ is a Cayley graph over an elementary abelian group or over a non-abelian simple group.

Corollary 1.2. $\mathrm{H}(m,2)$ *and its complement are the only 2-arc-transitive graphs admitting a group of automorphisms of product action type.*

The hypothesis in Theorem 1.1 on the decomposition of the vertex set of Γ as a cartesian product is very important. In fact, recently Li and Seress [7] have obtained several intricate examples of 2-distance transitive graphs Γ with $V\Gamma$ not admitting an $\mathrm{Aut}(\Gamma)$-invariant cartesian decomposition. However, in these remarkable examples $V\Gamma$ has a $\mathrm{Aut}(\Gamma)$-invariant partition $\mathcal{B}$, and the quotient graph $\Gamma/\mathcal{B}$ does admit a cartesian decomposition.

Finally, the definition of "product action type" that we use in this paper (see Definition 2.1) is inspired by [11], which is a complete treatment of permutation groups that preserve a cartesian decomposition of the underlying point-set.

2. Notation and basic examples

Let H be a permutation group acting on the set Δ, let T be a transitive normal subgroup of H and let K be a transitive subgroup of the symmetric group $\mathrm{Sym}(n)$ on $\{1,\ldots,n\}$ with $n \geq 2$. We let W denote the wreath product $H \,\mathrm{wr}\, K$ acting on the cartesian product $\Omega = \Delta^n$. We recall that, for $\sigma \in K$ and $h_1,\ldots,h_n \in H$, the group element $g = \sigma(h_1,\ldots,h_n)$ of W acts on the element $(\delta_1,\ldots,\delta_n)$ of Ω by

$$(\delta_1,\ldots,\delta_n)^g = (\delta_{1^{\sigma^{-1}}}^{h_1},\ldots,\delta_{n^{\sigma^{-1}}}^{h_n}).$$

In other words, σ permutes the n coordinates of Ω and the n-tuple $(h_1,\ldots,h_n)$ acts coordinate-wise on Ω.

For each $i \in \{1,\ldots,n\}$, we denote by T_i the ith coordinate subgroup of T^n, that is, $T_i = \{(t_1,\ldots,t_n) \in T^n : t_j = 1 \text{ for each } j \in \{1,\ldots,n\} \text{ with } j \neq i\}$. As H normalizes T, the group W acts by conjugation on the set $\{T_1,\ldots,T_n\}$ and the action of W on $\{T_1,\ldots,T_n\}$ is permutation equivalent to the action of K on $\{1,\ldots,n\}$. Furthermore, the normal subgroup $N = T_1 \times \cdots \times T_n$ of W acts transitively on Ω.

Definition 2.1. We say that a subgroup G of W is of *product action type* if

(i) T is not regular on Δ,
(ii) $N \leq G$, and
(iii) the action of G on $\{T_1,\ldots,T_n\}$ is transitive.

For each $i \in \{1,\ldots,n\}$, consider the group $G_i = N_G(T_i)$. If $g = \sigma(h_1,\ldots,h_n) \in G_i$, then $i^\sigma = i$ and the function $\pi_i : G_i \to H$ mapping g to h_i defines a group homomorphism. As G is transitive on $\{1,\ldots,n\}$, for each $i,j \in \{1,\ldots,n\}$, the group H^{π_i} is conjugate to H^{π_j}. In particular, replacing H by the image of π_i if necessary, we may assume that each π_i is surjective, for each i.

In this paper, we assume that G is of product action type and is a group of automorphisms of a connected (directed or undirected) graph Γ with vertex set $V\Gamma = \Omega$. We let $A\Gamma$ denote the arcs of Γ and, for a vertex v of Γ, we let $\Gamma^+(v)$ (respectively $\Gamma^-(v)$) denote the out-neighbours (respectively in-neighbours) of v and we write

$$A^2\Gamma = \{(u,v) \in V\Gamma \times V\Gamma : u, v \text{ are non-adjacent and } \Gamma^+(u) \cap \Gamma^+(v) \neq \emptyset\}$$

(equivantly, $(u,v) \in A^2\Gamma$ if u and v are non-adjacent and $u,v \in \Gamma^-(w)$, for some $w \in V\Gamma$).

We are concerned with the case that G acts 2-*distance-transitively* on Γ in the following sense: G acts transitively on $A\Gamma$ and on $A^2\Gamma$. (We say that Γ is 2-distance-transitive.)

If Γ is undirected, then our definition coincides with the natural definition of 2-distance-transitive graphs. For digraphs, this is not the most natural definition of 2-distance-transitivity, but it is the definition to which most of our arguments naturally generalize from the undirected to the directed case. We have two more motivations (aside from feasibility) for studying this situation. First, this sort of 2-distance-transitivity was investigated by Praeger, Saxl and Yokoyama in [10] (see for example [10, Proposition 2.4]). Their analysis of G and Γ heavily depends upon the Classification of Finite Simple Groups (though they state without proof that they can avoid this in the undirected case). Since our main results generalize theirs in the case of product action, we will be producing a CFSG-free proof of this part of result. Second, our definition of 2-distance-transitivity covers the special case where G acts transitively on each of the three sorts of pairs of vertices at distance 2, namely

$$\{(u,v) \in V\Gamma \times V\Gamma : u, v \text{ are non-adjacent and } \Gamma^+(u) \cap \Gamma^+(v) \neq \emptyset\},$$

$$\{(u,v) \in V\Gamma \times V\Gamma : u, v \text{ are non-adjacent and } \Gamma^+(u) \cap \Gamma^-(v) \neq \emptyset\}$$

and

$$\{(u,v) \in V\Gamma \times V\Gamma : u, v \text{ are non-adjacent and } \Gamma^-(u) \cap \Gamma^-(v) \neq \emptyset\}$$

(which is a very natural definition of 2-distance-transitivity on digraphs). There is one more remark we wish to make in this direction. (Given a digraph Γ, denote by Γ^{opp} the digraph with $V\Gamma^{opp} = V\Gamma$ and with $A\Gamma^{opp} = \{(u,v) : (v,u) \in A\Gamma\}$.) If G acts transitively on $A\Gamma$ and on $\{(u,v) \in V\Gamma \times V\Gamma : u, v \text{ are non-adjacent and } \Gamma^-(u) \cap \Gamma^-(v) \neq \emptyset\}$, then our arguments apply immediately to G and to Γ^{opp}.

We stress that in Definition 2.1 we assume that T does not act regularly on Δ; this condition is imposed in order to avoid the case that Γ is a Cayley graph on N.

In Section 2.1, we briefly describe the strategy for our proof of Theorem 1.1. Throughout the rest of the paper, we let Δ, n, Ω, T, H and W be as above. Furthermore, let G be a product action type subgroup of W and let Γ be a connected digraph with $\Omega = V\Gamma$ and with G acting 2-distance-transitively on Γ.

Remark 2.2. The action of a group G of product action type always preserves the Hamming distance between vertices. This is easy to verify, but very important.

We fix, once and for all, δ an element of Δ, $\alpha = (\delta, \dots, \delta) \in V\Gamma$ and $\beta = (\delta_1, \dots, \delta_n) \in \Gamma^-(\alpha)$.

Remark 2.3. Since N is transitive on $V\Gamma$, we have $G = NG_\alpha$. However, as N acts trivially by conjugation on $\{T_1, \dots, T_n\}$ and G acts transitively on $\{T_1, \dots, T_n\}$, we see that G_α acts transitively by conjugation on $\{T_1, \dots, T_n\}$. We use this remark repeatedly.

2.1. Structure of the paper.

Our proof is divided in various cases, depending upon the Hamming distance between α and β. The case that α and β are at Hamming distance 1 is studied in Section 4. In Section 5, we study the case that α and β are at Hamming distance ≥ 2 and we show that $n = 2$ (in particular, α and β are at Hamming distance 2). In Section 6, we conclude the case that α and β are at Hamming distance 2: in Subsection 6.1 we study the case that Γ is undirected and in Subsection 6.2 we study the case that Γ is directed.

3. Examples of 2-distance-transitive digraphs that admit product action

In this section, we explain how to construct the graphs and digraphs that are listed in Theorem 1.1.

First we give the definition of orbital graph. This will be required in the construction of some of the examples that follow.

Definition 3.1. Let G be a transitive permutation group on the set Ω and let α and β be elements of Ω. The *orbital graph* $(\beta, \alpha)^G$ is the graph with vertex set Ω and with arc set $\{(\beta^g, \alpha^g) : g \in G\}$. The group G acts transitively on the arcs of $(\beta, \alpha)^G$. Moreover the in-neighbourhood of α is β^{G_α} and the out-neighbourhood of β is α^{G_β}.

In the next example we describe the Hamming distance and the well-known Hamming graphs.

Example 3.2. We say that the elements $\omega = (\delta_1, \ldots, \delta_n)$ and $\omega' = (\delta'_1, \ldots, \delta'_n)$ of $\Omega = \Delta^n$ are at *Hamming distance* k if ω and ω' agree in all but k coordinates, that is, $k = |\{i \in \{1, \ldots, n\} : \delta_i \neq \delta'_i\}|$.

Write $m = |\Delta|$. Let $\mathrm{H}(m,n)$ be the graph with vertex set Ω and with ω adjacent to ω' if ω and ω' are at Hamming distance 1. Clearly, the group $W = \mathrm{Sym}(\Delta) \,\mathrm{wr}\, \mathrm{Sym}(n)$ acts transitively on the vertices of $\mathrm{H}(m,n)$. The stabilizer in W of the vertex $\alpha = (\delta, \ldots, \delta)$ of $\mathrm{H}(m,n)$ is $\mathrm{Sym}(\Delta \setminus \{\delta\}) \,\mathrm{wr}\, \mathrm{Sym}(n)$ and acts transitively on the neighbourhood of α in $\mathrm{H}(m,n)$ and on the vertices at distance 2 from α in $\mathrm{H}(m,n)$. Therefore W acts 2-distance-transitively on $\mathrm{H}(m,n)$.

When $n = 2$, the graph $\mathrm{H}(m,2)$ has diameter 2 and so the complement of $\mathrm{H}(m,2)$ is also a 2-distance-transitive graph.

The directed graphs arising in the next example show some remarkable properties which (to the best of our knowledge) have not been noticed previously.

Example 3.3 (The directed graphs X_q and $X_q(n)$)**.** Let q be a power of a prime with $q \equiv 3 \mod 4$ and $q \geq 7$, and let $H = \mathrm{SL}(2,q)$ be the special linear group of rank 2. Note that as $q \equiv 3 \mod 4$, the element -1 of $\mathbb{F}_q$ is not a square. Let $V = \mathbb{F}_q \times \mathbb{F}_q$ be the vector space of dimension 2 of row vectors over the field $\mathbb{F}_q$ of size q. Let Δ be the set of orbits of the group of diagonal matrices

$$C = \left\{ \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix} | x \in \mathbb{F}_q, x \neq 0 \right\}$$

acting on the set of non-zero vectors $V^* = V \setminus \{(0,0)\}$. Since C acts semiregularly on V^* and $|C| = (q-1)/2$, each orbit of C on V^* has size $(q-1)/2$. As $|V^*| = q^2 - 1$, we obtain that Δ contains $2(q+1)$ elements. For $(a,b) \in V^*$, we denote by $[a,b]$ the element of Δ containing (a,b). Since -1 is not a square, we see that $\Delta = \{[a, \pm 1], [\pm 1, 0] : a \in \mathbb{F}_q\}$.

The only non-identity proper normal subgroup of H is the centre $Z = \langle z \rangle$, with

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the orbits of Z on Δ are $\{[a,1], [-a,-1]\}$ (for each $a \in \mathbb{F}_q$) and $\{[1,0], [-1,0]\}$. In particular, H acts faithfully on Δ. Furthermore, the action of H on the Z-orbits of Δ is the natural 2-transitive action of $H/Z = \mathrm{PSL}(2,q)$ on the $q+1$ points of the projective line. The stabilizer in H of the element $[1,0]$ is the subgroup

$$H_{[1,0]} = \left\{ \begin{pmatrix} x^2 & 0 \\ y & x^{-2} \end{pmatrix} : x, y \in \mathbb{F}_q, x \neq 0 \right\}, \tag{1}$$

which has 4 orbits on Δ, namely $\{[1,0]\}$, $\{[-1,0]\}$, $\{[a,1] : a \in \mathbb{F}_q\}$ and $\{[a,-1] : a \in \mathbb{F}_q\}$, of size $1, 1, q$ and q, respectively.

We let X_q be the H-orbital graph $([1,0],[0,1])^H$. It is an easy computation (using the fact that -1 is not a square) to see that there is no matrix $h \in H$ such that $([1,0],[0,1])^h = ([0,1],[1,0])$. Therefore X_q is a directed graph of in- and out-valency q. For example, $X_q^+([1,0]) = \{[a,1] : a \in \mathbb{F}_q\}$ and $X_q^-([0,1]) = \{[1,a] : a \in \mathbb{F}_q\}$. By applying the matrix

$$\iota = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

to the set $X_q^-([0,1])$, we obtain $X_q^-([1,0]) = \{[a,-1] : a \in \mathbb{F}_q\}$. This gives that

$$VX_q = \{[1,0],[-1,0]\} \cup X_q^+([1,0]) \cup X_q^-([1,0])$$

and $[-1,0]$ is the unique vertex of X_q not adjacent to $[1,0]$. Now vertex transitivity shows that, for each vertex v, there exists a unique vertex which is not adjacent to v (namely v^z).

We have

$$(X_q^+([1,0]))^z = \{[-a,-1] : a \in \mathbb{F}_q\} = \{[a,-1] : a \in \mathbb{F}_q\} = X_q^-([1,0])$$

and similarly $(X_q^-([1,0]))^z = X_q^+([1,0])$. Therefore, for each vertex v, we have

$$X^+(v) = X^-(v^z) \quad \text{and} \quad X^-(v) = X^+(v^z).$$

Therefore $X^+(v) \cap X^+(v^z) = \emptyset$ and $A^2X = \emptyset$.

Consider the matrix

$$o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the map $\circ : \Delta \to \Delta$ defined by $v \mapsto v^o$. It is an easy computation to show that $\circ$ determines a graph isomorphism from X_q to X_q^{opp}. So, $X_q \cong X_q^{opp}$.

Let $n \geq 2$, let $W = H \operatorname{wr} \operatorname{Sym}(n)$ and let $\alpha = ([1,0],[1,0],\ldots,[1,0])$ and $\beta = ([0,1],[1,0],\ldots,[1,0])$ be in Δ^n. We denote by $X_q(n)$ the orbital graph $(\beta,\alpha)^W$. Clearly, $X_q(1) = X_q$.

In Example 3.3, we exclude the case that $q = 3$. In fact, for $q = 3$, the graphs $X_q(n)$ are still well-defined, but, since the socle of the group $H = \mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$ acts regularly on Δ, we get that $X_q(n)$ is a Cayley graph (recall that we are not concerned with Cayley graphs in this paper).

The following lemma explains the relevance of the graphs $X_q(n)$ to our investigation. (The group W and the graphs $X_q(n)$ are as in Example 3.3.)

Lemma 3.4. *The group W acts 2-distance-transitively on $X_q(n)$.*

Proof. In order to simply the notation, for this proof we write $Y = X_q(n)$. Let α and β be as in Example 3.3. We have $W_\alpha = H_{[1,0]} \operatorname{wr} \operatorname{Sym}(n)$. Since Y is a W-orbital graph, Y is arc-transitive. In fact, using (1) we have

$$\begin{aligned} Y^-(\alpha) = \beta^{W_\alpha} &= \{([a,1],[1,0],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \{([1,0],[a,1],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \cdots \cup \{([1,0],\ldots,[1,0],[a,1]) : a \in \mathbb{F}_q\}. \end{aligned} \tag{3}$$

As X_q is connected, so is Y. Set $g = (\iota, 1, \ldots, 1)$ and write $\gamma = \alpha^g = ([0,-1],[1,0],\ldots,[1,0])$ (for the definition of ι recall (2)). Note that $\alpha = \beta^g$. As $\beta \in Y^-(\alpha)$, we have $\alpha = \beta^g \in (Y^-(\alpha))^g = Y^-(\gamma)$. Therefore $\gamma \in Y^+(\alpha)$ and

$$\begin{aligned} Y^+(\alpha) = \gamma^{W_\alpha} &= \{([a,-1],[1,0],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \{([1,0],[a,-1],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \cdots \cup \{([1,0],[1,0],\ldots,[1,0],[a,-1]) : a \in \mathbb{F}_q\}. \end{aligned} \tag{4}$$

Since $\beta = \alpha^{g^{-1}}$, from (4) we have

$$\begin{aligned} Y^+(\beta) = Y^+(\alpha)^{g^{-1}} &= \{([1,a],[1,0],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \{([0,1],[a,-1],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \cdots \cup \{([0,1],[1,0],\ldots,[1,0],[a,-1]) : a \in \mathbb{F}_q\} \end{aligned} \tag{5}$$

and from (3) we have

$$\begin{aligned} Y^-(\beta) = Y^-(\alpha)^{g^{-1}} &= \{([-1,a],[1,0],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \{([0,1],[a,1],[1,0],\ldots,[1,0]) : a \in \mathbb{F}_q\} \\ &\cup \cdots \cup \{([0,1],[1,0]\ldots,[1,0],[a,1]) : a \in \mathbb{F}_q\}. \end{aligned} \tag{6}$$

FIGURE 1. The graph X_3

We now show that W is transitive on A^2Y. Let (u_1,v_1) and (u_2,v_2) be in A^2Y. Choose $w_i \in Y^+(u_i) \cap Y^+(v_i)$, for $i \in \{1,2\}$. Since W acts transitively on the arcs of Y, replacing u_1 and

u_2 by β and w_1 and w_2 by α if necessary, we may assume that $\beta = u_1 = u_2$ and $\alpha = w_1 = w_2$, that is, $v_1, v_2 \in Y^-(\alpha)$. Assume now that, for $i = 1$ or for $i = 2$, we have $v_i = ([a,1],[1,0],\ldots,[1,0])$, for some $a \in \mathbb{F}_q \setminus \{0\}$. If $a = b^2$ (for some $b \neq 0$), then $[a,1] = [1,a^{-1}]$ and $v_i \in Y^+(\beta)$ from (5). If $a = -b^2$ (for some $b \neq 0$), then $[a,1] = [-1,b^{-2}]$ and $v_i \in Y^-(\beta)$ from (6). In either case, β is adjacent to v_i in Y, contradicting the fact that $(\beta, v_i) \in A^2Y$. Using (3), this shows that the first coordinate of both v_1 and v_2 is $[1,0]$.

Note that $W_{\alpha,\beta} = H_{[1,0],[0,1]} \times (H_{[1,0]} \operatorname{wr} \operatorname{Sym}(\{2,\ldots,n\}))$. Therefore, $W_{\alpha,\beta}$ acts transitively on the $n-1$ coordinates of Δ^n distinct from the first. Finally, from (1) and (3) we see that v_1 and v_2 are conjugate by an element of $W_{\alpha,\beta}$. $\square$

4. α and β are at Hamming distance 1

In this section we prove Theorem 1.1 when α and β are at Hamming distance 1. We start by recalling the definition of a *tournament*, which surprisingly is necessary in our arguments. A tournament is a directed graph obtained by assigning a direction to each edge in an undirected complete graph (that is, every pair of vertices is connected by a single directed edge). A tournament is called *symmetric* if its automorphism group is transitive on the arcs. A finite symmetric tournament $\mathscr{T}$ has an odd number of vertices, say $|V\mathscr{T}| = 1 + 2k$, and every vertex has k in-neighbours and k out-neighbours. The *Payley tournament* $\mathscr{T}_q$ is the tournament with vertices the elements of the finite field $\mathbb{F}_q$, where $q \equiv 3 \mod 4$, and with an arc from a to b exactly when $b - a$ is a non-zero square in $\mathbb{F}_q$ (that is, $b - a = x^2$ for some $x \in \mathbb{F}_q \setminus \{0\}$).

Lemma 4.1. *Let $\mathscr{T}$ be a finite symmetric tournament and let H be a group of automorphisms acting transitively on the arcs of $\mathscr{T}$. Then $\mathscr{T} \cong \mathscr{T}_q$, for some $q \equiv 3 \mod 4$, and the socle of T acts regularly on the vertices of $\mathscr{T}$.*

Proof. The main result of Berggren [1] shows that if $\mathscr{T}$ is a finite symmetric tournament, then $\mathscr{T}$ is isomorphic to $\mathscr{T}_q$ for some $q \equiv 3 \mod 4$. In particular, we may assume that $\mathscr{T} = \mathscr{T}_q$.

Moreover, [1, Theorem A] gives that the automorphism group $\operatorname{Aut}(\mathscr{T}_q)$ of $\mathscr{T}_q$ is the group of all affine permutations of $\mathbb{F}_q$ of the form $\tau_{\sigma,x^2,c} : a \mapsto x^2a^\sigma + c$, where c ranges over all elements of $\mathbb{F}_q$, x over all non-zero elements of $\mathbb{F}_q$, and σ over all field automorphisms of $\mathbb{F}_q$. Using this description of $\operatorname{Aut}(\mathscr{T}_q)$, it is easy to see that if H acts transitively on the arcs of $\mathscr{T}_q$, then $A = \{\tau_{id,x^2,c} : x, c \in \mathbb{F}_q, x \neq 0\}$ is a subgroup of H (where id denotes the identity Galois automorphism of $\mathbb{F}_q$). Now the socle of A is $\{\tau_{id,1,c} : c \in \mathbb{F}_q\}$ and coincides with the socle of $\operatorname{Aut}(\mathscr{T}_q)$. Clearly $T = \{\tau_{id,1,c} : c \in \mathbb{F}_q\}$ acts regularly on the vertices of $\mathscr{T}_q$. $\square$

Before proceeding, we need the following important definition. (The normal quotient technique is a very important idea introduced in [9] which has proven useful in the investigation of many families of graphs [5, 8].)

Definition 4.2. Let G be a group acting transitively on the digraph Γ, and let C be a normal subgroup of G with C intransitive on $V\Gamma$. Let α^C denote the C-orbit containing $\alpha \in V\Gamma$. Then the *normal quotient* Γ_C is the graph whose vertices are the C-orbits on $V\Gamma$, with an arc between distinct vertices α^C and β^C if and only if there is an arc of Γ between α' and β', for some $\alpha' \in \alpha^C$ and some $\beta' \in \beta^C$.

The following proposition is the most substantial result of this section. The proof is quite long and involved, however, it is elementary and combinatorial, and in particular we do not make use of the Classification of the Finite Simple Groups. The corollary that follows it will complete the proof of Theorem 1.1 in the case where α and β are at Hamming distance 1.

Proposition 4.3. *Let X be a connected H-orbital graph and let δ' be an arbitrary vertex of X (so $\delta' \in \Delta$). Assume that any two vertices of $X^-(\delta')$ are adjacent. Then either X is the complete graph, or $X = X_q$, for some $q \equiv 3 \mod 4$.*

Proof. Fix δ_0 a vertex of X. Suppose that X is undirected. As $X^-(\delta_0) = X(\delta_0)$ is a complete graph and as every vertex of $X(\delta_0)$ is adjacent to δ_0, we obtain that $\{\delta_0\} \cup X(\delta_0)$ is a connected component of X. Since X is connected, we see that X is complete. Therefore, in the rest of the proof we may assume that X is a digraph. Let q be the in-valency of X.

Suppose that $|VX| = 1 + 2q$, that is, $VX = \{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0)$. Since H acts transitively on the vertices of X, we see that any two vertices of X are adjacent. In particular, X is a tournament. Since we are assuming that T does not act regularly on VX, from Lemma 4.1 we obtain a contradiction. In particular, in the rest of the proof we may assume that $|VX| > 1 + 2q$. If $q = 1$, then X is a directed cycle and its automorphism group is a cyclic group. The socle of $\mathrm{Aut}(X)$ acts regularly on VX which again contradicts our hypothesis on T. Thus $q > 1$.

As H_{δ_0} acts transitively on $X^-(\delta_0)$ and as any two vertices of $X^-(\delta_0)$ are adjacent, we see that the induced subgraph of X on $X^-(\delta_0)$ is a symmetric tournament. In particular, q is odd. Now we prove eight claims from which the proposition will follow.

Claim 1. The induced subgraph of X on $X^+(\delta_0)$ is a symmetric tournament.

Let δ' be in $X^+(\delta_0)$, let δ'' be in $X^-(\delta_0)$ and let $Y = X^+(\delta'') \cap X^-(\delta_0)$ be the out-neighbours of δ'' in $X^-(\delta_0)$. Since the induced subgraph of X on $X^-(\delta_0)$ is a tournament, we have $|Y| = (q-1)/2$. As (δ'', δ_0) and (δ_0, δ') are arcs of X and H is transitive on AX, there exists $h \in H$ with $(\delta'', \delta_0)^h = (\delta_0, \delta')$. We obtain $Y^h = X^+(\delta_0) \cap X^-(\delta')$ and δ' has $|Y^h| = (q-1)/2$ in-neighbours in $X^+(\delta_0)$. Since H_{δ_0} is transitive on $X^+(\delta_0)$, the induced subgraph of X on $X^+(\delta_0)$ has out-valency $(q-1)/2$ and in-valency $(q-1)/2$ and hence it is a symmetric tournament. ■

Claim 2. Let δ' be in $X^+(\delta_0)$. Then δ' has exactly $(q-1)/2$ in-neighbours in $X^+(\delta_0)$ and exactly $(q-1)/2$ in-neighbours in $X^-(\delta_0)$.

As $X^+(\delta_0)$ is a symmetric tournament, δ' has $(q-1)/2$ in-neighbours in $X^+(\delta_0)$, that is $|X^-(\delta') \cap X^+(\delta_0)| = (q-1)/2$. Set $Y = X^-(\delta') \setminus (\{\delta_0\} \cup X^+(\delta_0))$ and let δ'' be in Y. As $X^-(\delta')$ is a symmetric tournament and as $\delta_0, \delta'' \in X^-(\delta')$, we have that δ_0 and δ'' are adjacent. Since $\delta'' \notin X^+(\delta_0)$, we get $\delta'' \in X^-(\delta_0)$. As δ'' is an arbitrary element of Y, we have $Y \subseteq X^-(\delta_0)$ and δ' has $|Y| = (q-1)/2$ in-neighbours in $X^-(\delta_0)$. ■

Claim 2 shows that for each $\delta' \in X^+(\delta_0)$, we have $X^-(\delta') \subseteq (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$. If, for every $\delta' \in X^+(\delta_0)$, we also have $X^+(\delta') \subseteq (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$, then (using the transitivity of H on VX together with a connectedness argument) we obtain $VX = \{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0)$, which contradicts $|VX| > 1 + 2q$. This shows that there exists $\delta' \in X^+(\delta_0)$ and $\delta_0^* \in X^+(\delta')$ with $\delta_0^* \notin (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$.

Claim 3. $X^-(\delta_0^*) = X^+(\delta_0)$.

Let δ'' be an out-neighbour of δ' in $X^+(\delta_0)$. Since $\delta_0^*, \delta'' \in X^+(\delta')$, by Claim 1 and by vertex transitivity, we obtain that δ_0^* and δ'' are adjacent. If $\delta_0^* \in X^-(\delta'')$, then by Claim 2 applied to δ'', we see that $\delta_0^* \in (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$, which contradicts our choice of δ_0^*. Therefore $\delta_0^* \in X^+(\delta'')$. Since δ'' is an arbitrary out-neighbour of δ' in $X^+(\delta_0)$, we see that every out-neighbour of δ' in $X^+(\delta_0)$ is an in-neighbour of δ_0^*. The vertex δ' was an arbitrary element of $X^+(\delta_0)$ in this argument, so since the induced subgraph of X on $X^+(\delta_0)$ is a symmetric tournament, every vertex of $X^+(\delta_0)$ is an in-neighbour of some element of $X^+(\delta_0) \cap X^-(\delta_0^*)$. Hence $X^+(\delta_0) \subseteq X^-(\delta_0^*)$. Since $q = |X^+(\delta_0)| = |X^-(\delta_0^*)|$, we have $X^+(\delta_0) = X^-(\delta_0^*)$. ■

Claim 4. $X^+(\delta_0^*) = X^-(\delta_0)$.

We first show that δ_0^* has at least one out-neighbour in $X^-(\delta_0)$. Fix an element w in $X^+(\delta_0)$ and write $U = X^-(w) \cap X^+(\delta_0)$ and $V = X^-(w) \cap X^-(\delta_0)$. From Claim 2, $|U| = |V| = (q-1)/2$. As $X^-(\delta_0)$ is a symmetric tournament, by H-transitivity, we have that $X^-(w) = \{\delta_0\} \cup U \cup V$ is a symmetric tournament. Let δ'' be in U. Now, as δ'' has $(q-1)/2$ out-neighbours in $X^-(w)$ and as δ_0 is not an out-neighbour of δ'' (because $\delta'' \in U \subseteq X^+(\delta_0)$), we obtain by the pigeon-hole principle that δ'' has an out-neighbour δ_0' in V, that is, $\delta_0' \in X^+(\delta'') \cap V$. As $\delta'' \in X^+(\delta_0) = X^-(\delta_0^*)$, we see that $\delta_0^* \in X^+(\delta'')$. Therefore, δ_0' and δ_0^* are both in $X^+(\delta'')$. From Claim 1 applied to δ'' we get that δ_0' and δ_0^* are adjacent. Since $\delta_0' \in V \subseteq X^-(\delta_0)$ and since $X^-(\delta_0^*) = X^+(\delta_0)$, we have $\delta_0' \notin X^-(\delta_0^*)$. Therefore we must have that $\delta_0' \in X^+(\delta_0^*)$.

Now that we have shown that δ_0^* has one out-neighbour δ_0' in $X^-(\delta_0)$, using an argument similar to the argument in the proof of Claim 3, we obtain that every element of $X^-(\delta_0)$ is an out-neighbour of δ_0^*. So $X^+(\delta_0^*) = X^-(\delta_0)$. ■

Claim 5. $VX = \{\delta_0, \delta_0^*\} \cup X^+(\delta_0) \cup X^-(\delta_0)$ and $|VX| = 2(1+q)$.

We show that every vertex v in the induced subgraph of X on $V = \{\delta_0, \delta_0^*\} \cup X^+(\delta_0) \cup X^-(\delta_0)$ has in-valency q, from which the claim follows by connectedness. If $v = \delta_0$, then there is nothing to prove. If $v = \delta_0^*$, then from Claim 3 we have $X^-(\delta_0^*) = X^+(\delta_0) \subseteq V$. Also, if $v \in X^+(\delta_0)$, then from Claim 2 we have $X^-(v) \subseteq V$. It remains to consider $v \in X^-(\delta_0)$. Applying the argument in Claim 2 with $v = \delta'$ and with $\delta_0^* = \delta_0$, we obtain that v has $(q-1)/2$ in-neighbours in $X^+(\delta_0^*)$ and

$(q-1)/2$ in-neighbours in $X^-(\delta_0^*)$. As δ_0^* is also an in-neighbour of v, we obtain $X^-(v) \subseteq V$ from Claims 3 and 4. ∎

Since H acts transitively on the vertices of X, we have that for every vertex v, there exists a *unique* vertex v^* with $X^+(v) = X^-(v^*)$ and $X^-(v) = X^+(v^*)$. In particular, the set $\mathscr{B} = \{\{v, v^*\} : v \in VX\}$ is a system of imprimitivity for the action of H on VX. Let C be the kernel of the action of H on $\mathscr{B}$.

Claim 6. Let v be in VX and let v' be in $VX \setminus \{v, v^*\}$. Then the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle.

From Claim 5, we see that v and v' are adjacent, so replacing (v, v') by (v', v) if necessary, we may assume that $v' \in X^+(v)$. As $v' \in X^+(v) = X^-(v^*)$, (v', v^*) is an arc of X. Since $(v')^* \neq v^*$ and since v is adjacent to every element different from v^*, we obtain that either $(v')^* \in X^+(v)$ or $(v')^* \in X^-(v)$. If $(v')^* \in X^+(v)$, then $v', (v')^* \in X^+(v)$ and so from Claim 1, v' and $(v')^*$ are adjacent, a contradiction. Therefore $((v')^*, v)$ is an arc of X. As $(v')^* \in X^-(v) = X^+(v^*)$, $(v^*, (v')^*)$ is also an arc of X. ∎

Claim 7. H contains a unique element of order 2 and $|C| = 2$.

As $|VX| = 2(q+1)$ is even and H acts transitively on X, the group H contains an element h of order 2. Assume that $h \in H \setminus C$. As $h \notin C$, there exists $v \in VX$ with $\{v, v^*\}^h \neq \{v, v^*\}$. Set $v' = v^h$. From Claim 6, the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle which is h-invariant because $h^2 = 1$. As the automorphism group of a directed cycle of length four is a cyclic group whose generator squares to an involution mapping v to v^*, we obtain $v' = v^h = v^*$, a contradiction. Therefore, every involution of H lies in C.

Since the blocks of $\mathscr{B}$ have size 2, we have that C is an elementary abelian 2-group. Let h be an element of $C \setminus \{1\}$ and assume that h fixes a vertex, v say, of X. Let v' be any vertex of X with $v' \notin \{v, v^*\}$. Now, from Claim 6, the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle which is h-invariant. As the automorphism group of a directed cycle is a cyclic group acting regularly and as h fixes v, we obtain that h fixes $v, v^*, v', (v')^*$. Since v' is an arbitrary element of VX with $v' \notin \{v, v^*\}$, we obtain $h = 1$. This shows that $|C| = 2$. ∎

We let z denote the generator of C. We have $v^z = v^*$, for every vertex $v \in VX$. Denote by T the socle of H.

Note that as T acts transitively on VX and since $|VX|$ is even, $C \leq T$. Let S be a Sylow 2-subgroup of T. Since H has a unique involution, the group S has a unique involution (namely z). It follows from [12, Proposition (4.4), p. 59] that S is either a cyclic group of order 2^a (for $a \geq 1$), or a quaternion group of order 2^a (for $a \geq 3$). It follows that either S/C is a cyclic group of order 2^{a-1}, or a dihedral group of order 2^{a-1} (if $a \geq 4$), or an elementary abelian 2-group of order 4 (if $a = 3$).

Since the group C acts transitively on $\{v, v^*\}$ (for each $v \in VX$), the system of imprimitivity $\mathscr{B}$ consists of the orbits of C on VX. In particular, the quotient graph X_C is a normal quotient. From Claim 5, X_C is an undirected complete graph with $q+1$ vertices. Write $\overline{H} = H/C$. Since H acts arc-transitively on X, the group $\overline{H}$ acts arc-transitively on X_C. In particular, $\overline{H}$ is a 2-transitive group of degree $q+1$. Let $\overline{T} = T/C$ be the socle of $\overline{H}$. By a celebrated theorem of Burnside [4, Theorem 4.1B], $\overline{T}$ is either a regular elementary abelian p-group (for some prime p), or a non-regular non-abelian simple group. Assume that $\overline{T}$ is abelian. Since $|\overline{T}| = q+1$, we have $|T| = 2(q+1)$ and T acts regularly on VX, a contradiction.

This shows that $\overline{T}$ is a non-regular non-abelian simple group whose Sylow 2-subgroup S/C is either a cyclic group, or a dihedral group or an elementary abelian group of order 4.

Claim 8. $T = \mathrm{SL}(2, r)$ for some odd r.

From [12, Corollary 2, p. 144], we see that the Sylow 2-subgroup of a simple group is not cyclic. If C splits over T (that is, C has a complement, L say, in T), then $T = L \times C$ for some finite non-abelian simple group L. As L has even order by the Odd order theorem, the group T has more than one involution, which contradicts Claim 7. Thus C does not split over T. Therefore T is a quotient of the universal covering group U of $\overline{T}$, that is, $T \cong U/Z$ for some central subgroup Z of U. We now show that $\overline{T} \cong \mathrm{PSL}(2, r)$, for some odd r.

Suppose that S/C is a dihedral group. From the classification of Gorenstein and Walter [6] of the non-abelian simple groups with a dihedral Sylow 2-subgroup, we see that either $\overline{T} \cong \mathrm{Alt}(7)$ or

$\overline{T} \cong \mathrm{PSL}(2,r)$ for some odd r with $r \equiv 1 \mod 8$ or $r \equiv 7 \mod 8$. From [3], we see that Alt(7) has only two 2-transitive permutation representations, one of degree 7 and one of degree 15. As $q+1$ is even, we obtain that $\overline{T} \not\cong \mathrm{Alt}(7)$.

We may now assume that S/C is an elementary abelian group of order 4. From the classification of Walter [14] of the non-abelian simple groups with an abelian Sylow 2-subgroup, we see that either $\overline{T} \cong \mathrm{PSL}(2,r)$ (for $r = 2^b$, or for some odd r with $r \equiv 3 \mod 8$ or $r \equiv 5 \mod 8$), or $\overline{T} = J_1$, or $\overline{T} = {}^2G_2(3^\ell)$ (for some odd $\ell > 1$). From [3, Table 5 and p. 36], we see that the universal covering groups of ${}^2G_2(3^\ell)$ and of J_1 are simple. Hence $T = U = \overline{T}$, a contradiction. Since a Sylow 2-subgroup of $\mathrm{PSL}(2,2^b)$ has order 2^b, we obtain that $b = 2$. Moreover, since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, we can include this case in the odd characteristic.

From the first line of [3, Table 5], we see that the universal covering group of $\mathrm{PSL}(2,r)$, with r odd, is $\mathrm{SL}(2,r)$, whose centre has order 2. Since $|T| = 2|\overline{T}|$, we see that $T = U = \mathrm{SL}(2,r)$ and C equals the centre of T. ■

We are now ready to conclude the proof. From [2, Table 7.4, p. 197], the group $\mathrm{PSL}(2,r)$ has only one 2-transitive permutation representation of even degree, namely the natural action of degree $r+1$ on the points of the projective line. In particular, $r = q$. Moreover, there exists $\{v, v^*\} \in \mathcal{B}$ such that the stabilizer $T_{\{v,v^*\}}$ is the Borel subgroup

$$\left\{ \left(\begin{array}{cc} x & 0 \\ b & x^{-1} \end{array} \right) : x, b \in \mathbb{F}_q, x \neq 0 \right\}.$$

Since $|\{v, v^*\}| = 2$, the stabilizer T_v has index 2 in $T_{\{v,v^*\}}$. As $T_{\{v,v^*\}} \cong \mathbb{F}_q \rtimes \mathbb{F}_q^*$, we see that $T_{\{v,v^*\}}$ has a unique subgroup of index 2 and hence

$$T_v = \left\{ \left(\begin{array}{cc} x^2 & 0 \\ b & x^{-2} \end{array} \right) : x, b \in \mathbb{F}_q, x \neq 0 \right\}.$$

Therefore the action of T on VX can be identified with the natural action of T on the set of right cosets T/T_v.

If $q \equiv 1 \mod 4$, then the centre C of T is contained in T_v and the action of T on VX is unfaithful, a contradiction. So $q \equiv 3 \mod 4$. Moreover, T_v has four orbits on T/T_v of size $1, 1, q$ and q respectively. Therefore, T gives rise to only two orbital graphs of out-valency q, namely X_q and X_q^{opp}. As $X_q \cong X_q^{opp}$, the proof is completed. □

As a consequence of Propositon 4.3 we obtain the following corollary.

Corollary 4.4. *If α and β are at Hamming distance 1, then either $\Gamma = \mathrm{H}(m,n)$ or $\Gamma = X_q(n)$.*

Proof. Since β is at Hamming distance 1 from α and since G_α acts transitively on $\{T_1, \ldots, T_n\}$, replacing β by a suitable conjugate under G_α if necessary, we may assume that $\beta = (\delta', \delta, \ldots, \delta)$, for some $\delta' \in \Delta \setminus \{\delta\}$. In particular,

$$\begin{aligned} \Gamma^-(\alpha) \quad = \quad & \beta^{G_\alpha} = \left(\delta'^{H_\delta} \times \{\delta\} \times \cdots \times \{\delta\}\right) \cup \left(\{\delta\} \times \delta'^{H_\delta} \times \{\delta\} \times \cdots \times \{\delta\}\right) \\ & \cup \cdots \cup \left(\{\delta\} \times \cdots \times \{\delta\} \times \delta'^{H_\delta}\right). \end{aligned} \tag{7}$$

We denote by $X_1, \ldots, X_n$ the n sets on the right hand side of β^{G_α} (for instance $X_1 = \delta'^{H_\delta} \times \{\delta\} \times \cdots \times \{\delta\}$). For each $i \in \{1, \ldots, n\}$, any two distinct vertices in X_i are at Hamming distance 1. Furthermore, for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and for each $x \in X_i, y \in X_j$, we have that x and y are at Hamming distance 2. As α and β are adjacent and at Hamming distance 1 and as G is transitive on $A^2\Gamma$, this shows that if $(u, v) \in A^2\Gamma$, then u and v are at Hamming distance 2. Furthermore, for each $i \in \{1, \ldots, n\}$, every two vertices u, v of X_i are adjacent because they are at Hamming distance 1 and $u, v \in \Gamma^-(\alpha)$.

Let X be the H-orbital graph $(\delta', \delta)^H$ and let X' be the connected component of X containing δ. Observe that $(\delta, \ldots, \delta, \nu, \delta, \ldots, \delta) \in \Gamma^-(\alpha)$ if and only if $(\nu, \delta) \in AX$. Set

$$Y = \underbrace{X' \times \cdots \times X'}_{n\text{-times}}$$

(seen as a subset of $V\Gamma = \Delta^n$). Let $u = (\varepsilon_1, \dots, \varepsilon_n)$ be an element of Y. With a computation similar to the case of the vertex α, we have

$$\begin{aligned}\Gamma^-(u) &= \{(\nu, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n) : (\nu, \varepsilon_1) \in AX\} \cup \{(\varepsilon_1, \nu, \varepsilon_3, \dots, \varepsilon_n) : (\nu, \varepsilon_2) \in AX\} \\ &\quad \cup \dots \cup \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, \nu) : (\nu, \varepsilon_n) \in AX\}.\end{aligned}$$

For each $i \in \{1, \dots, n\}$ and $\nu \in VX$ with $(\nu, \varepsilon_i) \in AX$, we have $\nu \in X'$ because $\varepsilon_i \in X'$ and X' is a connected component of X. Therefore we obtain $\Gamma^-(u) \subseteq Y$. As u is an arbitrary vertex in Y and Γ is connected, we have $Y = \Delta^n$. Hence $VX' = \Delta$, that is, X is connected.

Let x and y be in $X^-(\delta) = \delta'^{H_\delta}$. From (7), the vertices $\gamma_x = (x, \delta, \dots, \delta)$ and $\gamma_y = (y, \delta, \dots, \delta)$ are in X_1 and hence are adjacent in Γ. In particular, it can be shown that any $g \in G_\alpha$ that takes γ_x to γ_y must fix the first coordinate, and hence that x and y must be adjacent in X. Since x and y are arbitrary elements of $X^-(\delta)$, we have that any two vertices in $X^-(\delta)$ are adjacent.

As X is connected, from Proposition 4.3 we have that either X is complete (and so $\Gamma = \mathrm{H}(m, n)$) or $X = X_q$ (and so $\Gamma = X_q(n)$). □

5. α and β are at Hamming distance at least 2

In this section, we start our analysis of the case in which α and β are at Hamming distance at least 2 from one another, by showing that if this occurs, then in fact $n = 2$ (so the Hamming distance must be exactly 2) as well as some other restrictions.

Lemma 5.1. *Assume that β is at Hamming distance $k \geq 2$ from α. If $(u, v) \in A^2\Gamma$, then u and v are at Hamming distance 1. Furthermore, $n = 2$ and neither δ_1 nor δ_2 is fixed by T_δ.*

Proof. Note that since α and β are at Hamming distance $k \geq 2$, $\beta \in \Gamma^-(\alpha)$, and G is arc-transitive, Remark 2.2 implies that any pair of adjacent vertices must be at Hamming distance $k \geq 2$. Suppose that N_α fixes $\Gamma^-(\alpha)$ point-wise. Since $N \trianglelefteq G$, we have that N_γ fixes $\Gamma^-(\gamma)$ point-wise for each vertex γ, and so by connectedness, $N_\alpha = 1$ and N acts regularly on the vertices of Γ. As we are assuming that N is not regular, we have a contradiction. Therefore N_α does not fix $\Gamma^-(\alpha)$ point-wise. Since $N_\alpha \trianglelefteq G_\alpha$, $\beta \in \Gamma^-(\alpha)$, and G_α is transitive on $\Gamma^-(\alpha)$, the group $N_\alpha = T_\delta^n$ does not fix β and hence there exists a coordinate δ_i of β with T_δ not fixing δ_i, for some $i \in \{1, \dots, n\}$. Using Remark 2.3 we see that, replacing β by a suitable conjugate under G_α if necessary, we may assume that $i = 1$. Let t be in $T_\delta \setminus T_{\delta_1}$, that is, $\delta^t = \delta$ and $\delta_1^t \neq \delta_1$. Now, $g = (t, 1, \dots, 1) \in N_\alpha \subseteq G_\alpha$ and $\gamma = \beta^g \in \Gamma^-(\alpha)$. Since β, γ are at Hamming distance 1 but any pair of adjacent vertices is at Hamming distance $k \geq 2$, we see that β and γ are not adjacent in Γ. Therefore since $\beta, \gamma \in \Gamma^-(\alpha)$, we have that $(\beta, \gamma) \in A^2\Gamma$. Since G acts transitively on $A^2\Gamma$, Remark 2.2 implies that all the pairs in $A^2\Gamma$ are at Hamming distance 1, which proves the first part of this lemma.

Suppose that β has only one entry not fixed by T_δ. Using Remark 2.3 we see that, replacing β by a suitable conjugate if necessary, we may assume that δ_1 (the first coordinate of β) is not fixed by T_δ and $\delta_2, \dots, \delta_n$ are point-wise fixed by T_δ. Therefore $T_\delta = T_{\delta_i}$ for each $2 \leq i \leq n$ and

$$N_\beta = T_{\delta_1} \times T_\delta \times \dots \times T_\delta \quad \text{with} \quad T_\delta \neq T_{\delta_1}. \tag{8}$$

Since N is transitive on the vertices of Γ and since $N_\alpha \trianglelefteq G_\alpha$, we have that for every vertex γ of Γ and for every $\nu \in \Gamma^-(\gamma)$, the group N_γ acts non-trivially only on one coordinate of ν. Since G_α acts transitively on $\{T_1, \dots, T_n\}$, there exists $x = \tau(h_1, \dots, h_n)$ in $G_\alpha \subseteq W_\alpha = H_\delta \operatorname{wr} K$ with $1^\tau = 2$. Consider the vertex

$$\begin{aligned}\gamma = \beta^x &= (\delta_{1^{\tau^{-1}}}^{h_1}, \delta_{2^{\tau^{-1}}}^{h_2}, \delta_{3^{\tau^{-1}}}^{h_3}, \dots, \delta_{n^{\tau^{-1}}}^{h_n}) \\ &= (\delta_{1^{\tau^{-1}}}^{h_1}, \delta_1^{h_2}, \delta_{3^{\tau^{-1}}}^{h_3}, \dots, \delta_{n^{\tau^{-1}}}^{h_n}) \in \Gamma^-(\alpha).\end{aligned} \tag{9}$$

Since $x \in G_\alpha$, we have $\gamma \in \Gamma^-(\alpha)$, so T_δ acts non-trivially on only one coordinate of γ. Since T_δ does not fix δ_1, $T_\delta \trianglelefteq H_\delta$ and $h_2 \in H_\delta$, we obtain that T_δ does not fix $\delta_1^{h_2}$. As T_δ fixes δ_2, we have $\delta_2 \neq \delta_1^{h_2}$. Moreover, as T_δ fixes $\delta_{1^{\tau^{-1}}}^{h_1}$, we have $\delta_1 \neq \delta_{1^{\tau^{-1}}}^{h_1}$. So, β and γ are at Hamming distance at least 2 (the first two coordinates of β and γ are distinct). Since $\beta, \gamma \in \Gamma^-(\alpha)$, we obtain from the previous paragraph that $(\beta, \gamma) \notin A^2\Gamma$. So β and γ are adjacent in Γ, that is, either $\gamma \in \Gamma^-(\beta)$ or $\beta \in \Gamma^-(\gamma)$. Suppose that $\gamma \in \Gamma^-(\beta)$. Thus N_β acts non-trivially on only one coordinate of

γ. Now, as T_δ does not fix $\delta_1^{h_2}$, we obtain from (8) that the second coordinate of γ is the only coordinate not fixed by N_β. Therefore, from (8) and (9), we have

$$T_{\delta_1} = T_{\delta^{h_1}_{1^{\tau^{-1}}}},\ T_\delta \neq T_{\delta_1{}^{h_2}} \text{ and } T_\delta = T_{\delta^{h_i}_{i^{\tau^{-1}}}} \quad \text{for } i \in \{3,\ldots,n\}. \tag{10}$$

Since $1^{\tau^{-1}} \neq 1$, we see from (8) that T_δ fixes $\delta_{1^{\tau^{-1}}}$ and, since $T_\delta \lhd H_\delta$ and $h_1 \in H_\delta$, we have

$$T_{\delta_1} = T_{\delta^{h_1}_{1^{\tau^{-1}}}} = (T_{\delta_1^{\tau^{-1}}})^{h_1} = T_\delta^{h_1} = T_\delta.$$

Thus T_δ fixes δ_1, contradicting (8). A similar contradiction (along these lines) is obtained by supposing that $\beta \in \Gamma^-(\gamma)$. Therefore β must have at least two entries not fixed by T_δ. Now, to conclude the proof, it suffices to show that $n = 2$.

Replacing G by a suitable conjugate in $H \operatorname{wr} \operatorname{Sym}(n)$ if necessary, we may assume that the first two coordinates of β (that is, δ_1 and δ_2) are not fixed by T_δ. Let $t' \in T_\delta \setminus T_{\delta_2}$ so $\delta_2^{t'} \neq \delta_2$, and set $n' = (t, t', 1, \ldots, 1) \in N_\alpha$. As $\gamma' = \beta^{n'}$ has exactly two entries different from β (namely in the first two coordinates), we see that $(\beta, \gamma') \notin A^2\Gamma$ (the pairs in $A^2\Gamma$ are at Hamming distance 1). Since $\beta, \gamma' \in \Gamma^-(\alpha)$, we see that β and γ' are adjacent and hence by Remark 2.2 since G acts arc-transitively, $k = 2$. In particular, as β is adjacent to α, we have $\beta = (\delta_1, \delta_2, \delta, \ldots, \delta)$.

Suppose that $n \geq 3$. Since G_α acts transitively on $\{1, \ldots, n\}$, there exists $x = \tau(h_1, \ldots, h_n) \in G_\alpha \leq W_\alpha = H_\delta \operatorname{wr} K$ with $3^\tau = 1$. Since the third coordinate of β is δ, we obtain that the first coordinate of $\gamma = \beta^x$ is δ. Since β and γ each have $n-2$ coordinates equal to δ and since the first coordinate of β is $\delta_1 \neq \delta$, we obtain that β and γ are at Hamming distance at least 2. Therefore $(\beta, \gamma) \notin A^2\Gamma$. As $\beta, \gamma \in \Gamma^-(\alpha)$, the vertices β and γ are adjacent in Γ and in particular are at Hamming distance $k = 2$. Since β and γ differ in the first coordinate and in the ith coordinate for some $i \in \{3, \ldots, n\}$ such that the ith coordinate of γ is not δ, we obtain that $\gamma = (\delta, \delta_2, \ldots)$, that is, the second coordinate of γ equals the second coordinate of β. Set $n'' = (1, t', 1, \ldots, 1) \in N_\alpha$. The vertex $\gamma^{n''}$ is adjacent to α and is at Hamming distance 3 from β, thus $(\beta, \gamma'') \notin A^2\Gamma$ and β and γ'' are adjacent, contradicting that $k = 2$. This yields $n = 2$. □

When G acts primitively on $V\Gamma$, the proof of Lemma 5.1 is much simplier. Indeed, H is primitive on Δ and hence δ is the unique element of Δ fixed by T_δ. Therefore the detailed analysis on the fixed points of T_δ in the proof of Lemma 5.1 becomes irrelevant and the argument is greatly simplified.

6. α and β are at Hamming distance 2

In this section we continue the analysis begun in Section 5 by considering the case that α and β are at Hamming distance exactly 2. This will conclude the proof of Theorem 1.1. Recall that from Lemma 5.1, we have $n = 2$, $\alpha = (\delta, \delta)$, $\beta = (\delta_1, \delta_2)$ and δ_1, δ_2 are not fixed by T_δ.

We start our analysis with a rather technical lemma. This tells us that if two rows (or columns) of Δ^2 are in the same H_δ-orbit, then there is an element of G that fixes α and takes the first row (or column) to the second (without exchanging the coordinates).

Lemma 6.1. *For each $\varepsilon_1, \varepsilon_2 \in \Delta$ and, for each $i \in \{1, 2\}$, we have $\pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)}) = H_{\varepsilon_i}$.*

Proof. Fix $i \in \{1, 2\}$ and write $R = \pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)})$. Since $G_{(\varepsilon_1, \varepsilon_2)} \leq W_{(\varepsilon_1, \varepsilon_2)}$, we see that $R \leq H_{\varepsilon_i}$. Also, as $T_{\varepsilon_1} \times T_{\varepsilon_2} \leq G_{(\varepsilon_1, \varepsilon_2)}$, we see that $T_{\varepsilon_i} \leq R$.

Since N is transitive on the vertices of Γ, we have $G = NG_{(\varepsilon_1, \varepsilon_2)}$. Furthermore, since $N \leq G_i$, from the "modular law", we obtain $G_i = NG_{(\varepsilon_1, \varepsilon_2)} \cap G_i = N(G_i \cap G_{(\varepsilon_1, \varepsilon_2)})$. Applying π_i on both sides of this equality, we get $H = \pi_i(G_i) = \pi_i(T)\pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)}) = TR$. Using again the "modular law", we see that $H_{\varepsilon_i} = H_{\varepsilon_i} \cap TR = (H_{\varepsilon_i} \cap T)R = T_{\varepsilon_i}R = R$. □

In what follows we use Lemma 6.1 with $\varepsilon_1 = \varepsilon_2 = \delta$ (except in the proof of Lemma 6.13, where we need it in its full generality).

The following two facts hardly deserve to be called lemmas, but will be used several times in what follows.

Lemma 6.2. *If $\gamma_1, \gamma_2 \in \Gamma^-(\gamma)$ and are at Hamming distance 2 from each other, then they must be adjacent.*

Proof. Either $(\gamma_1, \gamma_2) \in A^2\Gamma$, or γ_1 is adjacent to γ_2. But Lemma 5.1 says that if $(\gamma_1, \gamma_2) \in A^2\Gamma$, they would be at Hamming distance 1, a contradiction. □

Lemma 6.3. *If γ, γ' are at Hamming distance 1, then they are not adjacent.*

Proof. This is simply a reminder of Remark 2.2. □

We require some information about neighbourhoods. These will be used in our proofs of both the undirected and directed cases. Since each of these results applies (with the same proof) to any one of $\Gamma(\alpha')$, $\Gamma^+(\alpha')$ and $\Gamma^-(\alpha')$ for the appropriate choice of α', we introduce the notation $\Gamma^*(\alpha')$. This notation will be used to indicate that the result holds when "Γ^*" is replaced by any one of "Γ", "Γ^+", or "Γ^-".

We have $\beta = (\delta_1, \delta_2) \in \Gamma^-(\alpha)$ (in the undirected case, this is $\Gamma(\alpha)$). Let $\gamma = (\delta_1', \delta_2') \in \Gamma^+(\alpha)$. Clearly:

$$\begin{aligned}\Gamma(\alpha) = \beta^{G_\alpha} \subseteq \beta^{W_\alpha} &= \left(\delta_1^{H_\delta} \times \delta_2^{H_\delta}\right) \cup \left(\delta_2^{H_\delta} \times \delta_1^{H_\delta}\right);\\ \Gamma^-(\alpha) = \beta^{G_\alpha} \subseteq \beta^{W_\alpha} &= \left(\delta_1^{H_\delta} \times \delta_2^{H_\delta}\right) \cup \left(\delta_2^{H_\delta} \times \delta_1^{H_\delta}\right);\ \text{and}\\ \Gamma^+(\alpha) = \gamma^{G_\alpha} \subseteq \gamma^{W_\alpha} &= \left((\delta_1')^{H_\delta} \times (\delta_2')^{H_\delta}\right) \cup \left((\delta_2')^{H_\delta} \times (\delta_1')^{H_\delta}\right).\end{aligned}$$

Thus $\Gamma(\alpha)$ (in the undirected case) or $\Gamma^-(\alpha)$ (in the directed case) is the disjoint union of N_α-orbits each of which is a "rectangle" with $|\delta_1^{H_\delta}|$ rows and $|\delta_2^{H_\delta}|$ columns or with $|\delta_2^{H_\delta}|$ rows and $|\delta_1^{H_\delta}|$ columns. If δ_1 and δ_2 are in the same H_δ-orbit then these two rectangles will instead be a single square. Similarly, $\Gamma^+(\alpha)$ has the same structure.

Lemma 6.4. *Fix any $\delta' \in \Delta$. For each $\varepsilon \in \delta'^{H_\delta}$, the number a of $\nu \in \Delta$ with $(\varepsilon, \nu) \in \Gamma^*(\alpha)$ is equal to the number of $\nu \in \Delta$ with $(\nu, \varepsilon) \in \Gamma^*(\alpha)$ and depends only on the H_δ-orbit δ'^{H_δ} (and not on the element ε).*

Furthermore, if $(\varepsilon_1, \varepsilon_2) \in \Gamma^*(\alpha)$ and $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$, then $|\Gamma^*(\alpha)| = a|\varepsilon_1^{H_\delta}|$.

Proof. Fix ε in $(\delta')^{H_\delta}$ and let h_1 be in H_δ with $\varepsilon^{h_1} = \delta'$. From Lemma 6.1, there exists $h_2 \in H_\delta$ such that $g = (h_1, h_2) \in G_\alpha$. In particular, applying the automorphism g we see that if $\nu_1, \ldots, \nu_a$ are the elements of Δ with $(\varepsilon, \nu_i) \in \Gamma^*(\alpha)$, then $\nu_1^{h_2}, \ldots, \nu_a^{h_2}$ are exactly the elements of Δ with $(\delta', \nu_i^{h_2}) \in \Gamma^*(\alpha)$. This shows that the number a does not depend on the choice of ε in $(\delta')^{H_\delta}$.

Now we show that there are exactly a elements ν in Δ with $(\nu, \delta') \in \Gamma^*(\alpha)$. Let $\nu_1, \ldots, \nu_a$ be the elements of Δ with $(\delta', \nu_i) \in \Gamma^*(\alpha)$. Since G_α is transitive on $\{T_1, T_2\}$, there exists $x = (1\ 2)(t_1, t_2) \in G_\alpha$. As $t_2 \in H_\delta$, from Lemma 6.1 there exists $h_1 \in H_\delta$ such that $y = (h_1, t_2^{-1}) \in G_\alpha$. Now, $z = xy = (1\ 2)(t_1 h_1, 1) \in G_\alpha$ and $(\delta', \nu_i)^z = (\nu_i^{t_1 h_1}, \delta')$, for $i \in \{1, \ldots, a\}$, are exactly the elements in $\Gamma^*(\alpha)$ with second coordinate δ'.

If $(\varepsilon_1, \varepsilon_2) \in \Gamma^*(\alpha)$ and $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$, then $\Gamma^*(\alpha) \subseteq \varepsilon_1^{H_\delta} \times \varepsilon_1^{H_\delta}$, and by Lemma 6.1, there are elements of $\Gamma^*(\alpha)$ with every possible first coordinate from $\varepsilon_1^{H_\delta}$. By the earlier part of this lemma, there are exactly a such elements for every possible first coordinate, making $|\Gamma^*(\alpha)| = a|\varepsilon_1^{H_\delta}|$, as claimed. □

6.1. **Γ is undirected.** We limit our attention to the undirected case first, which will prove easiest to complete.

In this subsection, we reserve the letters a and b to denote the numbers defined in Lemma 6.4 that come from choosing $\delta' = \delta_1$ and $\delta' = \delta_2$, respectively. As T_δ does not fix either δ_1 or δ_2, we have $a, b \geq 2$.

A subset $X \subseteq V\Gamma$ is said to be *independent* if any two elements of X are non-adjacent.

Lemma 6.5. *If γ is any vertex of Γ and $(\varepsilon_1, \varepsilon_2) = \gamma' \in \Gamma(\gamma)$, then $\{(\nu_1, \nu_2) \in \Gamma(\gamma) : \nu_1 = \varepsilon_1\}$ and $\{(\nu_1, \nu_2) \in \Gamma(\gamma) : \nu_2 = \varepsilon_2\}$ are the only maximal independent sets in $\Gamma(\gamma)$ containing γ'. Moreover, the cardinalities of these sets are a and b (respectively).*

Proof. Write $X_1 = \{(\nu_1, \nu_2) \in \Gamma(\gamma) : \nu_1 = \varepsilon_1\}$ and $X_2 = \{(\nu_1, \nu_2) \in \Gamma(\gamma) : \nu_2 = \varepsilon_2\}$. Lemma 6.3 shows that X_1 and X_2 are independent sets (both containing γ'), and Lemma 6.2 implies that no other vertex of $\Gamma(\gamma)$ is independent from γ', so these independent sets are maximal and there are no others. The cardinality follows from Lemma 6.4. □

Corollary 6.6. *If $b \neq a$, then for each vertex γ, the neighbourhood $\Gamma(\gamma)$ can be uniquely decomposed into a disjoint union of independent sets of cardinality b.*

Proof. Since $b \neq a$, Lemma 6.5 says that every neighbour of γ lies in a unique maximal independent set of cardinality b. The uniqueness means that these sets must be disjoint. The result follows. $\square$

Lemma 6.7. *The vertices of* $\Gamma(\beta) \setminus \Gamma(\alpha)$ *are:* α, (δ, ν) *for every* ν *such that* $(\delta_1, \nu) \in \Gamma(\alpha)$ *and,* (ν, δ) *for every* ν *such that* $(\nu, \delta_2) \in \Gamma(\alpha)$.

Proof. From Lemmas 6.2 and 6.3, we see that the elements of $\Gamma(\beta) \setminus (\Gamma(\alpha) \cup \{\alpha\})$ are of the form (δ, ν) or (ν, δ), for some $\nu \in \Delta \setminus \{\delta\}$. Let ν be in Δ with $(\delta, \nu) \in \Gamma(\beta)$. We need to show that $(\delta_1, \nu) \in \Gamma(\alpha)$. We argue by contradiction, so we assume that $(\delta_1, \nu) \notin \Gamma(\alpha)$. Write $X_\alpha = \{\eta \in \Delta : (\eta, \nu) \in \Gamma(\alpha)\}$ and $X_\beta = \{\eta \in \Delta : (\eta, \nu) \in \Gamma(\beta)\}$.

From Lemma 6.5, we have that $|X_\alpha|$ and $|X_\beta|$ are each either a or b. Now, replacing a by b if necessary, we may assume that $b \geq a$. Since α and β are adjacent, Lemmas 6.2 and 6.3 yield that

$$X_\alpha \setminus \{\delta_1\} \subseteq X_\beta \quad \text{and } X_\beta \setminus \{\delta\} \subseteq X_\alpha.$$

(Because if $(\mu, \nu) \in \Gamma(\alpha)$ and $\mu \neq \delta_1$, then since $(\delta, \nu) \in \Gamma(\beta)$ we have $\nu \neq \delta_2$, so $\beta \in \Gamma(\alpha)$ implies $(\mu, \nu) \in \Gamma(\beta)$.) As we are assuming that $\delta_1 \notin X_\alpha$, we get $X_\beta = X_\alpha \cup \{\delta\}$ so $|X_\beta| = |X_\alpha| + 1$. Since $b \geq a$, we obtain $|X_\beta| = b$, $|X_\alpha| = a$ and $b = a + 1$.

Write $X_\alpha = \{x_1, \dots, x_a\}$, $\gamma = (\delta, \nu)$ and $\gamma_i = (x_i, \nu)$, for $i \in \{1, \dots, a\}$. From the previous paragraph, $\{\gamma_1, \dots, \gamma_a\}$ is a maximal independent set of $\Gamma(\alpha)$ of size a and $\{\gamma, \gamma_1, \dots, \gamma_a\}$ is a maximal independent set of $\Gamma(\beta)$ of size b. So, from Lemma 6.5 applied to α and γ_i (for each $i \in \{1, \dots, a\}$), we see that there exists $Y_i \subseteq \Delta$ of size b such that $V_i = \{(x_i, y) : y \in Y_i\}$ is a maximal independent set of $\Gamma(\alpha)$. If $\delta_2 \notin Y_i$, then by Lemmas 6.2 and 6.3 the set V_i is contained in $\Gamma(\beta)$. Therefore V_i and $\{\gamma, \gamma_1, \dots, \gamma_a\}$ are both independent sets of $\Gamma(\beta)$ of size b containing γ_i, which contradicts Lemma 6.5. Thus $\delta_2 \in Y_i$, for each $i \in \{1, \dots, a\}$, so $(x_i, \delta_2) \in \Gamma(\alpha)$ for every $i \in \{1, \dots, a\}$.

Now, V_1 and $\{\beta, (x_1, \delta_2), \dots, (x_a, \delta_2)\}$ are independent sets of $\Gamma(\alpha)$ of size b both containing (x_1, δ_2), contradicting again Lemma 6.5. This final contradiction gives that (δ_1, ν) must be in $\Gamma(\alpha)$.

The proof for the neighbours of β of the form (ν, δ) is entirely symmetric. $\square$

Lemma 6.8. Γ *has diameter* 2.

Proof. Towards a contradiction, suppose that γ is a vertex at distance 3 from α. Then γ must have a neighbour α' that is at distance 2 from α. By Lemma 5.1, α' is at Hamming distance 1 from α.

Using Lemma 6.7, we can determine that the mutual neighbours of α and α' can be found in the following manner: first, choose any vertex $\tau \in \Gamma(\alpha)$ that is at Hamming distance 1 from α'; then, any vertex $\sigma \in \Gamma(\alpha)$ that is at Hamming distance 1 from τ but at Hamming distance 2 from α', will be a mutual neighbour of α and α'.

Since γ is at distance 3 from α, and any such σ is adjacent to α and to α', it must be the case that σ is at distance 2 from γ. Then by Lemma 5.1, the Hamming distance between γ and any such σ must be 1.

We have either a or b choices for τ (vertices that are in $\Gamma(\alpha)$ at Hamming distance 1 from α'). For each choice of τ, we have either $b - 1$ or $a - 1$ choices for σ (vertices that are in $\Gamma(\alpha)$, at Hamming distance 1 from τ and at Hamming distance 2 from α'). Thus, there are either $b(a - 1)$ or $a(b - 1)$ possible choices for σ. However every such σ is in $\Gamma(\alpha')$ together with γ. By Lemmas 6.3 and 6.5, there can be at most $a - 1 + b - 1 = a + b - 2$ choices for σ. The inequality $b(a - 1) \leq a + b - 2$ can be solved only if $a = 2$, while the inequality $a(b - 1) \leq a + b - 2$ can be solved only if $b = 2$. So, replacing a by b if necessary, we may assume that $a = 2$. In particular, $b \geq a$. Furthermore, without loss of generality, we can assume that there are b choices for τ, and for each of these there is a unique choice for σ. (We denote by σ_τ the choice of σ determined by τ.)

Using 2-distance-transitivity, we may assume that $\alpha' = (\delta, \delta')$ (recall that $\alpha = (\delta, \delta)$). Notice that the b choices for τ must all have the same value in their 2$^{\text{nd}}$ entry because they are at Hamming distance 2 from α and at Hamming distance 1 from α'. So they must have b distinct values in their 1$^{\text{st}}$ entry. However, since σ_τ is at Hamming distance 2 from α' and at Hamming distance 1 from τ, we obtain that τ and σ_τ have the same 1$^{\text{st}}$ entry. Moreover, each σ_τ is at Hamming distance 1 from γ only if σ_τ has the same 2$^{\text{nd}}$ entry as γ, for each τ. But in this case, these b choices for σ_τ, together with γ, form an independent set of cardinality $b + 1$ in $\Gamma(\alpha')$, contradicting Lemma 6.5. $\square$

The proof of Theorem 1.1 when Γ is undirected is now easy.

Corollary 6.9. *Γ is isomorphic to the complement of* $\mathrm{H}(m,2)$.

Proof. From Lemma 6.8, we conclude that there are no vertices at distance 3 from α. Since Γ is connected, every vertex is at distance 1 or 2 from α. Since G is 2-distance transitive and preserves Hamming distance, every vertex at Hamming distance 2 from α is adjacent to α in Γ, and every vertex at Hamming distance 1 from α is at distance 2 from α in Γ. But then Γ is isomorphic to the complement of the Hamming graph $\mathrm{H}(m,2)$. $\square$

6.2. Γ **is directed.** In this subsection we assume that Γ is directed and we conclude the proof of Theorem 1.1. To complete the proof in this case, we need a number of additional results.

We let k denote the number of out-neighbours of any vertex (so the total valency of a vertex is $2k$). Note that for any vertex v, we have $\Gamma^+(v)\cap\Gamma^-(v)=\emptyset$, since otherwise by arc-transitivity Γ is undirected. Furthermore, we fix $\gamma=(\delta_1',\delta_2')\in\Gamma^+(\alpha)$.

The first ideas we will need lead to a strengthening of Lemma 6.4.

Lemma 6.10. *For η and ν in Δ, we have $|\eta^{H_\nu}|=|\nu^{H_\eta}|$.*

Proof. A double counting gives $|\Delta||\nu^{H_\eta}|=|(\eta,\nu)^H|=|\Delta||\eta^{H_\nu}|$ (see [15, Theorem 16.3]). Since Δ is finite and nonempty, this concludes our proof. $\square$

We will need the following fact in a few places.

Lemma 6.11. *We have $\delta_1^{H_\delta}=\delta_2^{H_\delta}$ if and only if $\delta_1'^{H_\delta}=\delta_2'^{H_\delta}$.*

Proof. If $\delta_1^{H_\delta}=\delta_2^{H_\delta}$, then there is some $h\in H_\delta$ with $\delta_1^h=\delta_2$. Arc-transitivity means that there is some $g\in G$ with $(\beta,\alpha)^g=(\alpha,\gamma)$. Now, $g=\sigma(h_1,h_2)$ with $h_1,h_2\in H$ and with $\sigma=1$ or $\sigma=(1\ 2)$. As $\alpha^g=\gamma$, we must have $\delta^{h_1}=\delta_1'$ and $\delta^{h_2}=\delta_2'$.

If $\sigma=1$, from $\beta^g=\alpha$ we also have $\delta_1^{h_1}=\delta$ and $\delta_2^{h_2}=\delta$. It is now clear that $h_2^{-1}h^{-1}h_1\in H_\delta$, and $(\delta_2')^{h_2^{-1}h^{-1}h_1}=\delta_1'$, completing the proof in this case.

If $\sigma=(1\ 2)$, from $\beta^g=\alpha$ we also have $\delta_2^{h_1}=\delta$ and $\delta_1^{h_2}=\delta$. It is clear that $h_1^{-1}h^{-1}h_2\in H_\delta$, and $(\delta_1')^{h_1^{-1}h^{-1}h_2}=\delta_2'$, completing the proof in this case.

The converse is analogous. $\square$

The next result nicely limits the cases that we need to consider.

Lemma 6.12. *If $(\varepsilon_1,\delta),(\delta,\varepsilon_2)\in\Gamma^-(\gamma)$ with $\varepsilon_1,\varepsilon_2\neq\delta$, then $\varepsilon_1^{H_\delta}=\varepsilon_2^{H_\delta}$. Moreover, $\delta_1'^{H_\delta}=\delta_2'^{H_\delta}$, and either $\varepsilon_1^{H_\delta}=\delta_1'^{H_\delta}$, or $\varepsilon_1^{H_\delta}=\delta_1^{H_\delta}$.*

Proof. We start by proving that if $\alpha_1=(\varepsilon_1,\delta)$, $\alpha_2=(\delta,\varepsilon_2)$ and $\alpha_1,\alpha_2\in\Gamma^-(\gamma)$, then $\varepsilon_1^{H_\delta}=\varepsilon_2^{H_\delta}$. As the vertices α_1 and α_2 are at Hamming distance 1 from α, we see from Lemma 6.3 that α_1 and α_2 are not adjacent to α, that is, $(\alpha,\alpha_1),(\alpha,\alpha_2)\in A^2\Gamma$. So, by 2-distance-transitivity, there must be some element $g\in G_\alpha$ with $\alpha_1^g=\alpha_2$. It is not hard to see that $g=(1\ 2)(h_1,h_2)$, $h_1,h_2\in H_\delta$ and $\varepsilon_1^{h_2}=\varepsilon_2$. Hence ε_1 and ε_2 are in the same H_δ-orbit, that is, $\varepsilon_1^{H_\delta}=\varepsilon_2^{H_\delta}$.

Certainly,

$$(\dagger)\qquad \Gamma^-(\gamma)\supseteq\alpha^{N_\gamma}=\delta^{T_{\delta_1'}}\times\delta^{T_{\delta_2'}}.$$

Let $\varepsilon_1\in\delta^{T_{\delta_1'}}\setminus\{\delta\}$ and $\varepsilon_2\in\delta^{T_{\delta_2'}}\setminus\{\delta\}$. (Note that ε_1 and ε_2 are well-defined because $T_{\delta_i'}$ does not fix δ.) As $(\varepsilon_1,\delta),(\delta,\varepsilon_2)\in\Gamma^-(\gamma)$, from the previous paragraph, we have $\varepsilon_1^{H_\delta}=\varepsilon_2^{H_\delta}$. Now, we also have from $(\dagger)$ that $\alpha'=(\varepsilon_1,\varepsilon_2)\in\Gamma^-(\gamma)$ is at Hamming distance 2 from α. So by Lemma 6.2, there must be an arc between α and α'. If $\alpha'\in\Gamma^+(\alpha)$, then by arc-transitivity, $\Gamma^+(\alpha)=\alpha'^{G_\alpha}\subseteq\alpha'^{W_\alpha}=\varepsilon_1^{H_\delta}\times\varepsilon_1^{H_\delta}$ (since ε_1 and ε_2 are in the same H_δ orbit). As $\gamma\in\Gamma^+(\alpha)$ and $\gamma=(\delta_1',\delta_2')$, we obtain $\delta_1',\delta_2'\in\varepsilon_1^{H_\delta}$ and $\delta_1'^{H_\delta}=\delta_2'^{H_\delta}$. On the other hand, if $\alpha'\in\Gamma^-(\alpha)$, then an analogous argument yields $\Gamma^-(\alpha)\subseteq\varepsilon_1^{H_\delta}\times\varepsilon_1^{H_\delta}$, so $\delta_1^{H_\delta}=\delta_2^{H_\delta}=\varepsilon_1^{H_\delta}$, and Lemma 6.11 completes the proof. $\square$

Now we can obtain an extension of Lemma 6.4.

Lemma 6.13. *Let $\gamma'=(\delta_1',\delta_1')$. For each $\varepsilon_1\in\delta_1'^{H_\delta}$ and $\varepsilon_2\in\delta^{H_{\delta_1'}}$, the cardinalities of the following sets are equal and do not depend on ε_1 or ε_2:*

(a) $\{\nu\in\Delta:(\varepsilon_1,\nu)\in\Gamma^+(\alpha)\}$;

(b) $\{\nu \in \Delta : (\nu, \varepsilon_1) \in \Gamma^+(\alpha)\}$;
(c) $\{\nu \in \Delta : (\varepsilon_2, \nu) \in \Gamma^-(\gamma')\}$;
(d) $\{\nu \in \Delta : (\nu, \varepsilon_2) \in \Gamma^-(\gamma')\}$.

Furthermore, $k = a|\delta_1'^{H_\delta}|$, *where* a *is the cardinality of each of these sets, and* $a \geq 2$.

Proof. The equality of the cardinalities of the sets in (a) and (b) follows immediately from Lemma 6.4. Replacing α by γ' in Lemma 6.4 shows the equality of the cardinalities of the sets in (c) and (d). Call the first of these cardinalities a, and the second a'. By Lemma 6.12 we have $\delta_1'^{H_\delta} = \delta_2'^{H_\delta}$, so since $\gamma = (\delta_1', \delta_2') \in \Gamma^+(\alpha)$, Lemma 6.4 tells us that $k = a|\delta_1'^{H_\delta}|$.

We will now find an in-neighbour of γ' whose second entry is δ, and whose first entry is in $\delta^{H_{\delta_1'}}$. Then Lemma 6.4 applied to this in-neighbour shows that $k = a'|\delta^{H_{\delta_1'}}|$. Since $|\delta_1'^{H_\delta}| = |\delta^{H_{\delta_1'}}|$ by Lemma 6.10, this will show that $a = a'$.

Let $t \in H_\delta$ such that $(\delta_2')^t = \delta_1'$ (we can do this since $(\delta_2')^{H_\delta} = (\delta_1')^{H_\delta}$ by Lemma 6.12). Let $\alpha'' = (\delta_1', \delta)$. By Lemma 6.1, there must be some $g = (t', t) \in G_{\alpha''}$. Now we apply g to α and γ. We have $\gamma^g = (\delta_1', \delta_1')$, and $\alpha^g = (\delta^{t'}, \delta)$. This shows that α^g is an in-neighbour of γ' with the desired form.

That $a \geq 2$ follows from the fact that $T_{\delta_1'}$ does not fix δ. □

For the rest of this subsection, we let a denote the constant defined in Lemma 6.13.

Lemma 6.14. *The sets* $\Gamma^+(\beta') \cap \Gamma^-(\alpha)$ *and* $\Gamma^-(\beta') \cap \Gamma^-(\alpha)$ *each have cardinality* $(k - 2a + 1)/2$ *for any* $\beta' \in \Gamma^-(\alpha)$.

Proof. If $\beta' \in \Gamma^-(\alpha)$ is at Hamming distance 1 from β, then by Lemma 6.3, there cannot be an arc between β and β'. However, if $\beta' \in \Gamma^-(\alpha)$ is at Hamming distance 2 from β, then by Lemma 6.2, there must be an arc between β and β'. Using the fact that there cannot be arcs in both directions between β and any other vertex, we conclude that β has precisely $k - 2a + 1$ arcs to or from other vertices in $\Gamma^-(\alpha)$. Since G is arc-transitive, every vertex of $\Gamma^-(\alpha)$ has the same number of out-neighbours in $\Gamma^-(\alpha)$ as every other vertex; also, every vertex of $\Gamma^-(\alpha)$ has the same number of in-neighbours in $\Gamma^-(\alpha)$ as every other vertex. This shows that the total number of arcs both of whose endpoints lie within $\Gamma^-(\alpha)$ is $k(k - 2a + 1)/2$ (we have to divide by two since each arc has been counted at both its start and end). Our conclusions are immediate. □

We can now generalize Lemma 6.2 to the case of vertices that share an in-neighbour.

Lemma 6.15. *If* γ_1, γ_2 *share an in-neighbour and are at Hamming distance 2 from each other, then they must be adjacent.*

Proof. Call the shared in-neighbour α'. We will show that γ_1 has $(k - 2a + 1)/2$ out-neighbours and $(k - 2a + 1)/2$ in-neighbours in $\Gamma^+(\alpha')$. Since there are $2a - 2$ vertices in $\Gamma^+(\alpha')$ that are at Hamming distance 1 from γ_1 (by Lemma 6.13), and γ_1 is not at Hamming distance 2 from itself, there must be $k - 2a + 1$ vertices in $\Gamma^+(\alpha')$ that are at Hamming distance 2 from γ_1, so this count will show that all of these vertices are adjacent to γ_1, which yields the conclusion.

By Lemma 6.14 and arc-transitivity, $|\Gamma^+(\alpha') \cap \Gamma^-(\gamma_1)| = (k - 2a + 1)/2$. Consider the induced subgraph on $\Gamma^+(\alpha')$. Since $G_{\alpha'}$ is transitive on this set, the in-valency and out-valency of every vertex is constant in this induced subgraph, so every vertex has in-valency and out-valency $(k - 2a + 1)/2$, since γ_1 has this in-valency. □

Lemma 6.16. *For any* $\gamma' \in \Gamma^+(\alpha)$, *the following sets have cardinality* $(k - 2a + 1)/2$:

(a) $\Gamma^-(\gamma') \cap \Gamma^+(\alpha)$;
(b) $\Gamma^+(\gamma') \cap \Gamma^+(\alpha)$;

Proof. Replacing $\Gamma^-(\alpha)$ by $\Gamma^+(\alpha)$ and β by γ' throughout the proof of Lemma 6.14, with Lemma 6.2 replaced by Lemma 6.15 yields the desired conclusion. □

Corollary 6.17. *If* $\beta' \in \Gamma^-(\alpha)$, *then* $\Gamma^+(\beta') \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$, *where* $HD_1(\alpha)$ *is the set of vertices at Hamming distance 1 from* α.

Also, if $\gamma' \in \Gamma^+(\alpha)$, *then* $\Gamma^-(\gamma') \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$.

Proof. We have $|\Gamma^+(\beta')| = k$. The sets $\Gamma^+(\alpha)$ and $\Gamma^-(\alpha)$ are disjoint since we are in the directed case, and Lemmas 6.14 and 6.16 together with arc-transitivity tell us that $|\Gamma^+(\beta') \cap \Gamma^+(\alpha)| = |\Gamma^+(\beta') \cap \Gamma^-(\alpha)| = (k - 2a + 1)/2$, so this accounts for all but $2a - 1$ of the out-neighbours of β'. But $\alpha \in \Gamma^+(\beta')$, and by Lemma 6.13 with vertex-transitivity, we see that for $i = 1, 2$ there must be precisely $a - 1$ other out-neighbours of β' that have the same entry as α in coordinate i. By Lemma 6.3, none of these vertices is in either $\Gamma^+(\alpha)$ or $\Gamma^-(\alpha)$, so these together with α itself form the remaining $2a - 1$ out-neighbours of β'.

The proof for γ' is analogous. □

Lemma 6.18. *Suppose that $\delta_1^{H_\delta} = \delta_1'^{H_\delta}$. Then for any $\delta' \in \delta_1^{H_\delta}$, we have $\delta^{H_{\delta'}} \setminus \{\delta\} = \delta'^{H_\delta} \setminus \{\delta'\}$.*

Proof. We will show that $\delta^{H_{\delta'}} = (\delta'^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$. Clearly $\delta \in \delta^{H_{\delta'}} \setminus \delta'^{H_\delta}$ and $\delta' \in \delta'^{H_\delta} \setminus \delta^{H_{\delta'}}$, and the cardinalities of the two orbits are equal (by Lemma 6.10), so if we can show that $\delta^{H_{\delta'}} \subset \delta'^{H_\delta} \cup \{\delta\}$, that will be sufficient.

Since $\gamma = (\delta_1', \delta_2') \in \Gamma^+(\alpha)$ and $\delta' \in \delta_1'^{H_\delta}$, Lemma 6.1 tells us that there is some element of G_α that fixes the first coordinate (and so fixes each of the two coordinates) and takes γ to some vertex γ' whose first entry is δ'. Clearly, γ' must also lie in $\Gamma^+(\alpha)$.

A similar argument shows that there is some in-neighbour of α whose first entry is any fixed element of $\delta_1^{H_\delta}$. Thus, there must be some in-neighbour of α and some out-neighbour of α whose first entries are any fixed element of $\delta_1^{H_\delta}$. In fact, Lemma 6.13 tells us that there must be a in-neighbours and a out-neighbours of α in each of these columns. These are in fact all of the in- and out-neighbours of α, since $k = a|\delta_1^{H_\delta}|$.

By Corollary 6.17, $\Gamma^-(\gamma') \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$. Now, we have just concluded that any neighbour of α must have its first entry in the set $\delta_1^{H_\delta}$. The in-neighbours of γ', therefore, must have their first entries in the set $(\delta_1^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$. But since α is an in-neighbour of γ', Lemma 6.1 tells us that there is an element of G that fixes γ', fixes the coordinates, and takes the column containing δ to the column indexed by any element of $\delta^{H_{\delta'}}$. So these indices must be elements of $(\delta_1^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$, meaning that we must have $\delta^{H_{\delta'}} \subset \delta'^{H_\delta} \cup \{\delta\}$, as desired. □

Lemma 6.19. *Suppose that $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1'^{H_\delta} = \delta_2'^{H_\delta}$. Then H is 2-transitive on Δ.*

Proof. We claim that if $\Delta \neq \delta_1'^{H_\delta} \cup \{\delta\}$, then Γ is disconnected. This will be a contradiction, so we conclude that $\Delta = \delta_1'^{H_\delta} \cup \{\delta\}$, which forces H to be 2-transitive on Δ, completing the proof.

If we can prove that whenever ε is in $(\delta_1'^{H_\delta} \cup \{\delta\}) \times (\delta_1'^{H_\delta} \cup \{\delta\})$ and has some in- or out-neighbour in this set, all of its in- and out-neighbours must be in this set, this will establish the claim we made in the preceding paragraph, and so complete the proof. Let $\varepsilon = (\delta_3, \delta_4)$ and $\mu = (\delta_3', \delta_4')$, where $\delta_3, \delta_4, \delta_3', \delta_4' \in \delta_1'^{H_\delta} \cup \{\delta\}$, and suppose that μ is either an in-neighbour or out-neighbour of ε. Let us suppose that $\mu \in \Gamma^-(\varepsilon)$. By arc-transitivity, we have $\Gamma^-(\varepsilon) = \mu^{G_\varepsilon} \subseteq \left((\delta_3')^{H_{\delta_3}} \times (\delta_4')^{H_{\delta_4}}\right) \cup \left((\delta_4')^{H_{\delta_3}} \times (\delta_3')^{H_{\delta_4}}\right)$. Using Lemma 6.18 it is straightforward to verify that this is a subset of $\left((\delta_1')^{H_\delta} \cup \{\delta\}\right) \times \left((\delta_1')^{H_\delta} \cup \{\delta\}\right)$. It remains to show that $\Gamma^+(\varepsilon)$ is also in this set. Notice that a similar argument shows that $\Gamma^+(\mu) \subseteq (\delta_1'^{H_\delta} \cup \{\delta\}) \times (\delta_1'^{H_\delta} \cup \{\delta\})$ since μ has one out-neighbour (namely ε) in this set. But by Lemma 6.16 and arc-transitivity, $\Gamma^+(\mu) \cap \Gamma^+(\varepsilon)$ has cardinality $(k - 2a + 1)/2$. Since Lemma 6.13 shows that a divides k and $a \geq 2$, we must have $(k - 2a + 1)/2 > 0$, so ε has at least one out-neighbour in $(\delta_1'^{H_\delta} \cup \{\delta\}) \times (\delta_1'^{H_\delta} \cup \{\delta\})$, from which a similar argument shows that all out-neighbours of ε are in this set.

The case in which $\mu \in \Gamma^+(\varepsilon)$ is precisely analogous to the above case. □

Lemma 6.20. *It is not possible to have $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1'^{H_\delta} = \delta_2'^{H_\delta}$.*

Proof. Towards a contradiction, suppose that $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1'^{H_\delta} = \delta_2'^{H_\delta}$. By Lemma 6.14 we have $(k - 2a + 1)/2 \in \mathbb{Z}$, so k must be odd. But $k = a|\delta_1^{H_\delta}|$ (by Lemma 6.13), so a and $|\delta_1^{H_\delta}|$ are both odd. Then by Lemma 6.19, $|\Delta| = |\delta_1^{H_\delta}| + 1$, so $|\Delta|$ must be even.

Now, T is a transitive group acting on Δ, so T must have even order. Hence T contains an involution t. Without loss of generality, we can assume that $\delta^t = \delta' \neq \delta$, and $\delta'^t = \delta$. Since $\Delta = \delta_1^{H_\delta} \cup \{\delta\}$, we have $\delta' \in \delta_1^{H_\delta} = (\delta_1')^{H_\delta}$. Now by Lemma 6.13, α has a out-neighbours whose first entry is δ'; we choose one of these, (δ', δ'').

Since H is 2-transitive on Δ (by Lemma 6.19), there exists $h \in H_\delta$ such that $\delta'^h = \delta''$. Now since $T \lhd H$, we have $h^{-1}th \in T$. We know that $T \times T = N \leq G$; consider the action of

$g = (t, h^{-1}th) \in G$ on α and on (δ', δ''). We have $(\delta, \delta)^{(t,h^{-1}th)} = (\delta', \delta'')$ since $h \in H_\delta$. And $(\delta', \delta'')^{(t,h^{-1}th)} = (\delta, (\delta')^{th}) = (\delta, \delta^h) = (\delta, \delta)$. So g reverses this arc, contradicting the fact that we are in the directed case. $\square$

By Lemma 6.12 we have $(\delta_1')^{H_\delta} = (\delta_2')^{H_\delta}$, so by Lemma 6.11 $\delta_1^{H_\delta} = \delta_2^{H_\delta}$ and the hypothesis eliminated in our next (and final) lemma is the only remaining possibility.

Lemma 6.21. *It is not possible to have* $\delta_1^{H_\delta} \neq \delta_1'^{H_\delta}$.

Proof. By Corollary 6.17, $\Gamma^-(\gamma) \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$. So by Lemma 6.12, we can conclude that either

$$\Gamma^-(\gamma) \cap HD_1(\alpha) \subseteq (\delta_1^{H_\delta} \times \{\delta\}) \cup (\{\delta\} \times \delta_1^{H_\delta}), \quad \text{or}$$
$$\Gamma^-(\gamma) \cap HD_1(\alpha) \subseteq (\delta_1'^{H_\delta} \times \{\delta\}) \cup (\{\delta\} \times \delta_1'^{H_\delta}).$$

We assume that the first of these possibilities is true; the proof in the other event is analogous.

Since $\Gamma^-(\gamma) \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$, we see that our assumption forces the rows and columns of $\delta_1'^{H_\delta} \times \delta_1'^{H_\delta}$ to be disjoint from the rows and columns of all other in-neighbours of γ. Now $\Gamma^+(\alpha) \subseteq \delta_1'^{H_\delta} \times \delta_1'^{H_\delta}$, and γ has either 0 or a in-neighbours in any of these rows or columns (Lemma 6.13 together with vertex-transitivity yield this conclusion), all of which must also be out-neighbours of α; and these are all of the in-neighbours of γ that are also out-neighbours of α. Hence we must have $|\Gamma^-(\gamma) \cap \Gamma^+(\alpha)| = ja$ for some j. But Lemma 6.16 tells us that $|\Gamma^-(\gamma) \cap \Gamma^+(\alpha)| = (k - 2a + 1)/2$. So we have $k - 2a + 1 = 2ja$, but a divides each of these values with the exception of 1, and we know $a \geq 2$ (by Lemma 6.13), a contradiction. $\square$

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