INDUCED FORESTS IN SOME DISTANCE-REGULAR GRAPHS

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Abstract. In this article, we study the order and structure of the largest induced forests in some families of graphs. First we prove a variation of the Delsarte-Hoffman ratio bound for cocliques that gives an upper bound on the order of the largest induced forest in a graph. Next we define a *canonical induced forest* to be a forest that is formed by adding a vertex to a coclique and give several examples of graphs where the maximal forest is a canonical induced forest. These examples are all distance-regular graphs with the property that the Delsarte-Hoffman ratio bound for cocliques holds with equality. We conclude with some examples of related graphs where there are induced forests that are larger than a canonical forest.

1. Introduction

In this paper we study both the cardinality and structure of the largest sets of vertices inducing forests in some distance-regular graphs. For a graph G, let $\tau(G)$ be the maximum number of vertices inducing a forest in G. The quantity $\tau(G)$ is called the *acyclic number of G*. Letting $\alpha(G)$ denote the independence number of G, the order of the largest coclique, it is clear that for any non-empty graph, $\tau(G) \geq \alpha(G) + 1$ as adding any vertex to an independent set will induce a forest. The main results of this article are to give bounds on $\tau(G)$ for certain distance-regular graphs and to identify graphs in which every maximum induced forest can be obtained by adding a single vertex to an independent set.

A number of other graph parameters and special kinds of vertex subsets bear some relationship to this acyclic number $\tau(G)$. An induced forest in a graph is complementary to a set of vertices whose removal induces an acyclic graph and this is sometimes known as a 'decycling set' of a graph, or a 'feedback vertex set'. Recall that a graph is k-degenerate if and only if every subgraph has a vertex of valency at most k. The notion of degeneracy arises in colouring problems and in the study of 'cores' of graphs which is related to connectivity properties of the graph. A graph is empty if and only if it is 0-degenerate, while a graph is a non-empty forest if and only if it is 1-degenerate. Thus, the largest coclique in a graph is the largest set of vertices that induce a 0-degenerate subgraph, while the largest

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induced forest can be thought of as the largest set of vertices inducing a 1-degenerate subgraph.

Alon, Kahn, and Seymour [2] showed that $\tau(G) \geq \sum_{v \in V} 2/(d(v)+1)$, where $d(v)$ denotes the valency of v. In fact, this is a special case of the general bound they prove for k-degenerate induced subgraphs. In the case of a d-regular graph on n vertices, this implies that $\tau(G) \geq 2n/(d+1)$. This bound is tight for a graph consisting of disjoint copies of K_{d+1}. Bondy, Hopkins and Staton [8] showed that if $d = 3$ and G is connected (so that the previous tight examples do not apply), then $\tau(G) \geq \frac{5n-2}{8}$ (again, n is the number of vertices). They also provided examples where their bound is tight. Further refinements have been given for regular graphs of large girth [21, 22, 24]. Bau, Wormald, and Zhou [6] showed that for random 3-regular graphs, asymptotically almost surely, $\tau(G) = n - \lceil (n+2)/4 \rceil = \lfloor (3n-2)/4 \rfloor$ and gave bounds for random r-regular graphs in general. Alon, Mubayi and Thomas [4] gave bounds on $\tau(G)$ in terms of the independence number and the maximum valency.

The largest induced forests and smallest decycling sets in specific families of graphs have been well-studied in the literature, for example: planar graphs [1], bipartite graphs [3, 10], hypercubes [5, 14, 29] and binomial random graphs [23]. Related work has concerned the largest induced trees [13, 15, 26, 27, 28] and the largest induced matchings [9, 11].

Let $G = (V, E)$ be a graph and S be a coclique of G. As noted previously, for any $v \in V \setminus S$, the set $S \cup \{v\}$ induces a forest, so that $\tau(G) \geq \alpha(G) + 1$. We define forests constructed in such a manner to be *canonical*.

Definition 1.1. *Let $G = (V, E)$ be a non-empty graph and let $F \subset V$ induce a forest. A set F is a canonical induced forest if there is a vertex $v \in F$ such that $F \setminus \{v\}$ is an independent set.*

We will refer to induced forests of maximum possible order as *maximum induced forests*. Often we drop the word induced and refer to these as just a canonical forest in G or maximum forests. We note that a canonical forest of order k in a graph G is a spanning subgraph of the complete multipartite graph $K_{1,\,k-1}$, where $k-1 \leq \alpha(G)$; further, any non-canonical induced forest is not such a subgraph, so this characterizes the canonical forests.

In this article, we deal with the following problems:

(A) Find families of graphs with $\alpha(G) + 1 = \tau(G)$, for every graph G in the family.
(B) Find families of graphs such that every maximum induced forest in a graph of the family is canonical.

The family of complete graphs is an easy example of a family satisfying the conditions in both (A) and (B). Another easy example is the family of complete bipartite graphs.

Another straight-forward example are the threshold graphs. A graph is a *threshold graph* if it can be formed by starting with a single vertex and repeatedly either adding an isolated vertex, or adding a dominating vertex to the graph. It is well-known that threshold graphs are exactly the graphs that do not contain, as an induced subgraph, either path on four vertices, P_4, or a cycle on four vertices, C_4 or two disjoint edges $2K_2$; thus the threshold graphs are exactly the $\{P_4, C_4, 2K_2\}$-free graphs (see [18]or [25] for a comprehensive treatise of this topic).

Proposition 1.2. *Let G be a threshold graph. Then $\alpha(G) + 1 = \tau(G)$ and every maximum induced forest in G is canonical.*

Proof. Any non-canonical forest in a graph must contain, as an induced subgraph, either P_4 or $2K_2$. If G is a threshold graph, then it is both P_4-free and $2K_2$-free, then every maximum induced forest is necessarily canonical. □

If G is a disconnected graph in which each component has at least two vertices (so no component is an isolated vertex), then a forest can be constructed by taking a maximum coclique and adding one vertex, not in the maximum coclique, from each component of the graph. So in this case, the bound $\tau(G) \geq \alpha(G) + \kappa(G)$ holds, where $\kappa(G)$ is the number of components of G. Since our goal in this paper is to find graphs for which condition (A) holds, we will only consider connected graphs.

The *join* of two graphs G and H is the graph formed by taking the disjoint union of G and H and adding all edges with one vertex in G and the other in H. This new graph is denoted by $G \vee H$.

Proposition 1.3. *If G and H are two graphs that satisfy both conditions (A) and (B), then $G \vee H$ also satisfy both conditions (A) and (B).*

Proof. First note that $\alpha(G \vee H) = \max\{\alpha(G), \alpha(H)\}$.

If F is a forest in $G \vee H$ then F cannot have two vertices in G and two vertices in H, as this would form a copy of C_4 in F. Thus, either the vertices of F are all vertices of G, or all vertices of H or F has exactly one vertex in G, or F has exactly one vertex in H.

Since condition (B) holds for both G and H, if either of the first two possibilities hold, then F is canonical. If either of the last two possibilities hold then F has one vertex in G (or H) and the vertices in H (respectively, G) must be a coclique, thus F is a canonical forest.

Note that this means that

$$\tau(G \vee H) = \max\{\alpha(G), \alpha(H)\} + 1 = \alpha(G \cup H) + 1,$$

so the graph $G \vee H$ satisfies both condition (A) and (B). □

The next natural step is to consider connected graphs which are neither P_4-free nor $2K_2$-free, and are not the join of two smaller graph. We start looking at regular graphs.

The following result, known as the Delsarte-Hoffman ratio bound, is a spectral graph theoretic method that has been used to bound the size of the maximum cocliques in many families of regular graphs, refinements of this theorem can be used to characterize the maximum cocliques in a graph (see [17] for examples).

Theorem 1.4. *(see* [19, Theorem 3.2]*) Let G be a k-regular graph on n vertices and let λ be the smallest eigenvalue of the adjacency matrix of G. Then*

$$\alpha(G) \leq \frac{n(-\lambda)}{k - \lambda}.$$

This result is an application of the Cauchy Interlacing Theorem (see [19, Theorem 2.1]). Applying the same technique, we will show the following spectral upper bound for the order of an induced forest in a regular graph.

Theorem 1.5. *Let G be a k-regular graph on n vertices and let λ be the smallest eigenvalue of the adjacency matrix of G. Then*

$$\tau(G) \leq \frac{n(2-\lambda) + \sqrt{n^2(2-\lambda)^2 - 8n(k-\lambda)}}{2(k-\lambda)} < \frac{-n\lambda}{k-\lambda} + \frac{2n}{k-\lambda}.$$

Below is an edge-counting argument, which provides an alternative bound on the order of an induced forest in a regular graph that is sometimes better than the spectral bound (see discussion after Lemma 4.1).

Theorem 1.6. *Let G be a k-regular graph on n vertices. Let f be the number of vertices and c the number of connected components in an induced forest of G. Then*

$$f \leq \frac{nk-2c}{2k-2} \leq \frac{nk-2}{2k-2}.$$

The first summand of the right-hand side of the inequality in Theorem 1.5 is equal to the Delsarte-Hoffman ratio bound on the independence number $\alpha(G)$. As we seek graphs G, for which $\alpha(G)-\tau(G)$ is "small", it is natural to focus our investigations on regular graphs for which the Delsarte-Hoffman bound is tight. Many distance-regular graphs satisfy the tightness of the Delsarte-Hoffman bound, for instance the Kneser graph is one such graph.

Let G be a graph and F be a non-canonical forest in G. In Lemma 3.1, we show that $|F| \leq 2+\eta(G)$, where $\eta(G)$ is the graph invariant defined in (1). If $2+\eta(G) \lneqq \alpha(G)+1$, then $\tau(G)=\alpha(G)+1$ and every maximum induced forest in G is canonical. This gives us a method to prove a graph satisfies both conditions (A) and (B). Further, in the case that G is a distance-regular graph for which the Delsarte-Hoffman ratio bound holds with equality, both $\alpha(G)$ and $\eta(G)$ can be determined from the intersection array of G. Thus our focus is on such graphs and below is a list of five families of distance-regular graphs in which every maximum induced forest is canonical.

Theorem 1.7. *In the following graphs, every maximum induced forest is a canonical forest:*

1. *the Kneser graph $K(n,k)$, for every $k \geq 2$ and $n \geq 2k^3$;*
2. *the q-Kneser graph $K_q(n,k)$, for $k \geq 2$, $n > 3k-2$ and q sufficiently large;*
3. *the non-collinearity graph on points in a generalized quadrangle with parameters (s,t) and $s > 3$;*
4. *$X_{m,n} = \otimes^m K_n$ with $m \geq 2$ and $n > 2m(m-1)$;*
5. *the complement of the block graph of an orthogonal array with parameters m,n with $n > 1+2m(m-1)$;*

We were able to make a few refinements in some subfamilies of the graphs mentioned in the above result. These can be found in Theorems 3.4, 3.6, and 3.9

These results can be found in Theorems 3.4, 3.6, and 3.9, along with a few refinements in some subfamilies of the graphs.

The theorems we referenced in this paragraph are the refinements of Thm 1.7 in the SRG case. I think we should stick to the previous (version above in red) of this paragraph.

We prove Theorems 1.5 and 1.6 in Section 2. In Section 3, we prove the results of Theorem 1.7, characterizing induced forests in some other families of graphs. In Section 4, we produce an infinite family of graphs with "large" maximum forests.

2. Upper bounds

We begin this section by proving Theorem 1.6.

Proof of Theorem 1.6. Let $G=(V,\ E)$ be a k-regular graph on n vertices, and let F be an induced forest of G with f vertices and c connected components.

Since it is a forest, F has $f-c$ edges. Since each of the f vertices of F has k incident edges and each of the $f-c$ edges of F is counted twice in the valency of vertices of F, there are $fk-2(f-c)=f(k-2)+2c$ edges of G that join vertices of F to vertices that are not in F. In total, this makes $f(k-1)+c$ edges of G that are incident with at least one vertex of F.

Clearly, the number of edges of G that are incident with at least one vertex of F cannot exceed the total number of edges of G, which by the Handshaking Lemma is $nk/2$. So

$$f(k-1) + c \leq nk/2.$$

Rearranging this inequality produces the given result, which is maximized when $c = 1$. □

We next work toward the proof of Theorem 1.5. Let $G = (V,\ E)$ be a k-regular graph on n vertices. Let $k = \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. The following result from [19] gives algebraic bounds for induced subgraphs. We include the proof for completeness.

Theorem 2.1. [19, Theorem 3.5] *Let G be a k-regular graph on n vertices and suppose that G has an induced subgraph G' with n' vertices and m' edges. Then*

$$\lambda_2 \geq \frac{2m'n - k(n')^2}{n'(n-n')} \geq \lambda_n.$$

Proof. Consider the partition $\pi = \{G',\ \overline{G'}\}$ of the vertex set. The corresponding quotient matrix is

$$\begin{pmatrix} \frac{2m'}{n'} & k - \frac{2m'}{n'} \\ \frac{n'k-2m'}{n-n'} & k - \frac{n'k-2m'}{n-n'} \end{pmatrix}.$$

The eigenvalues of this matrix are k and $\frac{2m'}{n'} - \frac{n'k-2m'}{n-n'} = \frac{2m'n-n'^2k}{n'(n-n')}$. The result follows by Cauchy's Interlacing Theorem (see [19, Theorem 2.1]). □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let G be a k-regular graph on n vertices with λ_n its least eigenvalue. Let F be an induced forest of G with f vertices with c connected components. Since F has exactly $f - c$ edges and f vertices, using the above result, we have

$$\frac{2(f-c)n - kf^2}{f(n-f)} \geq \lambda_n,$$

and thus

$$(k-\lambda_n)f^2 + n(\lambda_n - 2)f + 2cn \leq 0.$$

As $c \geq 1$, we have $(k-\lambda_n)f^2 + n(\lambda_n-2)f + 2n \leq 0$, and thus $f \leq \frac{n(2-\lambda_n)+\sqrt{n^2(2-\lambda_n)^2-8n(k-\lambda_n)}}{2(k-\lambda_n)}$. □

We now use Theorem 1.5 to find the acyclic number of a small graph.

Example 1. Consider the complement $\mathcal{P}'(9)$ of the Paley graph on 9 vertices. The vertex set of this graph is the field $\mathbb{F}_9$ of size 9; and two elements $a, b \in \mathbb{F}_9$ are adjacent if and only if $a - b$ is not a quadratic residue in $\mathbb{F}_9$. We identify $\mathbb{F}_9 \cong \mathbb{F}_3[x]/\langle x^2+1\rangle$ and the set of quadratic residues is $S = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{x},\ \overline{2x}\}$. The induced subgraph $\mathbb{F}_3 \cup \{\overline{x+1},\ \overline{x+2}\}$ is a path on 5 vertices, in $\mathcal{P}'(9)$. This construction implies that $\tau(\mathcal{P}'(9)) \geq 5$. It is well-known that $\mathcal{P}'(9)$ is a strongly-regular graph whose spectrum is $(4,\ 1,\ -2)$. Using Theorem 1.5, we have $\tau(\mathcal{P}'(9)) < 6$. We note that Theorem 1.6 gives us the same upper bound. Thus we have $\tau(\mathcal{P}'(9)) = 5$.

We were not able to extend this to other Paley graphs. In Section 4, we present some observations (on the acyclic number) stemming from computations on small order Paley graphs.

3. Graphs whose maximum induced forests are canonical.

In this section, we characterize maximum induced forests in some families of regular graphs. In particular, we will prove Theorem 1.7 using a counting method for each graph.

Let G be a regular graph. We recall that the order $\tau(G)$ of a maximum induced forest satisfies $\tau(G) \geq \alpha(G) + 1$. To show that every maximum induced forest in G is canonical, it suffices to show that $|F| < \alpha(G) + 1$ for every non-canonical induced forest F. Note that an induced forest F in G is not canonical if and only if F contains either a copy of P_4 or a copy of $2K_2$ as an induced subgraph. We now find an upper bound on the order of an induced forest F that does not contain either a P_4 or a $2K_2$.

Given a pair (a, b) of adjacent vertices in G, by $N(a, b)$, we denote the set of vertices in G that are not adjacent to either of a or b; and by $\eta(a, b)$, we denote $|N(a, b)|$. We denote the maximum such value by

$$\eta(G) = \max \{\eta(a, b) \mid a, b \in G \text{ and } a \sim b\}. \tag{1}$$

Lemma 3.1. *If F is a non-canonical forest in a graph G, then $|F| \leq 2 + 2\eta(G)$.*

Proof. First assume that F contains a path on four vertices; call this subgraph P. Since F is a forest, every $v \in F \setminus P$ is adjacent to at most one vertex of P. Therefore, every $v \in F \setminus P$ is non-adjacent to at least one leaf and the neighbour of that leaf in P. Suppose that P is made up of vertices $\{a, b, c, d\}$ with $a \sim b$, $b \sim c$ and $c \sim d$. Then we see that $F \subset N(a, b) \cup N(c, d) \cup \{b, c\}$, completing the proof in this case.

Next assume that F does not contain a path on four vertices but has an induced subgraph Q that is isomorphic to $2K_2$. Let Q be made up of vertices $\{a, b, c, d\}$ such that $a \sim b$ and $c \sim d$. Since F is a forest that does not contain a path on four vertices, every $v \in F \setminus Q$ is adjacent to at most one vertex of Q, so is non-adjacent to a pair of adjacent vertices of Q. Thus we have $F \subseteq N(a, b) \cup N(c, d)$. □

This lemma is particularly applicable to strongly-regular graphs since the value of $\eta(\alpha, \beta)$ is the same for all pairs (α, β) of adjacent vertices. We now recall that given $n, k, a, c \in \mathbb{N}$, a strongly-regular graph with parameters $(n, k : a, c)$ is a k-regular graph on n vertices such that (i) every pair of adjacent vertices have exactly a neighbours in common; and (ii) every pair of non-adjacent vertices have exactly c neighbours in common. Using inclusion-exclusion on the parameters of a strongly-regular graph we see that $\eta(\alpha, \beta) = n - 2k + a$. Now Lemma 3.1 yields the following result.

Corollary 3.2. *Let G be a strongly-regular graph with parameters $(n, k : a, c)$. If*

$$1 + 2(n - 2k + a) < \alpha(G),$$

then every maximum induced forest is a canonical induced forest.

This corollary can be used to prove that for $n \geq 17$ the maximum forests in $K(n, 2)$ are canonical (we omit this proof, since Theorem 3.4 gives a stronger result).

In the following subsections, we apply Lemma 3.1 to show that maximum induced forests in some families of strongly-regular graphs must be canonical.

3.1. Kneser Graphs.

In this section we consider the Kneser graphs $K(n, k)$ with $n \geq 2k$. The graph $K(2k, k)$ consists of exactly $\frac{1}{2}\binom{2k}{k}$ disjoint edges and is itself a forest, so we will only consider $n > 2k$. It is well known from the Erdős-Ko-Rado Theorem [12] that the order of a maximum coclique in $K(n, k)$ is $\binom{n-1}{k-1}$ and that the Delsarte-Hoffman ratio bound holds with equality. Thus a canonical forest has order $\binom{n-1}{k-1} + 1$. We will show for n large relative to k that this is the largest possible induced forest.

Theorem 3.3. *For every $k \geq 2$ and $n \geq 2k^3$, we have*

$$\tau(K(n,k)) = \binom{n-1}{k-1} + 1.$$

Moreover, every maximum induced forest is a canonical induced forest.

Proof. Let γ and δ be a pair of adjacent vertices in $K(n,k)$ (so γ and δ are any two disjoint k-sets from $[n]$). Elementary counting arguments (overcounting sets whose intersection with γ or δ has cardinality greater than 1) show that there are at most $k^2\binom{n-2}{k-2}$ k-subsets of $[n]$ intersecting both γ and δ. Thus in this case, we have $\eta(K(n,k)) \leq k^2\binom{n-2}{k-2}$.

By Lemma 3.1, if F is a non-canonical induced forest, then $|F| \leq 2 + 2k^2\binom{n-2}{k-2}$. In the case $n \geq 2k^3$, we have

$$2 + 2k^2\binom{n-2}{k-2} < 1 + \binom{n-1}{k-1}.$$

Therefore non-canonical induced forests are smaller than the canonical induced forests. □

We consider one special case of Kneser graphs, in which the same sort of counting can be done more precisely.

Theorem 3.4. *For $n \geq 5$*

$$\tau(K(n,2)) = \max\{n, 7\}.$$

If $n > 7$, every maximum induced forest in $K(n,2)$ is canonical.

Proof. For any n, a canonical forest in $K(n,2)$ has order n. Further, $\tau(K(n,2)) \geq 7$ for any $n \geq 5$, this is seen by taking the following vertex set:

$$\{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\},\{1,5\}\}.$$

Recall that any non-canonical forest contains either a copy of P_4 or a copy of $2K_2$.

Assume F is an induced forest in $K(n,2)$. If F contains a copy of P_4, then the vertices of this P_4 must be the sets $\{a,b\},\{c,d\},\{a,e\},\{b,c\}$ for some a,b,c,d,e. Any other vertex in F is adjacent to at most one of these vertices. There are only 5 vertices that are adjacent to at most one vertex of the path, namely the elements of $T := \{\{a,d\},\{c,e\},\{b,d\},\{a,c\},\{b,e\}\}$. Therefore any vertices of F that are not in the P_4 must lie in T, and T itself induces a 5-cycle (in the order given above). Furthermore, every vertex of T except $\{a,c\}$ is adjacent to some vertex of the P_4. If F were to contain at least four vertices of T then these vertices would induce a second P_4, and at least three of these vertices would have neighbours in the first P_4, leading to a cycle in F, a contradiction. Thus $|F \cap T| \leq 3$, and therefore $|F| \leq 7$.

Similarly, if F contains a copy of $2K_2$, then the vertices of this subgraph must be the sets $\{a,b\},\{c,d\},\{a,c\},\{b,d\}$ for some a,b,c,d. If F has no P_4, then F cannot include any vertex of the form $\{a,e\},\{b,e\},\{c,e\}$, or $\{d,e\}$ (for any $e \notin \{a,b,c,d\}$). Since any vertex in F must be nonadjacent to at least 2 of the vertices of the $2K_2$, this implies that the elements of the 2-set defining the vertex must lie entirely in $\{a,b,c,d\}$, so there are only 2 other vertices that can be added: $\{a,d\}$ and $\{b,c\}$. So any such F contains at most 6 vertices.

Therefore any induced forest that is not canonical contains no more than 7 vertices and the result follows. □

3.2. **q-Kneser Graphs.** The next family we consider is the q-Kneser graphs. Let n, k be positive integers with $n \geq 2k$, and q be a power of a prime. The vertex set of the graph $K_q(n,k)$ is the set of all k-dimensional subspaces of $\mathbb{F}_q^n$; two vertices are adjacent if and only if they intersect trivially. It is well known that the cardinality of a coclique in this graph is $\binom{n-1}{k-1}_q$, and that the Delsarte-Hoffman ratio bound holds with equality (see [16] or [17, Chapter 9] for notation and details). The canonical induced forests have $1 + \binom{n-1}{k-1}_q$ vertices. We obtain the following characterization of maximum induced forests in q-Kneser graphs.

Theorem 3.5. *For $k \geq 2$, $n > 3k-2$ and q sufficiently large, we have*

$$\tau(K_q(n,k)) = \binom{n-1}{k-1}_q + 1.$$

Moreover, every maximum induced forest is canonical.

Proof. Let γ and δ be two adjacent vertices in $K(n,k)_q$. If ω is a k-subspace intersecting non-trivially with both γ and δ, then it contains a subspace of the form $\langle x \rangle + \langle y \rangle$, where $x \in \gamma \setminus \{0\}$ and $y \in \delta \setminus \{0\}$. A subspace of the form $\langle x \rangle + \langle y \rangle$ can be chosen in $\binom{k}{1}_q^2$ ways. It is a well known fact that there are $\binom{n-2}{k-2}_q$ subspaces of dimension k, which contain a specific 2-dimensional subspace. Thus we have $\eta(K_q(n,k)) \leq \binom{k}{1}_q^2 \binom{n-2}{k-2}_q$.

If F is a non-canonical induced forest, then by Lemma 3.1, we have

$$|F| \leq 2 + 2\binom{k}{1}_q^2 \binom{n-2}{k-2}_q.$$

We will now show that, provided $n > 3k-2$ and q sufficiently large, this upper bound is smaller than $1 + \binom{n-1}{k-1}_q$. Since $\binom{n-1}{k-1}_q = \dfrac{\binom{n-1}{1}_q}{\binom{k-1}{1}_q} \binom{n-2}{k-2}_q$, we have

$$1 + \binom{n-1}{k-1}_q - 2 - 2\binom{k}{1}_q^2 \binom{n-2}{k-2}_q = \binom{n-2}{k-2}_q \left(\frac{\binom{n-1}{1}_q - 2\binom{k-1}{1}_q \binom{k}{1}_q^2}{\binom{k-1}{1}_q} \right) - 1.$$

Expanding the q-binomial coefficients gives that

$$\binom{n-1}{1}_q - 2\binom{k-1}{1}_q \binom{k}{1}_q^2 = \frac{q^{n-1}-1}{q-1} - 2\left(\frac{q^{k-1}-1}{q-1}\right)\left(\frac{q^k-1}{q-1}\right)^2,$$

and, provided that $n-2 > 3k-4$, this is a monic polynomial of degree $n-2$ and hence positive for a sufficiently large q. So for $n > 3k-2$ and q sufficiently large, the order of any forest is bounded above by $1 + \binom{n-1}{k-1}_q$, and this bound is met by only canonical forests. □

As in the case of Kneser graphs, we consider the special case of strongly-regular q-Kneser graphs with $k = 2$, in which the same sort of counting can be done more precisely. In particular, the following result gives a complete characterization of maximum forests in $K_q(n,2)$ provided $n \geq 4$.

Theorem 3.6. *For $n \geq 4$*

$$\tau(K_q(n,2)) = \max\left\{ \binom{n-1}{1}_q + 1, 8 \right\}.$$

If $(n,q) \neq (4,2)$, then every maximum induced forest in $K(n,2)$ is canonical.

Proof. Let F be a non-canonical forest in $K_q(n,2)$. Then F contains either a copy of P_4 or a copy of $2K_2$ as an induced subgraph.

First assume that F has four vertices $\{X, Y, V, W\}$ inducing a path, with $X \sim Y$, $Y \sim V$, and $V \sim W$. From the discussion prior to Lemma 3.1, we have $F \subset N(X,Y) \cup N(V,W) \cup \{Y, V\}$. As $K_q(n,2)$ is edge-transitive, the graph induced by $N(X,Y)$ is isomorphic to the graph induced by $N(V,W)$. We now have $|F| \leq 2 + 2\tau(N(V,W))$, where $\tau(N(V,W))$ is the order of a maximum forest induced in the graph $N(V,W)$. Similarly, if F has four vertices $\{X, Y, V, W\}$ inducing a disjoint union of two edges, with $X \sim Y$ and $V \sim W$, then $|F| \leq 2\tau(N(V,W))$. Therefore the order of a non-canonical forest is bounded above by $2 + 2\tau(N(V,W))$. We will now determine the structure of $N(V,W)$.

As V and W are adjacent, V and W are disjoint 2-subpaces of $\mathbb{F}_q^n$, and thus any $U \in N(V,W)$ is completely determined by $U \cap V$ and $U \cap W$. Let $\mathbb{P}^1(V)$ denote the set of one dimensional subspaces of V, and let $\mathbb{P}^1(W)$ denote the set of one dimensional subspaces of W. Setting $\alpha_U := U \cap V$ and $\beta_U := U \cap W$, we note that the map $U \mapsto (\alpha_U,\ \beta_U)$, is a bijection between $N(V,W)$ and $\mathbb{P}^1(V) \times \mathbb{P}^1(W)$. We also observe that $(\alpha,\ \beta) \mapsto \alpha \oplus \beta$ is the inverse of the above mentioned bijection. Thus given $U_1, U_2 \in N(V,W)$, we have $U_1 \cap U_2 \neq \{0\}$ if and only if, for $i = 1,2$, there exist $x_i \in \alpha_{U_i}$, $y_i \in \beta_{U_i}$, such that $x_1 + y_1 = x_2 + y_2 \neq 0$. For $i = 1,2$, we have $\alpha_{U_i} \subset V$ and $\beta_{U_i} \subset W$. Since, $V \cap W = \{0\}$, given $x_1, x_2 \in V$, and $y_1, y_2 \in W$, we have $x_1 + y_1 = x_2 + y_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$. We can now conclude that $U_1, U_2 \in N(V,W)$ intersect non-trivially if and only if either $\alpha_{U_1} = \alpha_{U_2}$ or $\beta_{U_1} = \beta_{U_2}$. We can now conclude that given $U_1, U_2 \in N(V,W)$, we have $U_1 \sim U_2$ if and only if $\alpha_{U_1} \neq \alpha_{U_2}$ and $\beta_{U_1} \neq \beta_{U_2}$. Now, using $|\mathbb{P}^1(V)| = |\mathbb{P}^1(W)| = \binom{2}{1}_q = q + 1$, we can conclude that the graph induced by $N(V,W)$ is the two fold tensor product of the complete graph K_{q+1}. We deal with these graphs in Subsection 3.4. Using the notation in Subsection 3.4, we have $N(V,W) \cong X_{2,q+1}$.

In Theorem 3.8 we will show, provided that $q \geq 3$, we have $\tau(X_{2,q+1}) = q+2$. Therefore if $q \geq 3$, the order of a non-canonical forest is bounded above by $2+2\tau(N(V,W)) = 2q+6$. The order of the largest canonical forest is $\alpha(K_q(n,2))+1 = \binom{n-1}{1}_q + 1$. Elementary algebra shows that $\binom{n-1}{1}_q + 1 > 2q + 6$ for all $n \geq 4$ and $q \geq 3$. Therefore, provided $q \geq 3$, every maximum forest in $K_q(n,2)$ is canonical.

We now shift our attention to $q = 2$. If V, W are two adjacent vertices in $K_2(n,2)$, then we have seen that $N(V,W) \cong X_{2,3} = K_3 \otimes K_3$. The algebraic bound Theorem 1.5 shows that $\tau(X_{2,3}) < 6$. We can check either by hand or computer that $X_{2,3}$ has an induced path with 5 vertices, and therefore $\tau(X_{2,3}) = 5$. Thus the order of a non-canonical forest F is bounded above by $2+2\tau(N(V,W)) = 12$. For $n > 4$, we have $\alpha(K_2(n,2))+1 = 2^{n-1} > 12$. Therefore provided $n > 4$, every maximum forest in $K_2(n,2)$ is canonical.

We are now left with the case of $K_2(4,2)$. With the help of a computer algebra system (such as Sage [30]), we can show that $\tau(K_2(4,2)) = \alpha(K_2(4,2)) + 1 = 8$. It can also be shown that there are paths on 8 vertices in $K_2(4,2)$. Thus not all maximum forests are canonical in this case.

□

3.3. **Non-collinearity Graphs of Generalized Quadrangles.** The next family we consider is the family of non-collinearity graphs on generalized quadrangles. Let $\mathcal{G}$ be a generalized quadrangle with parameters s, t. Let $X_{\mathcal{G}}$ let denote the graph whose vertices are the points of $\mathcal{G}$, in which two points are adjacent if and only if they are not collinear. It is well known that X_G is a strongly-regular graph with $\left\{s^2t,\ -s,\ t\right\}$ as the set of distinct eigenvalues (see [17, Section 5.6]). By the Delsarte-Hoffman ratio bound for cocliques

(Theorem 1.4),

$$\alpha(X_G) \leq \frac{(s+1)(st+1)s}{s^2t+s} = s+1.$$

The set of all points on a line form a coclique, so this bound is tight. We obtain the following characterization of maximum induced forests in $X_{\mathcal{G}}$.

Theorem 3.7. *Let $\mathcal{G}$ be a generalized quadrangle with parameters (s,t) and let $X_{\mathcal{G}}$ be the non-collinearity graph on points in $\mathcal{G}$. Suppose that $s > 3$, then,*

$$\tau(X_{\mathcal{G}}) = s+2.$$

Moreover, every maximum induced forest in $X_{\mathcal{G}}$ is canonical.

Proof. Consider an induced forest F which contains a path $\mathcal{P}$ on 4 vertices as an induced subgraph. Let $\{A, B, C, D\}$ be the vertices inducing $\mathcal{P}$, with $A \sim B$, $B \sim C$, and $C \sim D$. As F is a forest, any $V \in F \setminus \mathcal{P}$ must be non-adjacent to at least three vertices in $\{A, B, C, D\}$. Suppose V is non-adjacent to each vertex in $\{A, C, D\}$. In other words, V is collinear with every point in $\{A, C, D\}$. Thus V must lie on both the lines $\overrightarrow{AC}$ and $\overrightarrow{AD}$. This implies that $V = A$, which is contrary to our assumption $V \in F \setminus \mathcal{P}$. By the same argument, V cannot be simultaneously non-adjacent to every vertex in $\{A, B, D\}$. Thus V must be adjacent to one of A or D. If V is adjacent to A, then as F is a forest, V must be collinear to every point in $\{B, C, D\}$. Similarly, if V is adjacent to D, then V must be collinear to every point in $\{A, B, C\}$. As $\mathcal{G}$ is a generalized quadrangle, given a line L and a point P not on L, there is a unique point on L, that is collinear with P. Let Q_1 be the unique point on $\overrightarrow{BD}$ that is collinear with C, and let Q_2 be the unique point on $\overrightarrow{AC}$ that is collinear with B. We can now conclude that $F \subseteq \{Q_1, A, B, C, D, Q_2\}$. We now claim that Q_1 and Q_2 are non-collinear. Let us assume the contrary, then we see that $\{Q_1, C, Q_2\}$ form a triangle (not in the graph) in $\mathcal{G}$. This is impossible as a generalized quadrangle cannot contain a triangle, and therefore Q_1 and Q_2 are not collinear. Thus $S := \{Q_1, A, B, C, D, Q_2\}$ induces a cycle on 6 vertices. As $F \subset \{Q_1, A, B, C, D, Q_2\}$ is a forest, we must have $|F| \leq 5$.

Now consider an induced forest F that contains a copy of $2K_2$. Let $\{P, Q\}$ and $\{R, S\}$ be two edges in distinct connected components of the forest. The points P, R, Q, S form vertices of a quadrilateral in $\mathcal{G}$. Suppose that $|F| > 4$, then any $V \in F \setminus \{P, R, Q, S\}$ must be non-adjacent to at least one point in both $\{P, Q\}$ and $\{R, S\}$. Without loss of generality, let V be non-adjacent with R and Q. We claim that V must be on the line $\overrightarrow{RQ}$. Assuming the contrary implies the existence of the triangle VRQ in $\mathcal{G}$, which is not possible as $\mathcal{G}$ is a generalized quadrangle. Again since $\mathcal{G}$ is a generalized quadrangle, R is the unique point on $\overrightarrow{RQ}$ collinear with P; and Q is the unique point on $\overrightarrow{RQ}$ collinear with S. Therefore $V \in \overrightarrow{RQ}$, must be simultaneously non-collinear with both P and S. Now the set $\{R, P, V, S\}$ induces a path on four vertices in F. By the argument in the above paragraph, existence of such a path implies that $|F| \leq 5$.

From the previous two paragraphs, we can conclude that the size of a non-canonical forest is at most 5. Since when $s > 3$ any canonical forest has $s + 2 \geq 6$ vertices, we have shown that if $s > 3$, the only maximum forests in $X_{\mathcal{G}}$ are the canonical ones. □

3.4. **Tensor powers of complete graphs.** We next consider a family of graphs in the Hamming scheme. Consider the complete graph on n vertices, K_n. By $X_{m,n}$, we denote the m-fold tensor product $\otimes^m K_n$. This is the mth graph in the Hamming Scheme $H(m, n)$. The vertex set can be considered as sequences of length m with entries from the additive group $\mathbb{Z}_n$, with two sequences adjacent if and only if they differ at every coordinate. This is an $(n-1)^m$-regular graph whose smallest eigenvalue is $-(n-1)^{m-1}$. Application of the Delsarte-Hoffman ratio bound (Theorem 1.4) shows that $\alpha(X_{m,n}) \leq n^{m-1}$. This bound is met by the subset of sequences whose first coordinate is 0.

If $m = 1$, then $X_{m,n} = K_n$ and any maximum forest is an edge which is a canonical maximum forest. Also, if $n = 1$ then $X_{m,n}$ is simply K_1, so trivially any maximum forest is canonical.

We obtain the following characterization of maximum induced forests in $X_{m,n}$.

Theorem 3.8. *Let m, n be positive integers with $m \geq 2$ and $n > 2m(m-1)$. Then*

$$\tau(X_{m,n}) = n^{m-1} + 1,$$

and every maximum induced forest in $X_{m,n}$ is canonical.

Proof. As before, we investigate the orders of non-canonical forests. A simple counting argument shows that $\eta \leq m(m-1)n^{m-2}$ and therefore by Lemma 3.1 a non-canonical forest has order at most $2 + 2m(m-1)n^{m-2}$. Thus canonical forests are the largest, provided that

$$2 + 2m(m-1)n^{m-2} < n^{m-1} + 1,$$

or, equivalently,

$$1 < n^{m-2}(n - 2m(m-1)).$$

If $m \geq 2$, then the above equation holds whenever $n > 2m(m-1)$. ◻

As in the case of the q-Kneser graphs and the Kneser graphs, we consider the special case of strongly regular tensor powers of complete graphs, in which the same sort of counting can be done more precisely. In particular, the following result gives a complete characterization of maximum forests in $X_{2,n}$ provided $n \geq 4$.

Theorem 3.9. *Given $n \geq 3$, we have $\tau(X_{2,n}) = max(\{5,\ n+1\})$. Moreover when $n \geq 4$, every maximum induced forest is canonical.*

Proof. Firstly, given an edge $\{A, B\}$ of $X_{2,n}$, we observe that $|N(A, B)| = 2$, where $N(A, B)$ is the set of vertices that are not adjacent to either A or B. Suppose that F is a forest, with an induced subgraph P_4 on the vertices $\{A, B, C, D\}$ with $A \sim B$, $B \sim C$ and $C \sim D$. Then from the discussion prior to Lemma 3.1, we know that $F \subset N(A, B) \cup N(C, D) \cup \{B, C\}$. Suppose that X and Y are vertices such that $N(A, B) = \{D, X\}$ and $N(C, D) = \{A, Y\}$. Without loss of generality, we may assume that $A = (a, b)$, $B = (c, d)$, $D = (a, d)$, and $C = (e, b)$, for some $a, b, c, d, e \in \mathbb{Z}_n$ (not necessarily distinct) such that $a \neq c$, $b \neq d$, $a \neq e$, and $e \neq c$. This forces $X = (c, b)$ and $Y = (e, d)$. Therefore X is adjacent to Y. Thus $N(A, B) \cup N(C, D) \cup \{B, C\}$ is a 6 cycle and $|F| \leq 5$.

If F is a forest with an induced copy of $2K_2$, then a similar argument shows that $|F| \leq 5$ (by adding a vertex that induces the same P_5 that arises if we start with a P_4 as above). This completes the proof. ◻

3.5. **Orthogonal Array Graphs.** We finally consider a family of strongly-regular graphs associated with orthogonal arrays. Let m and n be positive integers with $m < n + 1$. An orthogonal array with parameters (m, n) is an $m \times n^2$ array with entries in $\mathbb{Z}_n$ with the property that every $2 \times n^2$ array consists of all n^2 possible ordered pairs from $\mathbb{Z}_n$. Given an orthogonal array O with parameters (m, n), by X_O, we denote the graph on the columns of O, where two columns are adjacent if and only if there are no rows in which they have the same entry. We note that the graph X_O is the complement of the block graph of the orthogonal array O. It is well known, see for example [17, Theorem 5.5.1], that this is a strongly-regular graph with valency $m(n-1)$ and least eigenvalue $m - n - 1$. Application of the Delsarte-Hoffman ratio bound (Theorem 1.4) shows that $\alpha(X_O) \leq n$. This bound is met by the set of columns of O whose first entry is 1.

Theorem 3.10. *Let m, n be positive integers with $n > 1 + 2m(m-1)$ and let O be an orthogonal array with parameters (m, n). Then*

$$\tau(X_O) = n + 1.$$

Moreover, every maximum induced forest in $X_{m,n}$ is canonical.

Proof. We now apply Lemma 3.1 to characterize the maximum independent sets in X_O. We note that $\eta(X_O)$ is the number of common neighbours of two non-adjacent vertices in the complement of X_O. By [17, Theorem 5.5.1], we see that $\eta(X_O) = m(m-1)$. By Lemma 3.1, if F is a non-canonical induced forest, we have $|F| \leq 2 + 2m(m-1)$.

We can now conclude that if $\alpha(X_O) + 1 = n + 1 > 2 + 2m(m-1)$, then every maximum induced forest is canonical. □

4. Kneser graphs with non-canonical maximum forests

As noted in Subsection 3.1, $K(2k, k)$ is a forest, so all of these graphs have non-canonical maximum forests. The logical next family of Kneser graphs to consider are the graphs $K(2k+1, k)$, these graphs also have non-canonical maximum forests.

Lemma 4.1. *If $k > 3$, the graph $K(2k+1, k)$ has a forest of order*

$$\binom{2k}{k} + 2k - 2.$$

hence the maximum forests are not canonical.

Proof. Let F_1 be the set of all vertices in $K(2k+1, k)$ that do not contain the element $2k+1$; F_1 is a set of $\frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k}$ disjoint edges.

For $i = 1, \ldots, 2k-2$, define $x_i = \{i, i+1, \ldots, i+k-3\}$ with the entries taken modulo $2k-1$. Define the set F_2 of vertices of the form $\gamma_i = x_i \cup \{2k, 2k+1\}$ with $i = 1, \ldots, 2k-2$. Clearly F_2 is a coclique and any vertex in F_2 is adjacent to at most one vertex in any edge of F_1 (specifically, the vertex that does not contain $2k$). Further, vertices γ_i and γ_j, have exactly one common neighbour in F_1 if $j = i+1$, and no common neighbours otherwise.

Thus $F_1 \cup F_2$ forms a forest of order $\binom{2k}{k} + 2k - 2$. □

The eigenvalue bound from Theorem 1.5 in this case is

$$\tau(K(2k+1, k)) < \frac{\binom{2k+1}{k}\binom{k}{k-1}}{\binom{k+1}{k} + \binom{k}{k-1}} + \frac{2\binom{2k+1}{k}}{\binom{k+1}{k} + \binom{k}{k-1}} = \frac{k+2}{k}\binom{2k}{k-1}.$$

This bound is larger than the forest given in Lemma 4.1. We can do better using the bound produced by Theorem 1.6, which is

$$\frac{\binom{2k+1}{k}\binom{k+1}{k} - 2}{2\binom{k+1}{k} - 2} = \frac{k+1}{2k}\binom{2k+1}{k} - \frac{1}{k},$$

but this is still significantly larger than the forest our construction produces.

The final case to consider is $K(7, 3)$, and in this case Theorem 1.6 tells us that an induced forest has order at most

$$\frac{4}{6}\binom{7}{3} - \frac{1}{3} = \frac{2(35) - 1}{3} = 23$$

which can be achieved by the forest consisting of all triples from $\{1, \ldots, 6\}$ along with $\{1, 2, 7\}$, $\{1, 3, 7\}$ and $\{2, 3, 7\}$.

5. Further Work

It would be interesting to have more examples of graphs G with $\alpha(G)$ very close to $\tau(G)$. We suspect that a characterization of the graphs with $\tau(G) = \alpha(G) + 1$ is unlikely, but perhaps we can find properties of a graph that would imply these two values are close. In a sense, any such graph would have large independent sets that are uniformly connected to the vertices in its complement. Specifically, any two adjacent vertices outside of the large independent set would have to be adjacent to at least one common vertex in the independent set, and non-adjacent vertices to at least two. This may lead to some structure conditions on a graph that imply that $\tau(G) = \alpha(G) + 1$. We also suspect that focusing the search on strongly-regular graphs may produce more interesting examples.

All the examples of graphs we considered in this paper are graphs whose maximum independent sets have been characterized. Maximum independent sets in Paley graph on a square number vertices were characterized by Blokhuis [7]. We will now discuss some computational results we obtained regarding induced forests in these graphs. Let q be a power of an odd prime. Let $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ be a fields of cardinality q and q^2, respectively. By $\mathcal{P}(q^2)$, we denote the Paley graph on q^2 vertices. The vertex set for $\mathcal{P}(q^2)$ is $\mathbb{F}_q^2$, and two vertices are adjacent if and only if their difference is a quadratic residue in the $\mathbb{F}_{q^2}$. It is well-known that the Paley graph is self-complementary. In this regard, we could consider the complement $\mathcal{P}'(q^2)$ of $\mathcal{P}(q^2)$. We do so because the maximum independent sets in the complement have the following natural characterization.

Theorem 5.1. *(Blokhuis* [7]*) Let q be a power of a prime and S be the set of non-zero squares in $\mathbb{F}_{q^2}$, then $\alpha(\mathcal{P}'(q^2)) = q$. Further, the $\{s\mathbb{F}_q + e \ : s \in S \text{ and } e \in \mathbb{F}_{q^2}\}$ is the set of all cocliques of size q.*

So the size of any canonical forest in $\mathcal{P}'(q^2)$ is $q + 1$. We will now use Theorem 1.5 to obtain an upper bound on the acyclic number. $\mathcal{P}'(q^2)$ is strongly-regular graph and its spectrum is well known to be $(\frac{q^2-1}{2}, \frac{q-1}{2}, -\frac{q+1}{2})$ (see [17, Section 5.8]). Using Theorem 1.5, we have

$$\tau(\mathcal{P}(q^2) < \frac{q^2(q^2+5)}{q^2+q} < q+4$$

In Example 1, we concluded that $\tau(\mathcal{P}'(9)) = 5$. From the discussion above, the size of a canonical forest in $\mathcal{P}'(9)$ is 4 and, in this case, maximum forests are not canonical. We will now consider two more Paley graphs of small order.

Example 2. By Theorem 1.5, a forest in $\mathcal{P}'(25)$ cannot have more than 8 vertices. We have $\mathbb{F}_{25} \cong \mathbb{F}_5[x]/\langle x^2+x+1\rangle$, and the set of quadratic residues is $S = \{\overline{0}\} \cup \{a,\ a\overline{x},\ a\left(\overline{x+1}\right) \mid a \in \mathbb{F}_5^*\}$. The induced subgraph $\mathbb{F}_5 \cup \{\overline{x+2},\ \overline{x+4}\}$ is a forest (in fact, a tree) of order 7 formed by adding two vertices to a maximum independent set. Since canonical forests have order 6, this cannot be a maximum forest. A computational search indicates that 7 is the order of a maximum forest in this graph.

Example 3. Consider the complement of Paley graph on 49 vertices. Theorem 1.5 implies a forest can have no more than 10 vertices. We have $\mathbb{F}_{49} \cong \mathbb{F}_7[x]/\langle x^2+1\rangle$. The set of quadratic residues is $S = \{\overline{0}\} \cup \{a,\ a\overline{x},\ a\left(\overline{x+1}\right),\ a\left(\overline{x-1}\right) \mid a \in \mathbb{F}_7^*\}$. The induced subgraph $\mathbb{F}_7 \cup \{\overline{x+2},\ \overline{x+5}\}$ is a forest (in fact, a tree) of order 9 formed by adding two vertices to a maximum independent set. Again computations indicate that 9 is the order of a maximum forest in this graph, and canonical forests have order 8.

In Examples 1- 3, maximum induced forests (which are, in fact, trees) were obtained by adding two vertices to a maximum independent set. Using Blokhuis's characterization (Theorem 5.1) of maximum cocliques, we used Sage [30] to search if similar constructions were possible in bigger Paley graphs. We checked for all prime powers $7 < q \leq 67$ that adding two vertices to a maximum independent set in $\mathcal{P}'(q^2)$, will not result in a forest. (Of course, maximum forests need not contain maximum independent sets as they have in these examples.) So the examples we found may be anomalies occurring for small values of q. We make the following conjecture.

Conjecture 5.2. *For $q > 7$ a prime power, $\tau(\mathcal{P}(q^2)) = q + 1$.*

Paley graphs on q vertices can be defined whenever q is a prime power with $q \equiv 1 \pmod 4$. Let $\mathcal{P}'(q)$ denote the graph on the field $\mathbb{F}_q$, in which two vertices are adjacent if and only if their difference is not a quadratic residue in $\mathbb{F}_q$. When q is an even power, we conjectured above that $\tau(\mathcal{P}'(q)) = \sqrt{q} + 1$. It is natural to ask the question of what happens when p is not an even power of a prime. Applying Theorem 1.5, we can show that $\tau(\mathcal{P}'(q)) < \sqrt{q} + 4$. In this case, the order of the maximum independent sets is not known in general, but it is bounded by $\sqrt{q}$, and can be significantly smaller. For instance, when q is a prime, [20] shows that $\alpha(\mathcal{P}'(q)) < \sqrt{\frac{q}{2}} + 1$. From our computer searches it seems even in this case $\tau(\mathcal{P}'(q))$ is close to $\sqrt{q}$, so sometimes the induced forests are much larger than $\alpha(\mathcal{P}'(q))$. Further, $\tau(\mathcal{P}'(q))$ seems to be non-decreasing with q, which is not the case for the size of an independent set, and close to the eigenvalue bound. This may just be the case for small values of q, so more computational results would be helpful. A key missing result is a construction of an induced forest of size close to $\sqrt{q}$. Forests are bipartite graphs, and so existence of an induced forest of size $\sqrt{q}$ in $\mathcal{P}'(q)$ implies the existence of independent sets of size at least $\sqrt{q}/2$. When q is not an ever power of a prime, there are no known constructions of such large independent sets in $\mathcal{P}'(q)$.

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