Hamiltonian cycles in circulant digraphs with jumps 2, 3, c [We need a real title???]

Abstract

[We need an abstract???]

1 Introduction

It is not known which circulant digraphs have hamiltonian cycles; this is a fundamental open question. However, the circulants of outdegree 3 are the smallest ones that need to be considered, because a classic result of R. A. Rankin (see Theorem 2.7) provides a nice characterization for circulants of outdegree 2 (and any strongly connected digraph of outdegree 1 is obviously hamiltonian).

S. C. Locke and D. Witte [1] found two infinite families of non-hamiltonian circulant digraphs of outdegree 3. One of the families includes the following examples, which require introducing a piece of notation.

Notation 1.1 *For* $S \subset \mathbb{Z}$*, we use* $\mathrm{Circ}(n; S)$ *to denote the circulant digraph whose vertex set is* $\mathbb{Z}_n$*, and with an arc from* v *to* $v + s$ *for each* $v \in \mathbb{Z}_n$ *and* $s \in S$*.*

Theorem 1.2 (Locke-Witte, cf. [1, Thm. 1.4])

(1) $\mathrm{Circ}(6m; 2, 3, 3m + 2)$ *is not hamiltonian if and only if* m *is even.*
(2) $\mathrm{Circ}(6m; 2, 3, 3m + 3)$ *is not hamiltonian if and only if* m *is odd.*

In this paper, we completely characterize which loopless digraphs of the form $\mathrm{Circ}(n; 2, 3, c)$ that have outdegree 3 are hamiltonian:

Theorem 1.3 *Assume* $c \not\equiv 0, 2, 3 \pmod{n}$*. The digraph* $\mathrm{Circ}(n; 2, 3, c)$ *is **not** hamiltonian iff*

(1) n *is a multiple of* 6*, so we may write* $n = 6m$*,*
(2) *either* $c = 3m + 2$ *or* $c = 3m + 3$*, and*
(3) c *is even.*

Preprint submitted to Elsevier Science *28 September 2006*

The direction (⇐) of Theorem 1.3 is a restatement of part of the Locke-Witte Theorem (1.2), so we need only prove the opposite direction.

Acknowledgements

The work of D.W.M. and J.M. was partially supported by research grants from Canada's National Science and Engineering Research Council.

2 Preliminaries

Our goal is to establish Theorem 1.3(⇒). We will prove the contrapositive.

Assumption 2.1 *Throughout the paper:*

(1) *We assume the situation of Theorem 1.3, so* $n, c \in \mathbb{Z}^+$*, and* $c \not\equiv 0, 2, 3 \pmod{n}$.
(2) *We may assume* $c \not\equiv 1, -1 \pmod{n}$. (*Otherwise,* $\mathrm{Circ}(n; 2, 3, c)$ *has a hamiltonian cycle consisting entirely of arcs of length* c.)
(3) *Since the vertices of* $\mathrm{Circ}(n; 2, 3, c)$ *are elements of* $\mathbb{Z}_n$*, we may assume* $3 < c < n$.
(4) *We assume* n *is divisible by* 6. (*Otherwise,* $\mathrm{Circ}(n; 2, 3, c)$ *has either a hamiltonian cycle consisting entirely of arcs of length* 2 *or a hamiltonian cycle consisting entirely of arcs of length* 3.)
(5) *We write* $n = 6m$.

Notation 2.2 *Let* H *be a subdigraph of* $\mathrm{Circ}(n; 2, 3, c)$*, and let* v *be a vertex of* H.

(1) *We let* $d^+_H(v)$ *and* $d^-_H(v)$ *denote the number of arcs of* H *directed out of, and into, vertex* v*, respectively.*
(2) *If* $d^+_H(v) = 1$*, and the arc from* v *to* $v + a$ *is in* H*, then we say that* v travels by a in H.

Notation 2.3 *Let* u, w *be integers representing vertices of* $\mathrm{Circ}(n; 2, 3, c)$. *If* $u - 1 \le w < u + n$*, let* $I(u, w) = \{u, u + 1, \ldots, w\}$ *be the interval of vertices from* u *to* w. (*Note that* $I(u, u) = \{u\}$ *and* $I(u, u - 1) = \emptyset$.)

Notation 2.4 *For* $v_1, v_2 \in \mathbb{Z}_n$ *and* $s \in \{2, 3, -c'\}$*, we use*

- $v_1 \xrightarrow{s} v_2$ *to denote the arc from* v_1 *to* $v_1 + s = v_2$*, and*
- $v_1 \overset{s}{\dashrightarrow} v_2$ *to denote a path of the form* $v_1, v_1 + s, v_1 + 2s, \ldots, v_1 + ks = v_2$.

We now treat two simple cases that do not conform to the more general structures that we deal with in later sections.

Lemma 2.5 *The digraph* $\mathrm{Circ}(6m; 2, 3, 6m-2)$ *has a hamiltonian cycle.*

PROOF. For $0 \le i \le m-1$, define the path Q_{6i} as follows:

$$6i \overset{2}{\dashrightarrow} 6i+4 \overset{3}{\rightarrow} 6i+7 \overset{6m-2}{\dashrightarrow} 6i+3 \overset{3}{\rightarrow} 6i+6.$$

Notice that this path starts at vertex $6i$, ends at vertex $6i+6$, and uses one vertex from every other equivalence class modulo 6. It is therefore straightforward to verify that the concatenation of the paths $Q_0, Q_6, \ldots, Q_{6m-6}$ is a hamiltonian cycle.

Lemma 2.6 *The digraph* $\mathrm{Circ}(6m; 2, 3, 6m-3)$ *has a hamiltonian cycle.*

PROOF. The following is a hamiltonian cycle in this digraph:

$$0 \overset{3}{\dashrightarrow} 6m{-}6 \overset{2}{\rightarrow} 6m{-}4 \overset{6m-3}{\dashrightarrow} 2 \overset{2}{\rightarrow} 4 \overset{3}{\dashrightarrow} 6m{-}5 \overset{2}{\dashrightarrow} 1 \overset{6m-3}{\rightarrow} 6m{-}2 \overset{2}{\rightarrow} 0.$$

The path from 0 to $6m-6$, together with $6m-3$ (which immediately follows $6m-5$) uses all of the vertices that are $0 \pmod 3$; the path from $6m-4$ to 2, together with $6m-1$ (which immediately follows $6m-3$) uses all of the vertices that are $2 \pmod 3$, and the path from 4 to $6m-5$, together with 1 and $6m-2$, uses all of the vertices that are $1 \pmod 3$.

Although we do not use it in this paper, we recall the following elegant result that was mentioned in the introduction:

Theorem 2.7 (R. A. Rankin, 1948, [2, Thm. 4]) *The circulant digraph* $\mathrm{Circ}(n; a, b)$ *of outdegree* 2 *has a hamiltonian cycle iff there exist* $s, t \in \mathbb{Z}^+$*, such that*

- $s + t = \gcd(n, a-b)$*, and*
- $\gcd(n, sa + tb) = 1$*.*

3 Most cases of the proof

In this section, we prove the following two results that cover most of the cases of Theorem 1.3:

Proposition 3.1 *If $c > 3m$, then* $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle.*

Proposition 3.2 *If $c \leq 3m$ and $c \not\equiv 3 \pmod 6$, then* $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle.*

Notation 3.3 *For convenience, let $c' = 6m - c$, so $1 \leq c' < 6m - 3$, and*

$$\mathrm{Circ}(6m; 2, 3, c) = \mathrm{Circ}(6m; 2, 3, -c').$$

In fact, since the cases $c = 6m - 3, 6m - 2$ and $c \equiv -1 \pmod n$ have already been addressed, we may assume that $c' > 3$.

Remark 3.4 *The use of c' is very convenient when c is large (so one should think of c' as being small — less than $3m$), but it can also be helpful in some other cases.*

Definition 3.5 *A subdigraph P of* $\mathrm{Circ}(n; 2, 3, -c')$ *is a* pseudopath *from u to w if P is the disjoint union of a path from u to w and some number (perhaps 0) of cycles. In other words, if v is a vertex of P, then*

$$d^+_P(v) = \begin{cases} 0 & \text{if } v = w; \\ 1 & \text{otherwise}; \end{cases} \qquad \text{and} \qquad d^-_P(v) = \begin{cases} 0 & \text{if } v = u; \\ 1 & \text{otherwise}. \end{cases}$$

Definition 3.6 *Let u, w be integers representing vertices of* $\mathrm{Circ}(n; 2, 3, c)$*. If $u + c' + 2 \leq w \leq u + 2c'$, let $P(u, w)$ be the pseudopath from $u + 1$ to $w - 1$ whose vertex set is $I(u, w)$, such that v travels by*

$$\begin{cases} 2, & \text{if } v \in I(u, w - c' - 3) \cup I(u + c' + 1, w - 2), \\ 3, & \text{if } v \in I(w - c' - 2, u + c' - 1), \\ -c', & \text{if } v \in \{u + c', w\}. \end{cases}$$

Notice that the range of values for w makes sense because $c' > 3$.

Lemma 3.7 *$P(u, w)$ is a path if any of the following hold:*

- $w - u \equiv 2c' \pmod 3$*; or*
- $w - u \equiv 2c' + 1 \pmod 3$ *and* $w - u \equiv c' \pmod 2$*; or*
- $w - u \equiv 2c' + 2 \pmod 3$ *and* $w - u \not\equiv c' \pmod 2$*.*

PROOF. To simplify the notation slightly, let us assume (without loss of generality) that $u = 0$.

Case 1 *Assume $w - c' \equiv c' \pmod 3$.*

Choose $\varepsilon \in \{1,2\}$, such that $w-c'-\varepsilon-1$ is even. The path in $P(u,w)$ is

$$1 \overset{2}{\text{------}} w-c'-\varepsilon \overset{3}{\text{------}} c'-\varepsilon+3 \overset{2}{\text{------}} w \overset{-c'}{—} w-c'$$
$$\overset{3}{\text{------}} c' \overset{-c'}{—} 0 \overset{2}{\text{------}} w-c'+\varepsilon-3 \overset{3}{\text{------}} c'+\varepsilon \overset{2}{\text{------}} w-1.$$

This contains both of the c-arcs, so there are no cycles in $P(u,w)$.

Case 2 *Assume there exists $\varepsilon \in \{1,2\}$, such that $w-c'-\varepsilon \equiv c' \pmod 3$ and $w-c'-\varepsilon-1$ is even.*

The path in $P(u,w)$ is

$$1 \overset{2}{\text{------}} w-c'-\varepsilon \overset{3}{\text{------}} c' \overset{-c'}{—} 0 \overset{2}{\text{------}} w-c'+\varepsilon-3$$
$$\overset{3}{\text{------}} c'-\varepsilon+3 \overset{2}{\text{------}} w \overset{-c'}{—} w-c' \overset{3}{\text{------}} c'+\varepsilon \overset{2}{\text{------}} w-1.$$

This contains both of the c-arcs, so there are no cycles in $P(u,w)$.

Lemma 3.8 *Let $k \in \mathbb{Z}$ be such that*

- $k \leq 6m$,
- $c'+3 \leq k \leq 2c'+2$, *and*
- $k+c' \not\equiv 3 \pmod 6$.

Let u,w be integers representing vertices of $\text{Circ}(n;2,3,c)$. *Then for all u,w with $u \leq w$ and $w-u+1=k$, the subgraph induced by $I(u,w)$ has a hamiltonian path that starts at $u+1$ and ends in $\{w-1,w\}$.*

PROOF. We consider three cases.

Case 1 *Assume $k \equiv 2c'+1 \pmod 3$.*

We have $w-u=k-1 \equiv 2c' \pmod 3$. Since $k-1 \notin \{2c'+1, 2c'+2\}$, we must have $w-u=k-1 \leq 2c'$). By Lemma 3.7, $P(u,w)$ is a hamiltonian path from $u+1$ to $w-1$.

Case 2 *Assume $k \equiv 2c'+2 \pmod 3$.*

Suppose, first, that $k \neq c'+3$ (so $k \geq c'+4$). Letting $w'=w-1$, then

$$w'-u = w-u-1 = k-2 \geq (c'+4)-2 = c'+2$$

and $w'-u=(w-1)-u=k-2 \equiv 2c' \pmod 3$. By Lemma 3.7, $P(u,w')$ is a hamiltonian path in $I(u,w')$ from $u+1$ to $w'-1$. Adding the 2-arc from $w'-1$ to $w'+1=w$ yields a hamiltonian path in $I(u,w)$ from $u+1$ to w.

Suppose instead that $k = c' + 3$. Then $w - u = k - 1 \equiv 2c' + 1 \pmod 3$ and $w - u = k - 1 = (c' + 3) - 1 \equiv c' \pmod 2$, so by Lemma 3.7, $P(u, w)$ is a hamiltonian path from $u + 1$ to $w - 1$.

Case 3 *Assume* $k \equiv 2c' \pmod 3$.

By assumption, we have $k + c' \equiv 2c' + c' \equiv 0 \pmod 3$. Since $k + c' \not\equiv 3 \pmod 6$, we must have $k + c' \equiv 0 \pmod 6$, so $k \equiv c' \pmod 2$. Then $w - u = k - 1 \equiv 2c' + 2 \pmod 3$ and $w - u \equiv k - 1 \not\equiv k \equiv c' \pmod 2$, so by Lemma 3.7, $P(u, w)$ is a hamiltonian path from $u + 1$ to $w - 1$.

It is now easy to prove Propositions 3.2 and 3.1.

PROOF OF PROPOSITION 3.2. As previously mentioned, we may assume $c > 3$. Since $3 < c \leq 3m$, we have $3m \leq c' < 6m - 3$, so

$$c' + 3 < 6m < 2c' + 2.$$

Furthermore, since $c \not\equiv 3 \pmod 6$, we have

$$6m + c' \equiv c' \not\equiv 3 \pmod 6.$$

Hence, Lemma 3.8 implies that the interval $I(0, 6m - 1)$ has a hamiltonian path from 1 to $6m - 2$ or to $6m - 1$. Inserting the 3-edge from $6m - 2$ to 1 or the 2-edge from $6m - 1$ to 1, yields a hamiltonian cycle. Since $I(0, 6m - 1)$ is the entire digraph, this completes the proof.

PROOF OF PROPOSITION 3.1. We have already dealt with the cases $c = 6m - 2$ (in Lemma 2.5), $c = 6m - 3$ (in Lemma 2.6), and the cases $6m \in \{2c' + 4, 2c' + 6\}$ are dealt with by Theorem 1.2. Furthermore, we noted earlier that the case $c = 6m - 1$ is clearly hamiltonian, so we may assume in what follows that $c < 6m - 3$ and $6m \notin \{2c' + 4, 2c' + 6\}$.

Let $\mathcal{K}$ be the set of integers k that satisfy the conditions of Lemma 3.8. Note that $c' < 3m$.

We claim that n can be written as a sum $n = k_1 + k_2 + \cdots + k_s$, with each $k_i \in \mathcal{K}$. To see this, begin by noting that $c' + 4 \in \mathcal{K}$ (and 5 of any 6 consecutive integers between $c' + 3$ and $2c' + 2$, inclusive, belong to $\mathcal{K}$). Thus, we may assume $n < 2(c' + 4) = 2c' + 8$, for otherwise it is easy to write n as a sum of integers in $\mathcal{K}$. Since n is even, and $n \notin \{2c' + 4, 2c' + 6\}$, we conclude that $n = 2c' + 2 \in \mathcal{K}$. So n is obviously a sum of elements of $\mathcal{K}$. This completes the proof of the claim.

The preceding paragraph implies that we may cover the vertices of $\mathrm{Circ}(n;2,3,c)$ by a disjoint collection of intervals $I(u_i,w_i)$, such that the number of vertices in $I(u_i,w_i)$ is k_i. By listing the intervals in their natural order, we may assume $u_{i+1}=w_i+1$. By Proposition 3.8, the vertices of $I(u_i,w_i)$ can be covered by a path P_i that starts at u_i+1 and ends in $\{w_i-1,w_i\}$. Since

$$(u_{i+1}+1)-w_i=(w_i+2)-w_i=2$$

and

$$(u_{i+1}+1)-(w_i-1)=(w_i+2)-(w_i-1)=3,$$

there is an arc from the terminal vertex of P_i to the initial vertex of P_{i+1}. Thus, by adding a number of 2-arcs and/or 3-arcs, we may join all of the paths $P_1,P_2,\ldots,P_s$ into a single cycle that covers all of the vertices of $\mathrm{Circ}(n;2,3,c)$. Thus, we have constructed a hamiltonian cycle.

4 The remaining cases

In this section, we prove the following result. Combining it with Propostions 3.1 and 3.2 (and Theorem 1.2) completes the proof of Theorem 1.3.

Proposition 4.1 *If $c\le 3m$ and $c\equiv 3 \pmod 6$, then $\mathrm{Circ}(6m;2,3,c)$ has a hamiltonian cycle.*

Definition 4.2 *Let t be any natural number, such that $0\le 6t\le c-9$.*

(1) *Let*

$$\begin{aligned}\ell_1&=c-5,\\ \ell_2&=\ell_2(t)=c-1+6t,\\ \ell_3&=c-2,\\ \ell_4&=c+3.\end{aligned}$$

(2) *Define subdigraphs Q_1,Q_2,Q_3 and Q_4 of $\mathrm{Circ}(6m;2,3,c)$ as follows:*

- *The vertex set of Q_i is $I(0,\ell_i+2)\cup\{\ell_i+5\}$.*
- *In Q_1, vertex v travels by* $\begin{cases} c, & \text{if } v=0;\\ 2, & \text{if } v=1 \text{ or } 2;\\ 3, & \text{if } v=3,4,\ldots,c-6.\end{cases}$
- *In Q_2, vertex v travels by* $\begin{cases} c, & \text{if } v=1 \text{ or } 6t+4;\\ 2, & \text{if } v=2 \text{ or } 6t+5\le v\le c-2;\\ 3, & \text{if } v=0 \text{ or } 3\le v\le 6t+3 \text{ or } c-1\le v\le c-2+6t.\end{cases}$
- *In Q_3, vertex v travels by* $\begin{cases} c, & \text{if } v=1 \text{ or } 3;\\ 2, & \text{if } v=1,2 \text{ or } 4\le v\le c-3.\end{cases}$

- *In* Q_4, *vertex* v *travels by* $\begin{cases} c, & \text{if } v = 2 \text{ or } 8; \\ 2, & \text{if } 9 \le v \le c-1; \\ 3, & \text{if } v = 0, 1 \text{ or } 3 \le v \le 7 \text{ or } c \le v \le c+2. \end{cases}$

Notation 4.3 *For ease of later referral, we also let* $\ell_i(t)$ *denote* ℓ_i *for* $i \in \{1, 3, 4\}$.

Lemma 4.4

(1) *Each* Q_i *is the union of four disjoint paths from* $\{0, 1, 2, 5\}$ *to* $\{\ell_i, \ell_i + 1, \ell_i + 2, \ell_i + 5\}$.
(2) *Indeed, if we*
- *let* $u_1 = 0$, $u_2 = 1$, $u_3 = 2$, $u_4 = 5$, *and* $w_{i,j} = \ell_i + u_j$, *and*
- *define permutations*

$$\sigma_1 = (1423),\ \sigma_2 = (234),\ \sigma_3 = (1324),\ \text{and } \sigma_4 = \text{identity},$$

then Q_i *contains a path from* u_j *to* $w_{i,\sigma_i(j)}$ *for* $j = 1, 2, 3, 4$.

PROOF. The paths in Q_1 are:

$$\begin{array}{l} 0 \stackrel{c}{\text{---}} c \quad (= \ell_1 + 5), \\ 1 \stackrel{2}{\text{---}} 3 \stackrel{3}{\text{- - -}} c - 3 \quad (= \ell_1 + 2), \\ 2 \stackrel{2}{\text{---}} 4 \stackrel{3}{\text{- - -}} c - 5 \quad (= \ell_1), \\ 5 \stackrel{3}{\text{- - -}} c - 4 \quad (= \ell_1 + 1). \end{array}$$

The paths in Q_2 are:

$$\begin{array}{l} 0 \stackrel{3}{\text{- - -}} 6t + 6 \stackrel{2}{\text{- - -}} c - 1 \stackrel{3}{\text{- - -}} c - 1 + 6t \quad (= \ell_2), \\ 1 \stackrel{c}{\text{---}} c + 1 \stackrel{3}{\text{- - -}} c + 1 + 6t \quad (= \ell_2 + 2), \\ 2 \stackrel{2}{\text{---}} 4 \stackrel{3}{\text{- - -}} 6t + 4 \stackrel{c}{\text{---}} c + 4 + 6t \quad (= \ell_2 + 5), \\ 5 \stackrel{3}{\text{- - -}} 6t + 5 \stackrel{2}{\text{- - -}} c \stackrel{3}{\text{- - -}} c + 6t \quad (= \ell_2 + 1). \end{array}$$

The paths in Q_3 are:

$$\begin{array}{l} 0 \stackrel{c}{\text{---}} c \quad (= \ell_3 + 2), \\ 1 \stackrel{2}{\text{---}} 3 \stackrel{c}{\text{---}} c + 3 \quad (= \ell_3 + 5), \\ 2 \stackrel{2}{\text{- - -}} c - 1 \quad (= \ell_3 + 1), \\ 5 \stackrel{2}{\text{- - -}} c - 2 \quad (= \ell_3). \end{array}$$

The paths in Q_4 are:

$$
\begin{aligned}
&0 \overset{3}{\text{------}} 9 \overset{2}{\text{------}} c \overset{3}{\text{---}} c+3 \quad (= \ell_4),\\
&1 \overset{3}{\text{------}} 10 \overset{2}{\text{------}} c+1 \overset{3}{\text{---}} c+4 \quad (= \ell_4 + 1),\\
&2 \overset{c}{\text{---}} c+2 \overset{3}{\text{---}} c+5 \quad (= \ell_4 + 2),\\
&5 \overset{3}{\text{---}} 8 \overset{c}{\text{---}} c+8 \quad (= \ell_4 + 5).
\end{aligned}
$$

The above lemma yields the following conclusion:

Lemma 4.5 *If, for some natural number k, there exist sequences*

- $I = (i_1, i_2, \ldots, i_k)$ *with each* $i_j \in \{1, 2, 3, 4\}$*, and*
- $T = (t_1, t_2, \ldots, t_k)$ *with* $0 \le 6t_j \le c - 9$*, for each* j*,*

such that

(i) $\sum_{j=1}^{k} \ell_{i_j}(t_j) = 6m$*, and*
(ii) *the permutation product* $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ *is a cycle of length* 4*,*

then $\mathrm{Circ}(6m; 2, 3, c)$ *has a hamiltonian cycle constructed by concatenating copies of* Q_1*,* Q_2*,* Q_3*, and* Q_4*.*

PROOF OF PROPOSITION 4.1. Since σ_4 is the identity and $\ell_4 = c+3$, we see that if $\mathrm{Circ}(6m; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating copies of Q_1, Q_2, Q_3, and Q_4, then $\mathrm{Circ}(6m+c+3; 2, 3, c)$ also has such a hamiltonian cycle. Thus, by subtracting some multiple of $c+3$ from $6m$, we may assume

$$2c - 6 \le 6m \le 3c - 9.$$

(For this modified c, it is possible that $c > 3m$.)

Recall that $0 \le 6t \le c - 9$, so $2c - 6 + 6t$ can be any multiple of 6 between $2c - 6$ and $3c - 15$. Since $\sigma_1\sigma_2 = (1243)$ and $\ell_1 + \ell_2(t) = 2c - 6 + 6t$, it follows that $\mathrm{Circ}(6m; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating one copy of Q_1 and one copy of Q_2 whenever $2c - 6 \le 6m \le 3c - 15$.

The only case that remains is when $6m = 3c - 9$. Now $\sigma_1\sigma_3^2 = (1324)$ and $\ell_1 + 2\ell_3 = 3c - 9$, so $\mathrm{Circ}(3c - 9; 2, 3, c)$ has a hamiltonian cycle constructed by concatenating one copy of Q_1 and two copies of Q_3.

References

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[2] R. A. Rankin: A campanological problem in group theory, *Proc. Cambridge Philos. Soc.* 44 (1948) 17–25. MR0022846