DIRECTED CYCLIC HAMILTONIAN CYCLE SYSTEMS OF THE COMPLETE SYMMETRIC DIGRAPH

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Abstract. In this paper, we prove that directed cyclic hamiltonian cycle systems of the complete symmetric digraph, K_n^*, exist if and only if $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ with p prime and $\alpha \geq 1$. We also show that directed cyclic hamiltonian cycle systems of the complete symmetric digraph minus a set of $n/2$ vertex-independent digons, $(K_n - I)^*$, exist if and only if $n \equiv 0 \pmod 4$.

1. Introduction

Throughout this paper, K_n will denote the complete graph on n vertices, and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is a set $\mathcal{C}$ of m-cycles in G whose edges partition the edge set of G. An m-cycle system is called *hamiltonian* if $m = |V(G)|$. Finally, I will denote a 1-factor (that is, a perfect matching) in a complete graph on an even number of vertices.

Several obvious necessary conditions for an m-cycle system $\mathcal{C}$ of a graph G to exist are immediate: $m \leq |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G. A survey on cycle systems is given in [16] and necessary and sufficient conditions for the existence of an m-cycle system of K_n and $K_n - I$ were given in [1, 19] where it was shown that a m-cycle system of K_n or $K_n - I$ exists if and only if $n \geq m$, every vertex of K_n or $K_n - I$ has even degree, and m divides the number of edges in K_n or $K_n - I$, respectively.

Let ρ denote the permutation $(0\ 1\ \ldots\ n-1)$, so $\langle \rho \rangle = \mathbb{Z}_n$. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is called *cyclic* if, for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. An n-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is called a *cyclic hamiltonian cycle system.* Finding necessary and sufficient conditions for cyclic m-cycle systems of K_n is an interesting problem and has attracted much attention (see, for example, [6, 7, 8, 9, 11, 12, 14, 15, 17, 21]). The obvious necessary conditions for a cyclic m-cycle system of K_n are the same as for an m-cycle system of K_n; that is, $n \geq m \geq 3$, n is odd (so that the degree of every vertex is even), and m must divide the number of edges in K_n. However, these conditions are no longer necessarily sufficient. For example, it is not difficult to see that there is no cyclic decomposition of K_{15} into 15-cycles. Also, if p is an odd prime and $\alpha \geq 2$, then K_{p^α} cannot be decomposed cyclically into p^α-cycles [9]. In [9], it is shown that a cyclic hamiltonian cycle system of K_n exists if and only if $n \neq 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$.

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These questions can be extended to the case when n is even by considering the graph $K_n - I$. In [13], the hamiltonian case is considered where it is shown that $K_n - I$ has a cyclic hamiltonian cycle system if and only if $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^\alpha$ with p prime and $\alpha \geq 1$.

For a graph G, let G^* denote the digraph obtained from G by replacing every edge $\{u, v\}$ of G with the arcs uv and vu. Thus, K_n^* denotes the complete symmetric digraph, that is, the digraph with n vertices and all possible arcs, and for n even, $(K_n - I)^*$ denotes the complete symmetric digraph on n vertices with a set of $n/2$ vertex-independent digons removed.

A *directed m-cycle system* of a digraph G is a set $\mathcal{C}$ of directed m-cycles in G whose arcs partition the arc set of G. A directed m-cycle system is called *hamiltonian* if $m = |V(G)|$. The necessary conditions for a directed m-cycle system of K_n^* are that $2 \leq m \leq n$ and $m \mid n(n-1)$. In [3], it is shown that these necessary conditions are sufficient if $m \in \{10, 12, 14\}$. In [5], it is shown that the necessary conditions are sufficient if $m \in \{4, 6, 8, 16\}$ and $(n, m) \neq (4, 4), (6, 6)$. The problem is further resolved in [5] for m even and a divisor of $n - 1$. In [20], it is shown that K_n^* can be decomposed into directed hamiltonian cycles if n is even and $n \neq 4, 6$. In [4] the case when $m = 3$ is completely settled and some results are given for other odd lengths. In [2], it is shown that for positive integers m and n, with $2 \leq m \leq n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if m divides the number of arcs in K_n^* and $(n, m) \neq (4, 4), (6, 3), (6, 6)$.

A directed m-cycle system $\mathcal{C}$ of a digraph G with vertex set $\mathbb{Z}_n$ is *cyclic* if, for every directed m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the directed m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. A directed n-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is called a *directed cyclic hamiltonian cycle system.* Although necessary and sufficient conditions for directed hamiltonian cycle systems are known [20], the constructions given in [20] are not cyclic. Thus, in this paper, we investigate directed cyclic hamiltonian cycle systems and prove the following result.

Theorem 1.1. *Let $n \geq 2$ be an integer.*

(a) *For n odd, there exists a directed cyclic hamiltonian cycle system of K_n^* if and only if $n \neq 15$ and $n \neq p^\alpha$ where p is an odd prime and $\alpha \geq 2$.*
(b) *There exists a directed cyclic hamiltonian cycle system of K_n^* if and only if $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$.*
(c) *There exists a directed cyclic hamiltonian cycle system of $(K_n - I)^*$ if and only if $n \equiv 0 \pmod 4$.*

Our methods involve circulant digraphs and difference constructions. In Section 2, we give some basic definitions and lemmas while the proof of Theorem 1.1 is given in Sections 3, 4, and 5. Section 3 deals with the decomposition of K_n^* when n is odd, Section 4 deals with the decomposition of K_n^* when $n \equiv 2 \pmod 4$, and Section 5 deals with the decomposition of $(K_n - I)^*$ when $n \equiv 0 \pmod 4$. In Lemma 4.1, we show that if there is a cyclic hamiltonian cycle system of K_n^*, then $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$. Lemma 4.2 provides the existence of a directed hamiltonian cycle system of K_n^* when $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ for some odd prime p and $\alpha \geq 1$. Lemma 5.1 provides the existence of a directed hamiltonian cycle system of $(K_n - I)^*$ in the case that $n \equiv 0 \pmod 4$, as well as non-existence when $n \equiv 2 \pmod 4$. Theorem 1.1 then follows.

2. Preliminaries

The proof of Theorem 1.1 uses circulant digraphs, which we now define. The *circulant digraph* $\overrightarrow{X}(n;S)$ is defined to be that digraph whose vertices are the elements of $\mathbb{Z}_n$, with an arc from vertex g to vertex h if and only if $h = g + s$ for some $s \in S$. We call S the *connection set*, and we may write $-s$ for $n-s$ when n is understood. For a set S of integers, the notation $\pm S$ will denote the set $\{\pm s \mid s \in S\}$.

Notice that in order for a digraph G to admit a directed cyclic m-cycle system, G must be a circulant digraph, so circulant digraphs provide a natural setting in which to construct directed cyclic m-cycle systems.

The digraph K_n^* is a circulant digraph, since $K_n^* = \overrightarrow{X}(n;\{1,2,\ldots,n-1\})$. For n even, $(K_n - I)^*$ is also a circulant digraph, since $(K_n-I)^* = \overrightarrow{X}(n;\{1,2,\ldots,n-1\}\setminus\{n/2\})$ (so the arcs of I are of the form $(i, i+n/2)$ for $i = 0,1,\ldots,n-1$). In fact, if $n = a'b$ and $\gcd(a',b) = 1$, then we can view $\mathbb{Z}_n$ as $\mathbb{Z}_{a'} \times \mathbb{Z}_b$, using the group isomorphism $\phi : \mathbb{Z}_n \to \mathbb{Z}_{a'} \times \mathbb{Z}_b$ defined by $\phi(k) = (k \pmod{a'}, k \pmod{b})$. We can therefore relabel both the vertices and the arc lengths of the circulant digraphs, using ordered pairs from $\mathbb{Z}_{a'} \times \mathbb{Z}_b$, rather than elements of $\mathbb{Z}_n$, by identifying elements of $\mathbb{Z}_n$ with their images under ϕ. This will prove to be a very useful tool in our results. Throughout Section 3, as n is even, we will use the isomorphism ϕ with $a' = 2a$ for some a, and b odd.

Let H be a subdigraph of a circulant digraph $\overrightarrow{X}(n;S)$. For a fixed set of arc lengths S', the notation $\ell(H)$ will denote the set of arc lengths belonging to H, that is,

$$\ell(H) = \{s \in S' \mid \{g, g+s\} \in E(H) \text{ for some } g \in \mathbb{Z}_n\}.$$

Many properties of $\ell(H)$ are independent of the choice of S'; in particular, neither of the two lemmas in this section depends on the choice of S'. The proofs of these two lemmas are omitted as they follow directly from the proofs of the corresponding lemmas given in [13] for the undirected case.

Let C be a directed m-cycle in $\overrightarrow{X}(n;S)$ and recall that the permutation ρ, which generates $\mathbb{Z}_n$, has the property that $\rho(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$. We can therefore consider the action of $\mathbb{Z}_n$ as a permutation group acting on the elements of $\mathcal{C}$. Viewing matters this way, the *length of the orbit of* C (under the action of $\mathbb{Z}_n$) can be defined as the least positive integer k such that $\rho^k(C) = C$. Observe that such a k exists since ρ has finite order; furthermore, the well-known orbit-stabilizer theorem (see, for example [10, Theorem 1.4A(iii)]) tells us that k divides n. Thus, if G is a digraph with a directed cyclic m-cycle system $\mathcal{C}$ with $C \in \mathcal{C}$ in an orbit of length k, then it must be that k divides $n = |V(G)|$ and that $\rho(C), \rho^2(C), \ldots, \rho^{k-1}(C)$ are distinct m-cycles in $\mathcal{C}$, where $\rho = (0\ 1\ \cdots\ n-1)$.

Lemma 2.1. *Let $\mathcal{C}$ be a directed cyclic m-cycle system of a digraph G of order n. If $C \in \mathcal{C}$ is in an orbit of length k, then $|\ell(C)| = mk/n$. Furthermore, if $\ell \in \ell(C)$, then C has n/k arcs of length ℓ.*

In the case that $m = n$, Lemma 2.1 implies that a cycle in an orbit of length k has exactly k distinct arc lengths. More generally, Lemma 2.1 also implies that n/k must divide m; therefore, we have that $(n/k) \mid \gcd(m,n)$.

Lemma 2.2. *Let C be a directed n-cycle in a digraph G with $V(G) = \mathbb{Z}_n$. If C is in an orbit of length $k > 1$, then for each $\ell \in \ell(C)$, we have that $k \nmid \ell$.*

Let X be a set of directed m-cycles in a digraph G with vertex set $\mathbb{Z}_n$ such that $\mathcal{C} = \{\rho^i(C) \mid C \in X, i = 0, 1, \ldots, n-1\}$ is a directed m-cycle system of G. Then X is called a *starter set* for $\mathcal{C}$ and the directed m-cycles in X are called *starter cycles*. Clearly, every directed cyclic m-cycle system $\mathcal{C}$ of a digraph G has a starter set X as we may always let $X = \mathcal{C}$. A starter set X is called a *minimum starter set* if $C \in X$ implies $\rho^i(C) \notin X$ for $1 \leq i \leq n-1$. Observe that if X is a minimum starter set for a directed cyclic m-cycle system $\mathcal{C}$ of the digraph $\overrightarrow{X}(n; S)$ and S is a set of arc lengths, then it must be that the collection of sets $\{\ell(C) \mid C \in X\}$ forms a partition of S.

When we explicitly construct individual cycles, which will only occur in the cases where n is even, the strategy we will adopt is as follows. We will choose integers a and b so that $n = 2ab$ with b odd and $\gcd(a, b) = 1$. We view K_n^* or $(K_n - I)^*$ (as appropriate) as a circulant digraph labelled by the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$. Let

$$S' = \{(i, j) \mid 0 \leq i \leq 2a-1, 0 \leq j \leq b-1\} \setminus \{(0, 0)\},$$

if $n \equiv 2 (\mathrm{mod}\, 4)$, or

$$S' = \{(i, j) \mid 0 \leq i \leq 2a-1, 0 \leq j \leq b-1\} \setminus \{(0, 0), (a, 0)\},$$

if $n \equiv 0 (\mathrm{mod}\, 4)$. Observe that $|S'| = 2ab-1 = n-1$ if $n \equiv 2 (\mathrm{mod}\, 4)$, and $|S'| = 2ab-2 = n-2$ if $n \equiv 0 (\mathrm{mod}\, 4)$. Thus $\overrightarrow{X}(n; S') = K_n^*$ when $n \equiv 2 (\mathrm{mod}\, 4)$ and $\overrightarrow{X}(n; S') = (K_n - I)^*$ when $n \equiv 0 (\mathrm{mod}\, 4)$.

Let $\hat{\rho} = \phi\rho\phi^{-1}$ and note that

$$\hat{\rho} = ((0, 0)\ (1, 1)\ (2, 2)\ \cdots\ (2a-1, b-1))$$

generates $\mathbb{Z}_{2a} \times \mathbb{Z}_b$, that is, $\langle\hat{\rho}\rangle = \mathbb{Z}_{2a} \times \mathbb{Z}_b$. Let $\mathcal{C}$ be a directed m-cycle system of K_n^* or $(K_n - I)^*$, where the vertices have been labelled by the elements of $\mathbb{Z}_{2a} \times \mathbb{Z}_b$ such that $C \in \mathcal{C}$ implies $\hat{\rho}(C) \in \mathcal{C}$. Then, clearly $\{\phi^{-1}(C) \mid C \in \mathcal{C}\}$ is a directed cyclic m-cycle system of K_n^* or $(K_n - I)^*$, whichever is appropriate.

Next observe that if $(e, f) \in S'$ has $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$, then $\overrightarrow{X}(n; \{\phi^{-1}((e, f))\})$, the subdigraph consisting of the arcs of length $\phi^{-1}((e, f))$, forms a directed n-cycle C with the property that $\rho(C) = C$. Let

$$T = \{(i, j) \in S' \mid \gcd(i, 2a) > 1 \text{ or } \gcd(j, b) > 1\}.$$

To find a directed cyclic hamiltonian cycle system of K_n^* or $(K_n - I)^*$, it suffices to find a set X of directed n-cycles such that $\{\ell(C) \mid C \in X\}$ contains every element of T exactly once. Then the collection

$$\mathcal{C} = \{\phi^{-1}(C), \rho(\phi^{-1}(C)), \ldots, \rho^{n-1}(\phi^{-1}(C)) \mid C \in X\} \cup \{\overrightarrow{X}\left(n; \{\phi^{-1}((e, f))\}\right) \mid (e, f) \in S' \setminus T\}$$

is a directed cyclic hamiltonian cycle system of K_n^* or $(K_n - I)^*$, again whichever is appropriate.

3. The case when n is odd

In this section, we prove Theorem 1.1 (a). The proof of the following lemma is similar to the proof of Lemma 3.1 from [13] and the "only if" part of the proof of Theorem 1.1 of [9].

Lemma 3.1. *For an odd integer $n \geq 3$, if there exists a cyclic hamiltonian cycle system of K_n^*, then $n \neq 15$ and $n \neq p^\alpha$ where p is an odd prime and $\alpha \geq 2$.*

Proof. Suppose first that $K_{15}^* = \overrightarrow{X}(15; \{1, 2, \ldots, 14\})$ has a directed cyclic hamiltonian cycle system $\mathcal{C}$. Let X be a minimum starter set for $\mathcal{C}$. Let $C \in X$ be in an orbit of length k with $3i \in \ell(C)$. Then Lemma 2.2 gives $k \neq 3$. Also $k \neq 1$, since the arcs of length $3i$ form three disjoint directed cycles of length 5. Since $k \mid 15$ and we have a total of 14 arc lengths available, it must be that $k = 5$. Hence, Lemma 2.1 gives $|\ell(C)| = 5$. Suppose $\ell(C) = \{3, 6, 9, 12, \ell\}$. Since C is in an orbit of length 5, we have $3+6+9+12+\ell \equiv 0 \pmod 5$ giving $\ell \equiv 0 \pmod 5$. Then $k \mid \ell$, contradicting Lemma 2.2. Therefore not all of the arc lengths 3, 6, 9, and 12 can belong to the same cycle in X. Hence there must exist at least two distinct cycles C_1 and C_2 in X in distinct orbits of length 5 with $\{3, 6, 9, 12\} \subseteq \ell(C_1) \cup \ell(C_2)$, and the above remarks give $|\ell(C_1) \cup \ell(C_2)| = 10$. In a similar fashion, there must exist two distinct cycles C_3 and C_4 in X in distinct orbits of length 3 with $\{5, 10\} \subseteq \ell(C_3) \cup \ell(C_4)$ and $|\ell(C_3) \cup \ell(C_4)| = 6$. Hence, $|\{\ell(C) \mid C \in X\}| \geq 16$, producing a contradiction. Therefore K_{15}^* has no directed cyclic hamiltonian cycle system.

It remains to show that $n \neq p^\alpha$ where p is an odd prime and $\alpha \geq 2$. Suppose, to the contrary, that $n = p^\alpha$ for some odd prime p and $\alpha \geq 2$. Choose $C \in X$ with $p^{\alpha-1} \in \ell(C)$. Suppose that C is in an orbit of length k. Then $k \mid p^\alpha$, and since K_n^* has $p^\alpha(p^\alpha - 1)$ arcs and each cycle of $\mathcal{C}$ has p^α arcs, we must have $|\mathcal{C}| = p^\alpha - 1$. It therefore follows that $1 \leq k < p^\alpha$. Hence, $k \mid p^{\alpha-1}$, and by Lemma 2.2, we must have $k = 1$. But if $k = 1$, then $\ell(C) = \{p^{\alpha-1}\}$ and since $\overrightarrow{X}(p^\alpha; \{p^{\alpha-1}\})$ consists of $p^{\alpha-1}$ p-cycles, we have a contradiction. Therefore, $n \neq p^\alpha$ where p is an odd prime and $\alpha \geq 2$. $\square$

To complete the proof of Theorem 1.1 (a), we need only remind the reader that a cyclic hamiltonian cycle system $\mathcal{C}$ of K_n is constructed in [9] for n odd and $n \notin \{15, p^\alpha \mid \alpha \geq 2\}$. Now, $\mathcal{C}' = \{\overrightarrow{C}, \overleftarrow{C} \mid C \in \mathcal{C}\}$ is a directed cyclic hamiltonian cycle system for K_n^*, where $\overrightarrow{C}$ and $\overleftarrow{C}$ are the directed cycles obtained by orienting C in each direction.

4. The case when $n \equiv 2 \pmod 4$

In this section, we prove Theorem 1.1 (b). We begin by determining the admissible values of n in Lemma 4.1. Next, for those admissible values of n, we construct directed cyclic hamiltonian cycle systems of K_n^*, in Lemma 4.2.

Lemma 4.1. *For an even integer $n \geq 4$, if there exists a directed cyclic hamiltonian cycle system of K_n^*, then $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ where p is odd prime and $\alpha \geq 1$.*

Proof. This proof is analogous to the proof of Lemma 3.1 from [13]. Suppose $K_n^* = \overrightarrow{X}(n; \{1, 2, \ldots, n-1\})$ has a directed cyclic hamiltonian cycle system $\mathcal{C}$. Let X be a minimum starter set for $\mathcal{C}$ and let $C \in X$. Then, as in the proof of Lemma 3.1 of [13], $\ell(C)$ must have an even number of even arc lengths, hence forcing an even number of even integers in the set $\{1, 2, \ldots, n-1\}$. Thus $n \equiv 2 \pmod 4$.

It remains to show that $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$. Suppose, to the contrary, that $n = 2p^\alpha$ for some prime p and $\alpha \geq 1$. Choose $C \in X$ with $2p^{\alpha-1} \in \ell(C)$. Suppose that C is in an orbit of length k. Then (as previously noted) $k \mid 2p^\alpha$, and since K_n^* has $2p^\alpha(2p^\alpha - 1)$ arcs and each cycle of $\mathcal{C}$ has $2p^\alpha$ arcs, we must have $|\mathcal{C}| = 2p^\alpha - 1$. It therefore follows that $1 \leq k < 2p^\alpha$. Hence, $k \mid 2p^{\alpha-1}$, and by Lemma 2.2, we must have $k = 1$. But if $k = 1$, then $\ell(C) = \{2p^{\alpha-1}\}$ and since $\overrightarrow{X}(2p^\alpha; \{2p^{\alpha-1}\})$ consists of $2p^{\alpha-1}$ p-cycles, we have a contradiction. Therefore, $n \neq 2p^\alpha$ where p is prime and $\alpha \geq 1$. $\square$

Before proceeding, let Φ denote the Euler-phi function, that is, for a positive integer a, $\Phi(a)$ denotes the number of integers n with $1 \leq n \leq a$ and $\gcd(n, a) = 1$. For a positive integer a, $\Phi(a)$ is easily computed from the prime factorization of a. Let $a = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where $p_1, p_2, \ldots, p_t$ are distinct primes and $k_1, k_2, \ldots, k_t$ are positive integers. Then

$$\Phi(a) = \prod_{i=1}^{t} p_i^{k_i - 1}(p_i - 1).$$

Lemma 4.2. *For $n \equiv 2 \pmod 4$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$, the digraph K_n^* has a directed cyclic hamiltonian cycle system.*

Proof. Suppose that $n \equiv 2 \pmod 4$ with $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$, say $n = 4q + 2$ for some positive integer q. If $q = 0$, then K_2^* is a directed hamiltonian cycle. Thus, we may assume that $n \geq 6$. Let $2q + 1 = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are all distinct primes with $p_1 < p_2 < \ldots < p_r$. Notice that since $n \neq 2p^\alpha$, we have $r \geq 2$.

Let $a = p_1^{k_1}$, and $b = (2q + 1)/a$; observe that $\gcd(a, b) = 1$. Notice also that since p_1 is the smallest odd prime divisor of n, $\gcd(3, b) = 1$. The set

$$S' = \{(i, j) \mid 0 \leq i \leq 2a - 1, 0 \leq j \leq b - 1\} \setminus \{(0, 0)\}$$

has the property that $\phi^{-1}(S') = \{1, 2, \ldots, n-1\}$, so we can think of the elements of S' as the arc lengths of the relabelled digraph.

Let $d_1, d_2, \ldots, d_t$ denote the integers with $1 \leq d_j < 2a$ and $\gcd(d_j, 2a) > 1$ and let $e_1, e_2, \ldots, e_{2a-1-t}$ denote the integers in the set $\{1, 2, \ldots, 2a - 1\} \setminus \{d_1, d_2, \ldots, d_t\}$. Hence $\gcd(e_i, 2a) = 1$ for $1 \leq i \leq 2a - 1 - t$. Since $\gcd(d_j, 2a) > 1$ gives $\gcd(2a - d_j, 2a) > 1$, it follows that t is odd. Also, we assume $d_1 < d_2 < \ldots < d_t$, and $e_1 < e_2 < \ldots < e_{2a-1-t}$ so that $e_1 = 1$.

We need to show that $2(2a - 1 - t) \geq t + 1$. First, $\Phi(2a)$ is the number of integers k with $1 \leq k \leq 2a$ and $\gcd(k, 2a) = 1$. Thus, $2a - 1 - \Phi(2a)$ is the number of integers ℓ with $1 \leq \ell < 2a$ and $\gcd(\ell, 2a) > 1$. Hence $t = 2a - 1 - \Phi(2a)$. Substituting $t = 2a - 1 - \Phi(2a)$ into $2(2a - 1 - t) \geq t + 1$, we obtain the inequality $\Phi(2a) \geq 2a/3$. Since $a = p_1^{k_1}$ and $p_1 \geq 3$, we have $\Phi(2a) \geq 2a/3$. Hence $2(2a - 1 - t) \geq t + 1$.

We will deal separately with the cases $b \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$.

Case 1. Let $b \equiv 1 \pmod 4$.

Then $b = 4m + 1$ for some positive integer m. Define the walks P and P' by

$$\begin{aligned} P \ : \ & (0,0),(0,1),(0,-1),(0,2),(0,-2),\ldots,(0,m),(0,-m),\\ & (a,m+1),(0,-(m+1)),(a,m+2),(0,-(m+2)),\ldots,(a,2m),(a+1,-2m),(a+2,0), \end{aligned}$$

and

$$\begin{aligned} P' \ : \ & (0,0),(0,-1),(0,1),(0,-2),(0,2),\ldots,(0,-m),(0,m),\\ & (a,-(m+1)),(0,m+1),(a,-(m+2)),(0,m+2),\ldots,(a,-2m),(0,2m),(e_1,0). \end{aligned}$$

For both P and P', note that the vertices, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$. Therefore, P and P' are paths. Next, the arc lengths of P, in the order they are encountered, are $(0,1)$, $(0,-2),\ldots,(0,-2m)$, $(a,-2m)$, $(a,2m-1),\ldots,(a,-2)$, $(1,1)$, $(1,2m)$, while the arc lengths of P', in the order they are encountered, are $(0,-1)$, $(0,2),\ldots,(0,2m)$, $(a,2m)$, $(a,-(2m-1)),\ldots,(a,-1)$, $(e_1,-2m)$.

Let

$$C = P \cup \hat{\rho}^b(P) \cup \hat{\rho}^{2b}(P) \cup \cdots \hat{\rho}^{(2a-1)b}(P),$$

and let

$$C' = P' \cup \hat{\rho}^b(P') \cup \hat{\rho}^{2b}(P') \cup \cdots \hat{\rho}^{(2a-1)b}(P').$$

Since the last vertex of P is $(a+2,0)$, the last vertex of P' is $(e_1,0)$, and $\gcd(a+2,2a) = \gcd(e_1,2a) = 1$ (since a is odd), we have that C and C' are directed n-cycles in orbits of length b. Furthermore,

$$\ell(C) = \{(0,1),(0,-2),\ldots,(0,-2m),(a,-2m),(a,2m-1),\ldots,(a,-2),(e_1,1),(e_1,2m)\}$$

(since $e_1 = 1$), and

$$\ell(C') = \{(0,-1),(0,2),\ldots,(0,2m),(a,2m),(a,-(m-1)),\ldots,(a,-1),(e_1,-2m)\}.$$

Now, define the walks P_i as follows for $i = 1,2,\ldots,(t-1)/2$ (recall that t is odd):

$$\begin{aligned} P_i \ : \ & (0,0),(d_i,1),(2d_i,-1),(3d_i,2),(4d_i,-2),\ldots,(2md_i,-m),\\ & ((2m-1)d_i,m+1),((2m-2)d_i,-(m+1)),\ldots,(d_i,2m),(0,-2m),(e_{i+1},0), \end{aligned}$$

and

$$\begin{aligned} P_i' \ : \ & (0,0),(d_i,-1),(2d_i,1),(3d_i,-2),(4d_i,2),\ldots,(2md_i,m),\\ & ((2m-1)d_i,-(m+1)),((2m-2)d_i,m+1),\ldots,(d_i,-2m),(0,2m),(e_{i+1},0). \end{aligned}$$

For $i = 1,2,\ldots,(t-1)/2$, the vertices of P_i, except for the first and the last, are distinct modulo b, while the first and the last vertices are distinct modulo $2a$; the same is true for P_i'. Therefore, P_i and P_i' are paths. Next, the arc lengths of P_i, in the order they are encountered, are $(d_i,1),(d_i,-2),\ldots,(d_i,-2m),(-d_i,-2m),\ldots,(-d_i,1),(e_{i+1},2m)$. Similarly, the arc lengths of P_i', in the order they are encountered, are $(d_i,-1)$, $(d_i,2),\ldots,(d_i,2m)$, $(-d_i,2m),\ldots,(-d_i,-1)$, $(e_{i+1},-2m)$. Let

$$C_i = P_i \cup \hat{\rho}^b(P_i) \cup \hat{\rho}^{2b}(P_i) \cup \cdots \hat{\rho}^{(2a-1)b}(P_i)$$

and

$$C_i' = P_i' \cup \hat{\rho}^b(P_i') \cup \hat{\rho}^{2b}(P_i') \cup \cdots \hat{\rho}^{(2a-1)b}(P_i').$$

Since the last vertex $(e_{i+1}, 0)$ of both P_i and P'_i has the property that $\gcd(e_{i+1}, 2a) = 1$, we have that C_i and C'_i are directed n-cycles in orbits of length b. Furthermore,

$$\ell(C_i) = \{(d_i, 1), (d_i, -2), \ldots, (d_i, -2m), (-d_i, -2m), \ldots, (-d_i, 1), (e_{i+1}, 2m)\},$$

and

$$\ell(C'_i) = \{(d_i, -1), (d_i, 2), \ldots, (d_i, 2m), (-d_i, 2m), \ldots, (-d_i, -1), (e_{i+1}, -2m)\}.$$

Since $d_1 < d_2 < \ldots < d_t$ and $\gcd(d_i, 2a) = \gcd(2a - d_i, 2a)$, we have $\{d_{(t+3)/2}, \ldots, d_t\} = \{-d_1, \ldots, -d_{(t-1)/2}\}$. Thus all arc lengths of the form (d_i, j), for $1 \le i \le t$ and $1 \le j \le b-1$ are used.

Define the set $A = \ell(C) \cup \ell(C') \cup \ell(C_1) \cup \ell(C'_1) \cup \cdots \cup \ell(C_{(t-1)/2}) \cup \ell(C'_{(t-1)/2})$. Now, A contains $t+1$ elements from the set $\{(e_i, 2m), (e_i, 2m+1) \mid 1 \le i \le 2a-1-t\}$ (recall $-2m \equiv 2m+1 \pmod{b}$). Also,

$$|\{(e_i, 2m), (e_i, 2m+1) \mid 1 \le i \le 2a-1-t\}| = 2(2a-1-t).$$

Since we have seen previously that $2(2a-1-t) \ge t+1$, it follows that $|A| = (t+1)b$.

Let $c_1, c_2, \ldots, c_x$ denote the integers with $1 \le c_j < b$ and $\gcd(c_j, b) > 1$ for $1 \le j \le x$. Fix j with $1 \le j \le x$ and for $i = 1, 2, \ldots, 2a-1-t$, consider the walk $P_{i,j} : (0,0), (e_i, c_j), (2e_i, b-1)$. Clearly, $P_{i,j}$ is a path and the arc lengths of $P_{i,j}$, in the order they are encountered, are $(e_i, c_j), (e_i, b-1-c_j)$. (It is possible that $b-1-c_j = c'_j$ for some $j \ne j'$ with $1 \le j, j' \le x$; if so, we include only one of $P_{i,j}$ and $P_{i,j'}$ among our paths.) Since $\gcd((b-1)/2, b) = 1$, it follows that $c_j \ne b-1-c_j$ for each j with $1 \le j \le x$. Hence $|\ell(P_{i,j})| = 2$. Let

$$C_{i,j} = P_{i,j} \cup \hat{\rho}^2(P_{i,j}) \cup \hat{\rho}^4(P_{i,j}) \cup \hat{\rho}^6(P_{i,j}) \cup \cdots \hat{\rho}^{2ab-2}(P_{i,j}).$$

Since $\gcd(e_i, a) = 1$, it follows that $C_{i,j}$ is a directed n-cycle in an orbit of length 2 and

$$\ell(C_{i,j}) = \{(e_i, c_j), (e_i, b-1-c_j)\}.$$

Define the set

$$B = \bigcup_{\substack{1 \le i \le 2a-1-t \\ 1 \le j \le x}} \ell(C_{i,j}),$$

so that

$$B = \{(e_i, c_j)(e_i, b-1-c_j) \mid 1 \le i \le 2a-1-t, 1 \le j \le x\}.$$

We want $A \cap B = \emptyset$. Suppose, to the contrary, $A \cap B \ne \emptyset$. For $1 \le j \le x$, since $\gcd(c_j, b) > 1$, we have $c_j \ne 1$, $c_j \ne b-2$, $c_j \ne 2m$, and $c_j \ne 2m+1$. Thus, if $A \cap B \ne \emptyset$, it must be the case that $b-1-c_k = 2m+1$ for some k with $1 \le k \le x$. Hence $c_k = 2m-1 = (b-3)/2$. But since $\gcd(3, b) = 1$, we have $\gcd(c_k, b) = 1$, producing a contradiction. Thus $A \cap B = \emptyset$.

Finally, consider the path $P'' : (0,0), (1,0), (-1,0), (2,0), (-2,0), \ldots, (a,0), (0,1)$ and let

$$C'' = P'' \cup \hat{\rho}^{2a}(P'') \cup \hat{\rho}^{4a}(P'') \cup \cdots \cup \hat{\rho}^{2a(b-1)}(P').$$

Since $\gcd(1, b) = 1$, we have that C'' is a directed n-cycle in an orbit of length $2a$ and

$$\ell(C'') = \{(1,0), (-2,0), \ldots, (2a-1,0), (a,1)\}.$$

Notice that $(a, 1)$ was not used as an arc length in P (or in any of the other previously-defined paths), nor was any arc length of the form $(k, 0)$, so $(A \cup B) \cap \ell(C'') = \emptyset$.

Let $T = S' \setminus (A \cup B \cup \ell(C'))$ and let $(e,f) \in T$. Then, it must be that $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$. Thus,

$$\begin{aligned} X &= \{\phi^{-1}(C), \phi^{-1}(C'), \phi^{-1}(C'')\} \cup \{\phi^{-1}(C_i), \phi^{-1}(C_i') \mid 1 \le i \le (t-1)/2\} \\ &\cup \{\phi^{-1}(C_{i,j}) \mid 1 \le i \le 2a-1-t, 1 \le j \le x\} \\ &\cup \{\overrightarrow{X}(n; \{\phi^{-1}((e,f))\}) \mid (e,f) \in T\} \end{aligned}$$

is a minimum starter set for for a directed cyclic hamiltonian cycle system of K_n^*.

Case 2. Let $b \equiv 3 \pmod 4$.

Most of this case is very similar to Case 1, with minor adjustments to notation, since we now have $b = 4m+3$ for some integer m. There are a few more significant differences, however. Specifically, one arc in each of P and P' must be changed in order to create a directed hamiltonian cycle. In order to accommodate this change, the directed cycles C_1 and C_1' are replaced by a number of directed cycles in orbits of length 2.

We begin with the paths P and P', defined as follows:

$$\begin{aligned} P \;:\; & (0,0),(0,1),(0,-1),(0,2),(0,-2),\ldots,(0,m+1), \\ & (a,-(m+1)),(0,m+2),(a,-(m+2)),\ldots,(0,2m+1),(-1,-(2m+1)),(1,0), \end{aligned}$$

and

$$\begin{aligned} P' \;:\; & (0,0),(0,-1),(0,1),(0,-2),(0,2),\ldots,(0,-(m+1)), \\ & (a,(m+1)),(0,-(m+2)),(a,(m+2)),\ldots,(0,-(2m+1)),(a,2m+1),(a+2,0). \end{aligned}$$

As before, it is straightforward to verify that P and P' are paths, and that since $\gcd(a+2, 2a) = 1$, we can create cycles C and C' from these paths, as in Case 1. We have

$$\ell(C) = \{(0,1),(0,-2),\ldots,(0,2m+1),(a,2m+1),(a,-2m),\ldots,(a,2),(-1,1),(2,2m+1)\},$$

and

$$\ell(C') = \{(0,-1),(0,2),\ldots,(0,-(2m+1)),(a,-(2m+1)),(a,2m),\ldots,(a,-1),(2,-(2m+1))\}.$$

In the next step, recall that $d_1 = 2$, and the arc lengths $(2, 2m+1)$ and $(2, -(2m+1))$ have already been used in C and C', above. We therefore omit the cycles that are analogous to C_1 and C_1' from Case 1, but our new cycles $C_1, \ldots, C_{(t-3)/2)}$, $C_1', \ldots, C_{(t-3)/2}'$ are analogous to the cycles $C_2, \ldots, C_{(t-1)/2}$, $C_2', \ldots, C_{(t-1)/2}'$ from Case 1. Define the walks P_i as follows, for $i = 1, 2, \ldots, (t-3)/2$:

$$\begin{aligned} P_i \;:\; & (0,0),(d_{i+1},1),(2d_{i+1},-1),(3d_{i+1},2),(4d_{i+1},-2),\ldots,((2m+1)d_{i+1},m+1), \\ & (2md_{i+1},-(m+1)),((2m-1)d_{i+1},m+2),\ldots,(d_{i+1},2m+1),(0,-(2m+1)),(e_{i+1},0), \end{aligned}$$

and

$$\begin{aligned} P_i' \;:\; & (0,0),(d_{i+1},-1),(2d_{i+1},1),(3d_{i+1},-2),(4d_{i+1},2),\ldots,((2m+1)d_{i+1},-(m+1)), \\ & (2md_{i+1},m+1),((2m-1)d_{i+1},-(m+2)),\ldots,(d_{i+1},-(2m+1)),(0,2m+1),(e_{i+1},0). \end{aligned}$$

As before, it is straightforward to verify that these are paths, and that appropriate rotations will transform them into cycles $C_1, \ldots, C_{(t-3)/2}$ and $C'_1, \ldots, C'_{(t-3)/2}$ with

$$\begin{aligned} \ell(C_i) &= \{(d_{i+1}, 1), (d_{i+1}, -2), \ldots, (d_{i+1}, 2m+1), \\ &\quad (-d_{i+1}, 2m+1), \ldots, (-d_{i+1}, 1), (e_{i+1}, 2m+1)\}, \end{aligned}$$

and

$$\begin{aligned} \ell(C'_i) &= \{(d_{i+1}, -1), (d_{i+1}, 2), \ldots, (d_{i+1}, -(2m+1)), \\ &\quad (-d_{i+1}, -(2m+1)), \ldots, (-d_{i+1}, -1), (e_{i+1}, -(2m+1))\}. \end{aligned}$$

As in Case 1, we have now used all arc lengths of the form (d_i, j), for $2 \le i \le t-1$ and $1 \le j \le b-1$.

Define the set $A = \ell(C) \cup \ell(C') \cup \ell(C_1) \cup \ell(C'_1) \cup \cdots \cup \ell(C_{(t-3)/2}) \cup \ell(C'_{(t-3)/2})$. This time, A contains $t-1$ elements from the set $\{(e_i, 2m+1), (e_i, 2m+2) \mid 1 \le i \le 2a-1-t\}$ (recall $-(2m+1) \equiv 2m+2 \pmod{b}$). As before,

$$|\{(e_i, 2m+1), (e_i, 2m+2) \mid 1 \le i \le 2a-1-t\}| = 2(2a-1-t).$$

Since we have seen previously that $2(2a-1-t) \ge t+1$, it follows that $|A| = (t-1)b$.

Define $c_1, c_2, \ldots, c_x$, the walks $P_{i,j}$, and the cycles $C_{i,j}$ exactly as in Case 1 so that $\ell(C_{i,j}) = \{(e_i, c_j), (e_i, b-1-c_j)\}$ for $1 \le i \le 2a-1-t$ and $1 \le j \le x$.

We now deal with the arc lengths $(2, j)$ and $(-2, j)$ for $1 \le j \le x$, as follows. For $1 \le j \le m$, define the paths $P'_{1,j}$ by

$$P'_{1,j} : (0,0), (2, j), (4, 2m+1).$$

Clearly, $P'_{1,j}$ is a path and the arc lengths of $P'_{1,j}$, in the order they are encountered, are $(2, j), (2, 2m+1-j)$. Similarly, for $m+1 \le j \le 2m$, define the paths $P'_{1,j}$ by

$$P'_{1,j} : (0,0), (2, m+2+j), (4, -(2m+1)).$$

Again, each $P'_{1,j}$ is a path, and the arc lengths of $P'_{1,j}$, in the order they are encountered, are $(2, m+2+j), (2, m-j)$. Let

$$C'_{1,j} = P'_{1,j} \cup \hat{\rho}^2(P'_{1,j}) \cup \hat{\rho}^4(P'_{1,j}) \cup \hat{\rho}^6(P'_{1,j}) \cup \cdots \hat{\rho}^{2ab-2}(P'_{1,j}).$$

Since $\gcd(4, 2a) = 2$, it follows that each $C'_{1,j}$ is a directed n-cycle in an orbit of length 2 and

$$\ell(C'_{1,j}) = \{(2, j), (2, 2m+1-j)\} \text{ for } 1 \le j \le m, \text{ and}$$

$$\ell(C'_{1,j}) = \{(2, m+2+j), (2, m-j)\} \text{ for } m+1 \le j \le 2m.$$

This uses all of the remaining arc lengths whose first coordinate is 2, except $(2, 0)$.

Let

$$P'_{2,1} : (0,0), (-2, 1), (-4, -2m).$$

For $2 \le j \le 2m+1$, let

$$P'_{2,j} : (0,0), (-2, j), (-4, 1).$$

These are clearly paths, and for $1 \le j \le 2m+1$, we form $C'_{2,j}$ in the usual manner from $P'_{2,j}$. As $b - 2m = 2m+3 = (b+3)/2$ and $\gcd(3, b) = 1$, $C'_{2,1}$ is a cycle; it is easy to see that $C'_{2,j}$ is a cycle, since $\gcd(2a-4, 2a) = 2$ for $2 \le j \le 2m+1$. Note that

$\ell(C'_{2,1}) = \{(-2,1),(-2,-(2m+1))\}$ and $\ell(C'_{2,j}) = \{(-2,j),(-2,1-j)\}$ for $2 \le j \le 2m+1$. This uses all of the arc lengths whose first coordinate is -2, except $(-2,0)$.

Define the set

$$B = \left(\bigcup_{\substack{1 \le i \le 2a-1-t \\ 1 \le j \le x}} \ell(C_{i,j}) \right) \cup \left(\bigcup_{1 \le j \le 2m} \ell(C'_{1,j}) \right) \cup \left(\bigcup_{1 \le j \le 2m+1} \ell(C'_{2,j}) \right).$$

We want $A \cap B = \emptyset$. Now, if $A \cap B \ne \emptyset$, then since $\gcd(c_j, b) > 1$ for every each j with $1 \le j \le x$, we have $c_j \ne 2m+1$ and $c_j \ne 2m+2$. Thus it must be the case that $b-1-c_k = 2m+2$ for some k with $1 \le k \le x$ (the arc length $(-1,1)$ cannot appear in B because of the bounds on c_k). However, in this case $c_k = 2m = (b-3)/2$. But, as previously observed, $\gcd(3,b) = 1$, so we have $\gcd(c_k, b) = 1$, producing a contradiction. Thus $A \cap B = \emptyset$.

Finally, define P'' and C'' precisely as in Case 1. Again, C'' is a directed n-cycle in an orbit of length $2a$ and

$$\ell(C'') = \{(1,0),(-2,0),\dots,(2a-1,0),(a,1)\}.$$

Notice that the arc length $(a,1)$ had not been previously used in any path nor was any arc length of the form $(k,0)$ so $(A \cup B) \cap \ell(C'') = \emptyset$.

Let $T = S' \setminus (A \cup B \cup \ell(C'))$ and let $(e,f) \in T$. Then once again, it must be that $\gcd(e, 2a) = 1$ and $\gcd(f, b) = 1$. Thus,

$$\begin{aligned} X \;=\; & \{\phi^{-1}(C), \phi^{-1}(C'), \phi^{-1}(C'')\} \cup \{\phi^{-1}(C_i), \phi^{-1}(C'_i) \mid 1 \le i \le (t-3)/2\} \\ & \cup \{\phi^{-1}(C_{i,j}) \mid 1 \le i \le 2a-1-t, 1 \le j \le x\} \\ & \cup \{\phi^{-1}(C'_{1,j}) \mid 1 \le j \le 2m\} \cup \{\phi^{-1}(C'_{2,j}) \mid 1 \le j \le 2m+1\} \\ & \cup \{\overrightarrow{X}(n; \{\phi^{-1}((e,f))\}) \mid (e,f) \in T\} \end{aligned}$$

is a minimum starter set for for a directed cyclic hamiltonian cycle system of K_n^*. □

Theorem 1.1 (b) now follows from Lemmas 4.1 and 4.2.

5. The case when $n \equiv 0 \pmod 4$

In this section, we prove Theorem 1.1 (c). Recall that for n even, if a directed cyclic hamiltonian cycle system of K_n^* exists, then the set of arc lengths $\{1, 2, \dots, n-1\}$ must contain an even number of even integers. Thus it must be that $n \equiv 2 \pmod 4$. In the case that $n \equiv 0 \pmod 4$, we can obtain an even number of even arc lengths by removing the arcs of length $n/2$ so that in this case, we find a directed cyclic hamiltonian cycle decomposition of $(K_n - I)^*$. The following lemma completely characterises when $(K_n - I)^*$ admits a cyclic hamiltonian cycle system.

Lemma 5.1. *For $n \ge 4$, the digraph $(K_n - I)^*$ has a directed cyclic hamiltonian cycle system if and only if $n \equiv 0 \pmod 4$.*

Proof. As in the first paragraph of the proof of Lemma 4.1, the requirement that there be an even number of even arc lengths in $\overrightarrow{X}(n; \{1, 2, \ldots, n-1\} \setminus \{n/2\})$ forces $n \equiv 0 \pmod 4$.

Suppose first that $n \equiv 4 \pmod 8$. By [13], there exists a cyclic hamiltonian cycle system $\mathcal{C}$ of $K_n - I$. Then $\mathcal{C}' = \{\overrightarrow{C}, \overleftarrow{C} \mid C \in \mathcal{C}\}$ is a directed cyclic hamiltonian cycle system for $(K_n - I)^*$, where $\overrightarrow{C}$ and $\overleftarrow{C}$ are the directed cycles obtained from orienting C in each direction.

Now suppose that $n \equiv 0 \pmod 8$, say $n = 2^\alpha a$ for some integer $\alpha \geq 3$ and odd integer a. We begin with the case that $a = 1$; that is, $n = 2^\alpha$. Then $(K_n - I)^* = \overrightarrow{X}(n; \pm\{1, 2, \ldots, (2^{\alpha-1} - 1)\})$. Let $m = 2^{\alpha-2}$. Define the path P by

$$P : 0, 1, -1, 3, -3, \ldots, -(m-1), m, -(m+2), m+2, \ldots, -(2m-2), 2m-2, 2m.$$

Note that the vertices of P, except for the first and the last, are distinct modulo $2m$. Therefore, P is a path. The arc lengths of P, in the order they are encountered, are $1, -2, 4, -6, \ldots, -(2m-2), 2m-1, 2m-2, -(2m-4), \ldots, -4, 2$. These are all distinct modulo n, and include all of the permissible even arc lengths.

Let $C = P \cup \rho^{2m}(P)$. It is straightforward to verify that C is a cycle in an orbit of length $2m$. Furthermore,

$$\ell(C) = \pm\{2, 4, \ldots, (2m-2)\} \cup \{1, 2m-1\}.$$

Let $T = \pm\{1, 2, \ldots, (2m-1)\} \setminus \ell(C)$. For any $t \in T$, we have that $\gcd(t, n) = 1$ since t is odd. Thus $\overrightarrow{X}(n; \{t\})$ is a directed hamiltonian cycle in $(K_n - I)^*$. Hence,

$$X \quad = \quad \{C\} \cup \{\overrightarrow{X}(n; \{t\}) \mid t \in T\}$$

is a minimum starter set for for a directed cyclic hamiltonian cycle system of $(K_n - I)^*$.

We now assume that $a > 1$. Since a is odd, we have $a = 2q + 1$ for some integer q. Now, $\mathbb{Z}_n \cong \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2q+1}$ and thus we will use ϕ to relabel the vertices of $(K_n - I)^* = \overrightarrow{X}(n; \pm\{1, 2, \ldots, (n-2)/2\})$ with the elements of $\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2q+1}$. The set

$$S' = \{(0, j), (2^{\alpha-1}, j) \mid 1 \leq j \leq 2q\} \cup \{(i, j) \mid 1 \leq i \leq 2^\alpha - 1 \text{ with } i \neq 2^{\alpha-1}, 0 \leq j \leq 2q\}$$

has the property that $\phi^{-1}(S') = \pm\{1, 2, \ldots, (n-2)/2\}$. Thus we can think of the elements of S' as the arc lengths of the relabelled graph. If q is even, say $q = 2m$ for some positive integer m, define the walks P and P', by

$$\begin{aligned} P \quad : \quad & (0,0), (0,1), (0,-1), (0,2), (0,-2), \ldots, (0,m), (0,-m), \\ & (2^{\alpha-1}, m+1), (0, -(m+1)), (2^{\alpha-1}, m+2), (0, -(m+2)), \ldots, (2^{\alpha-1}, q), (0, -q), (-1, 0), \end{aligned}$$

and

$$\begin{aligned} P' \quad : \quad & (0,0), (0,-1), (0,1), (0,-2), (0,2), \ldots, (0,-m), (0,m), \\ & (2^{\alpha-1}, -(m+1)), (0, m+1), (2^{\alpha-1}, -(m+2)), (0, m+2), \ldots, (2^{\alpha-1}, -q), (0, q), (-1, 0). \end{aligned}$$

Note that the vertices of P and P', except for the first and the last, are distinct modulo $2q + 1$ while the first and the last vertices are distinct modulo 2^α. Therefore, P and P' are paths. Next, the arc lengths of P, in the order they are encountered, are $(0, 1), (0, -2), (0, 3), (0, -4), \ldots, (0, -q)$, $(2^{\alpha-1}, -q), (2^{\alpha-1}, q-1), \ldots, (2^{\alpha-1}, 1), (-1, q)$ while the arc lengths of P', in the order they are encountered are $(0,-1),(0,2),(0,-3),(0,4),\ldots,(0,q)$, $(2^{\alpha-1},q),(2^{\alpha-1},-(q-1)),\ldots,(2^{\alpha-1},-1),(-1,-q)$.

If q is odd, say $q=2m+1$ for some positive integer m, define the walks P and P' by

$$\begin{aligned} P \;:\; & (0,0),(0,1),(0,-1),(0,2),(0,-2),\ldots,(0,m),(0,-m),(0,m+1),\\ & (2^{\alpha-1},-(m+1)),(0,m+2),(2^{\alpha-1},-(m+2)),\ldots,(0,q),(2^{\alpha-1},-q),(2^{\alpha-1}-1,0),\end{aligned}$$

and

$$\begin{aligned} P' \;:\; & (0,0),(0,-1),(0,1),(0,-2),(0,2),\ldots,(0,-m),(0,m),(0,-(m+1)),\\ & (2^{\alpha-1},m+1),(0,-(m+2)),(2^{\alpha-1},m+2),\ldots,(0,-q),(2^{\alpha-1},q),(2^{\alpha-1}-1,0).\end{aligned}$$

Note that the vertices of P and P', except for the first and the last, are distinct modulo $2q+1$ while the first and the last vertices are distinct modulo 2^α. Therefore, P and P' are paths. Next, the arc lengths of P, in the order they are encountered, are $(0,1),(0,-2),(0,3),(0,-4),\ldots,(0,q)$, $(2^{\alpha-1},q),(2^{\alpha-1},-(q-1)),\ldots,(2^{\alpha-1},1),(-1,q)$ while the arc lengths of P', in the order they are encountered are $(0,-1),(0,2),(0,-3),(0,4),\ldots,(0,-q)$, $(2^{\alpha-1},-q),(2^{\alpha-1},q-1),\ldots,(2^{\alpha-1},-1),(-1,-q)$.

Let

$$C = P\cup\hat\rho^{2q+1}(P)\cup\hat\rho^{4q+2}(P)\cup\cdots\cup\hat\rho^{(2^\alpha-1)(2q+1)}(P)$$

and

$$C' = P'\cup\hat\rho^{2q+1}(P')\cup\hat\rho^{4q+2}(P')\cup\cdots\cup\hat\rho^{(2^\alpha-1)(2q+1)}(P').$$

Then, clearly C and C' are directed n-cycles in orbits of length $2q+1$ with

$$\ell(C)\cup\ell(C')=\{(0,j),(0,-j),(2^{\alpha-1},j),(2^{\alpha-1},-j)\mid 1\le j\le q\}\cup\{(-1,q),(-1,-q)\}.$$

For each i with $1\le i\le 2^{\alpha-3}$, define the walks P_i and P_i' by

$$P_i \;:\; (0,0),(2i,1),(0,-1),(2i,2),(0,-2),\ldots,(2i,q),(0,-q),(2i-1,0),$$

and

$$P_i' \;:\; (0,0),(2i,-1),(0,1),(2i,-2),(0,2),\ldots,(2i,-q),(0,q),(2i-1,0).$$

For each i with $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$, define the walks P_i and P_i' exactly as above, but with the final vertex changed to $(2i+1,0)$.

Again, note that the vertices of P_i and P_i', except for the first and the last, are distinct modulo $2q+1$ while the first and the last vertices are distinct modulo 2^α. Therefore, P_i and P_i' are paths. For each i with $1\le i\le 2^{\alpha-3}$ and q even, the arc lengths of P_i, in the order they are encountered, are $(2i,1),(-2i,-2),(2i,3),(-2i,-4),\ldots,(-2i,-q)$, $(2i,-q),(-2i,q-1),\ldots,(-2i,1)$, $(2i+1,q)$ with the final length replaced by $(2i-1,q)$ if $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$. For each i with $1\le i\le 2^{\alpha-3}$ and q odd, the arc lengths of P_i, in the order they are encountered, are $(2i,1),(-2i,-2),(2i,3),(-2i,-4),\ldots,(2i,q)$, $(-2i,q),(2i,-(q-1)),\ldots,(-2i,1),(2i+1,q)$, with the final length replaced by $(2i-1,q)$ if $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$. For each i with $1\le i\le 2^{\alpha-3}$ and q even, the arc lengths of P_i', in the order they are encountered, are $(2i,-1),(-2i,2),(2i,-3)$, $(-2i,4),\ldots,(-2i,q)$, $(2i,q),(-2i,-(q-1)),\ldots,(-2i,-1),(2i+1,-q)$, with the final length replaced by $(2i-1,-q)$ if $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$. For each i with $1\le i\le 2^{\alpha-3}$ and q odd, the arc lengths of P_i', in the order they are encountered, are $(2i,-1),(-2i,2),(2i,-3),(-2i,4),\ldots,(2i,-q)$,

$(-2i,-q),(2i,q-1),\dots,(-2i,-1),(2i+1,-q)$ if q is odd and $1\le i\le 2^{\alpha-3}$, with the final length replaced by $(2i-1,-q)$ if $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$.

For each i with $1\le i\le 2^{\alpha-2}-1$, let

$$C_i=P_i\cup\hat\rho^{2q+1}(P_i)\cup\hat\rho^{4q+2}(P_i)\cup\cdots\cup\hat\rho^{(2^\alpha-1)(2q+1)}(P_i)$$

and

$$C_i'=P_i'\cup\hat\rho^{2q+1}(P_i')\cup\hat\rho^{4q+2}(P_i')\cup\cdots\cup\hat\rho^{(2^\alpha-1)(2q+1)}(P_i').$$

Then, clearly C_i and C_i' are directed n-cycles in orbits of length $2q+1$ with

$$\ell(C_i)\cup\ell(C_i')=\{(2i,j),(2i,-j),(-2i,j),(-2i,-j)\mid 1\le j\le q\}\cup\{(2i+1,q),(2i+1,-q)\},$$

for $1\le i\le 2^{\alpha-3}$, and

$$\ell(C_i)\cup\ell(C_i')=\{(2i,j),(2i,-j),(-2i,j),(-2i,-j)\mid 1\le j\le q\}\cup\{(2i-1,q),(2i-1,-q)\},$$

for $2^{\alpha-3}+1\le i\le 2^{\alpha-2}-1$. Let $A=\ell(C)\cup\ell(C')\cup\ell(C_1)\cup\ell(C_1')\cup\ell(C_2)\cup\ell(C_2')\cup\cdots\cup\ell(C_{2^{\alpha-2}-1})\cup\ell(C'_{2^{\alpha-2}-1})$ so that

$$\begin{aligned}A\ =\ &\{(2i,j)\mid 0\le i\le 2^{\alpha-1}-1,1\le j\le 2q\}\cup\{(-1,q),(-1,q+1)\}\\&\cup\{(2i-1,q),(2i-1,q+1)\mid 1\le i\le 2^{\alpha-2}\}\setminus\{(2^{\alpha-2}+1,q),(2^{\alpha-2}+1,q+1)\}.\end{aligned}$$

Now, let $d_1,\dots,d_t$ denote the integers with $0<d_j<2q$ and $\gcd(d_j,2q+1)>1$. Note that if $t>1$, then $q\ge 4$. If $d_{j'}=2q-d_j$ for some j,j' with $1\le j,j'\le t$, then omit $d_{j'}$ from $\{d_1,d_2,\dots,d_t\}$. For each i with $0\le i\le 2^{\alpha-1}-1$ and for each j with $1\le j\le t$, consider the walk $P_{i,j}:(0,0),(2i+1,d_j),(4i+2,2q)$. Clearly, $P_{i,j}$ is a path and the arc lengths of $P_{i,j}$, in the order they are encountered, are $(2i+1,d_j),(2i+1,2q-d_j)$. Let

$$C_{i,j}=P_{i,j}\cup\hat\rho^{2}(P_{i,j})\cup\hat\rho^{4}(P_{i,j})\cup\hat\rho^{6}(P_{i,j})\cup\cdots\hat\rho^{2^{\alpha-1}(2q+1)}(P_{i,j}).$$

Then $C_{i,j}$ is a directed n-cycle in an orbit of length 2 and

$$\ell(C_{i,j})=\{(2i+1,d_j),(2i+1,2q-d_j)\}.$$

Note also that if $j\ne k$ with $1\le j,k\le t$, then $\ell(C_{i,j})\cap\ell(C_{i,k})=\emptyset$.

Define the set

$$B=\bigcup_{\substack{0\le i\le 2^{\alpha-1}-1\\1\le j\le t}}\ell(C_{i,j}).$$

We want $A\cap B=\emptyset$. If so, we are done, and note that

$$B=\{(2i+1,d_j),(2i+1,2q-d_j)\mid 0\le i\le 2^{\alpha-1}-1,1\le j\le t\}.$$

Otherwise, if $A\cap B\ne\emptyset$, then since $\gcd(d_j,2q+1)>1$ for each j with $1\le j\le t$, it follows that $d_j\ne q$ and $d_j\ne q+1$. Thus, if $A\cap B\ne\emptyset$, then it must be that $2q-d_k=q+1$ for some k with $1\le k\le t$. This implies that $d_k=q-1$. Therefore, in this case, for $i=0,1,\dots,2^{\alpha-2}-1$, redefine $P_{i,k}:(0,0),(2i+1,d_k),(4i+2,q)$ and for $i=2^{\alpha-2},\dots,2^{\alpha-1}-1$, redefine $P_{i,k}=(0,0),(2i+1,d_k),(4i+2,2q)$. Create the cycle $C_{i,k}$ as before. Observe that $\ell(C_{i,k})=\{(2i+1,d_k),(2i+1,1)\}$ for $0\le i\le 2^{\alpha-2}-1$ and $\ell(C_{i,k})=\{(2i+1,d_k),(2i+1,q+1)\}$

for $2^{\alpha-2} \leq i \leq 2^{\alpha-1}-1$. Also $C_{i,k}$ will be a directed n-cycle in an orbit of length 2. In this case, we have

$$\begin{aligned} B &= \{(2i+1, d_j), (2i+1, 2q-d_j) \mid 0 \leq i \leq 2^{\alpha-1}-1, 1 \leq j \neq k \leq t\} \\ &\quad \cup \{(2i+1, q-1), (2i+1, 1) \mid 0 \leq i \leq 2^{\alpha-2}-1\} \\ &\quad \cup \{(2i+1, q-1), (2i+1, q+1) \mid 2^{\alpha-2} \leq i \leq 2^{\alpha-1}-1\} \end{aligned}$$

Therefore, $A \cap B = \emptyset$.

Finally, consider the path

$$\begin{aligned} P'' \;:\; & (0,0), (1,0), (-1,0), (2,0), (-2,0), (3,0), (-3,0), \ldots, (-2^{\alpha-2}+1, 0), (2^{\alpha-2}, 0), \\ & (-2^{\alpha-1}+1, 2q-1), (2^{\alpha-1}, 2q-1), (-2^{\alpha-1}+2, 2q-1), (2^{\alpha-1}-1, 2q-1), (-2^{\alpha-1}+3, 2q-1), \\ & (2^{\alpha-1}-2, 2q-1), \ldots, (2^{\alpha-2}+1, 2q-1), (0, 2q) \end{aligned}$$

and let

$$C'' = P \cup \hat{\rho}^{2^\alpha}(P) \cup \hat{\rho}^{2\cdot 2^\alpha}(P) \cup \hat{\rho}^{3\cdot 2^\alpha}(P) \cup \cdots \hat{\rho}^{2q\cdot 2^\alpha}(P).$$

Since $\gcd(2q-1, 2q+1) = 1$, it follows that C'' is a directed n-cycle in an orbit of length 2^α and

$$\begin{aligned} \ell(C'') &= \{(1,0), (-2,0), (3,0), (-4,0), \ldots, (2^{\alpha-1}-1, 0), (2^{\alpha-2}+1, 2q-1), \\ &\quad (-1,0), (2,0), (-3,0), \ldots, (-2^{\alpha-1}+1, 0), (-2^{\alpha-2}-1, 1)\}. \end{aligned}$$

Now, $\ell(C'') \cap A = \emptyset$ since every $(c,d) \in A$ has $d \geq 1$, and $(2^{\alpha-2}+1, 2q-1), (-2^{\alpha-2}-1, 1) \notin A$. Similarly, $\ell(C'') \cap B = \emptyset$ since every $(c,d) \in B$ has $d \geq 1$ and $q \geq 4$ implies that $(2^{\alpha-2}+1, 2q-1), (-2^{\alpha-2}-1, 1) \notin B$. Thus, let $T = S' \setminus (A \cup B \cup \ell(C''))$ and let $(e,f) \in T$. Then it must be that e is odd and $\gcd(f, 2q+1) = 1$. Therefore,

$$\begin{aligned} X &= \{\phi^{-1}(C), \phi^{-1}(C'), \phi^{-1}(C'')\} \\ &\quad \cup \{\phi^{-1}(C_1), \phi^{-1}(C_1'), \phi^{-1}(C_2), \phi^{-1}(C_2'), \ldots, \phi^{-1}(C_{2^{\alpha-2}-1}), \phi^{-1}(C'_{2^{\alpha-2}-1})\} \\ &\quad \cup \{\phi^{-1}(C_{i,j}) \mid 1 \leq i \leq 2^{\alpha-1}-1, 1 \leq j \leq t\} \\ &\quad \cup \{\overrightarrow{X}(n; \{\phi^{-1}((e,f))\}) \mid (e,f) \in T\} \end{aligned}$$

is a minimum start set for a directed cyclic hamiltonian cycle system of $(K_n - I)^*$. □

Theorem 1.1 (c) now follows from Lemma 5.1.

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