Flows that are sums of hamiltonian cycles in Cayley graphs on abelian groups

To Brian Alspach on his sixty-fifth birthday

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Abstract

If X is any connected Cayley graph on any finite abelian group, we determine precisely which flows on X can be written as a sum of hamiltonian cycles. (This answers a question of B. Alspach.) In particular, if the degree of X is at least 5, and X has an even number of vertices, then the flows that can be so written are precisely the even flows, that is, the flows f, such that $\sum_{\alpha \in E(X)} f(\alpha)$ is divisible by 2. On the other hand, there are examples of degree 4 in which not all even flows can be written as a sum of hamiltonian cycles. Analogous results were already known, from work of B. Alspach, S. C. Locke, and D. Witte, for the case where X is cubic, or has an odd number of vertices.

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1 Introduction

If C is any cycle in a graph X, then providing C with an orientation naturally defines a flow on X. (See §2 for definitions and notation used here in the

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introduction.) Conversely, it is well known that every flow can be written as a sum of cycles. Brian Alspach (personal communication) has asked which flows can be written as a sum of *hamiltonian* cycles.

Notation 1.1. Suppose X is a graph. Then

- $\mathcal{F} = \mathcal{F}(X)$ denotes the space of all integral flows on X, that is, the $\mathbf{Z}$-valued flows on X,
- $\mathcal{E} = \mathcal{E}(X)$ denotes the additive subgroup of $\mathcal{F}$ consisting of the even flows, that is, the flows f such that the sum of the edge-flows of f is even, and
- $\mathcal{H} = \mathcal{H}(X)$ denotes the additive subgroup of $\mathcal{F}$ generated by the oriented hamiltonian cycles.

Note that $\mathcal{F} = \mathcal{E}$ if and only if X is bipartite. On the other hand, $\mathcal{H} \subseteq \mathcal{E}$ whenever X has even order.

Locke and Witte [LW] showed that if X is a connected Cayley graph on a finite abelian group of odd order, then every flow on X can be expressed as a sum of hamiltonian cycles (except for flows on one particular graph, the cartesian product $K_3 \mathbin{\square} K_3$ of two cycles of length 3).

Theorem 1.2 (Locke-Witte [LW, Thm. 4.1])**.** *If $X = \mathrm{Cay}(G;S)$ is a connected Cayley graph on a finite, abelian group G of odd order, then $\mathcal{H} = \mathcal{F}$, unless $X \cong K_3 \mathbin{\square} K_3$, in which case, $\mathcal{F}/\mathcal{H} \cong \mathbf{Z}_3$.*

They also settled the case where X is cubic.

Observation 1.3. A connected, cubic Cayley graph on a finite, abelian group is of one of two types: a Möbius ladder, or a prism over a cycle.

Theorem 1.4 (Locke-Witte [LW, Prop. 3.3])**.** *(1) If X is a Möbius ladder, then:*
- *(a) $\mathcal{E}/\mathcal{H} \cong \mathbf{Z}_{n/2}$ if X is bipartite, where n is the number of vertices of X;*
- *(b) $\mathcal{H} = \mathcal{E}$ if X is not bipartite.*

(2) If X is a prism over a cycle of length n, then:
- *(a) $\mathcal{E}/\mathcal{H} \cong \mathbf{Z}_{n-1}$ if X is bipartite;*
- *(b) $\mathcal{E}/\mathcal{H} \cong \mathbf{Z} \oplus \mathbf{Z}_{n-1}$ if X is not bipartite.*

We now complete this work, by calculating $\mathcal{H}$ for all the remaining Cayley graphs on finite abelian groups.

Theorem 1.5. *If $X = \mathrm{Cay}(G;S)$ is a connected, non-cubic Cayley graph on a finite, abelian group G of even order, then $\mathcal{H} = \mathcal{E}$, unless*

(1) X is the square of a cycle, in which case $\mathcal{E}/\mathcal{H} \cong \mathbf{Z}_{n-1}$, where $n = |G|/2$;

or

(2) X has degree 4, X is not bipartite, and $|G|$ is not divisible by 4, in which case $\mathcal{E}/\mathcal{H} \cong \mathbf{Z}_2$ (unless X is the square of a cycle, in which case, (1) *applies).*

In the exceptional cases of Theorem 1.5 (that is, in those cases where $\mathcal{H} \neq \mathcal{E}$), the results of Section 8 determine precisely which flows are in $\mathcal{H}$. The analogous results of Locke and Witte [LW] for the exceptional cases of Theorems 1.2 and 1.4 are recalled in Section 4.

In the special case of bipartite graphs, the theorem can be restated as follows.

Corollary 1.6. *Let $X = \mathrm{Cay}(G; S)$ be a connected Cayley graph on a finite, abelian group G. If X is bipartite, and X is not cubic, then every flow on X can be written as a sum of hamiltonian cycles.*

Analogous results for $\mathbf{Z}_2$-flows (in which the coefficients are taken modulo 2) were obtained by Alspach, Locke, and Witte [ALW] (see 5.1). We rely heavily on the results of [ALW,LW], and, to a large extent, we also use the same techniques. Thus, the reader may find it helpful to look at the proofs in those papers, especially because they provide drawings of many of the hamiltonian cycles that appear here.

Here is an outline of the paper. Section 2 presents notation and definitions. It also states our standing assumption, which holds everywhere except here in the introduction, that G has even order. Section 3 presents some useful observations on involutions. Section 4 briefly recalls the results of [LW] that calculate $\mathcal{H}$ in the exceptional cases of Theorems (1.2) and (1.4). Section 5 develops the main tools to be used in an inductive proof of our main theorem. Section 6 shows that $\mathcal{H}$ often contains certain basic 4-cycles. Section 7 proves that if X has degree 4, and is not one of the exceptional cases, then $\mathcal{H} = \mathcal{E}$. Section 8 treats the exceptional graphs of degree 4. Section 9 presents a somewhat lengthy proof that was omitted from Section 8. Section 10 shows, for many graphs of degree at least 5, that $\mathcal{H}$ contains all of the basic 4-cycles. Section 11 provides an induction step for the proof of the main theorem, under the assumption that the generating set S contains a redundant generator. Section 12 deals with two cases that are not covered by our other results. Section 13 proves that if the degree of X is at least 5, then $\mathcal{H} = \mathcal{E}$.

The statement of Theorem 1.5 merely combines the conclusions of (7.1), (8.1), (8.2) and (13.3) into a single assertion.

Remark. Although we discuss only integer flows, it is explained in [LW, §5] that these results are universal. They determine which A-valued flows are linear combinations of hamiltonian cycles, for any abelian group A.

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2 Preliminaries

Definition 2.1. Suppose S is a subset of a finite group G.

- S is a *symmetric generating set* for G if
  - S generates G, that is, no proper subgroup of G contains S, and
  - we have $s^{-1} \in S$, for every $s \in S$.
- The *Cayley graph* $\mathrm{Cay}(G;S)$ *of* S *on* G is the graph defined as follows:
  - the vertices of $\mathrm{Cay}(G;S)$ are the elements of G, and
  - there is an edge from g to gs, for every $g \in G$ and $s \in S$.

Notation 2.2. Throughout this paper,

- G is a finite abelian group (usually written multiplicatively);
- e is the identity element of G;
- S is a symmetric generating set for G, such that $e \notin S$; and
- $X = \mathrm{Cay}(G;S)$.

Assumption 2.3. Throughout the remainder of this paper,

$$|G| \text{ is even.}$$

Notation 2.4. When $X = \mathrm{Cay}(G;S)$, and some element s of S has been chosen, we let

- $S' = S \smallsetminus \{s, s^{-1}\}$;
- $G' = \langle S' \rangle$;

- $X' = \mathrm{Cay}(G'; S')$;
- $\mathcal{F}' = \mathcal{F}(X')$;
- $\mathcal{E}' = \mathcal{E}(X')$; and
- $\mathcal{H}' = \mathcal{H}(X')$.

Notation 2.5. We use
$$[v](t_1, t_2, \ldots, t_n),$$
where $v \in G$ and $t_i \in S$, to denote the path (or cycle) in $\mathrm{Cay}(G; S)$ that visits the vertices
$$v, vt_1, vt_1t_2, \ldots, vt_1t_2 \cdots t_n.$$
(When $v = e$, we usually write simply $(t_1, t_2, \ldots, t_n)$.)

We use a superscript to denote the concatenation of copies of the same sequence, and the symbol $\sharp$ denotes truncation of the last term of a sequence. For example,
$$\Big((s^2, t)^3\sharp, u\Big) = (s, s, t, s, s, t, s, s, u)$$
and
$$\Big((s^2, t)^0, u\Big) = (u).$$
Note that the notation s^m in such a sequence always denotes repetitions of the generator s, not a single occurrence of the group element s^m. We will always give a new name (such as $t = s^m$) if we wish to use the group element s^m in such a path or cycle.

The following illustrates another useful notation:
$$\left(\left(s^2, t_i\right)_{i=1}^{3}, u, \left(s^2, t_i\right)_{i=1}^{0}, u\right) = (s, s, t_1, s, s, t_2, s, s, t_3, u, u).$$

Remark.

- We do not usually distinguish between a cycle $C = [v](s_1, \ldots, s_m)$ and the corresponding element of $\mathcal{F}$.
- $[vs](s^{-1})$ and $-[v](s)$ each represent the same edge as $[v](s)$, but with the opposite orientation. Orientations serve two purposes: they arise in the definition of a flow (see Definition 2.6), and they may be used to indicate that a path traverses a certain edge in a certain direction.
- If $(s_1, \ldots, s_m)$ is a cycle, then $-(s_1, \ldots, s_m) = (s_m^{-1}, \ldots, s_1^{-1})$. In particular,
$$-(s, t, s^{-1}, t^{-1}) = (t, s, t^{-1}, s^{-1}).$$

Suggestion. Some hamiltonian cycles in X, such as that in Eq. (E1) on p. 13, depend on a path $(t_i)_{i=1}^{m}$ in a subgraph or quotient graph of X. For simplicity, the reader may find it helpful to assume that the path is simply (t^m), so that the subscripts can be ignored. For example, the above-mentioned hamiltonian cycle simplifies to
$$H = \Big((s^{m-1}, t, s^{-(m-1)}, t)^{r/2}, (t^{n-r-1}, s, t^{-(n-r-1)}, s)^{m/2}\Big).$$

(In order to facilitate the simplification process, we consistently begin our indices at 1, even when a different starting point would yield less complicated formulas in the subscripts.) As soon as the simpler cycle is understood, it should be clear that there is an analogous hamiltonian cycle that includes subscripts. Thus, the subscripts are essentially a formality, so the correctness of an overall proof can usually be verified without checking that the authors have calculated the subscripts correctly.

Definition 2.6.

- A *flow* on the Cayley graph $X = \mathrm{Cay}(G; S)$ is a function $f \colon G \times S \to \mathbf{Z}$, such that
  - $f(v, s) = -f(vs, s^{-1})$, for all $v \in G$ and $s \in S$, and
  - $\sum_{s \in S} f(v, s) = 0$, for all $v \in G$.

  We usually refer to $f(v, s)$ as the *edge-flow* of f on the oriented edge $[v](s)$.
- A *weighting* of X is a function $\phi \colon G \times S \to \mathbf{Z}$, such that $\phi(v, s) = -\phi(vs, s^{-1})$, for all $v \in G$ and $s \in S$. We usually refer to $\phi(v, s)$ as the *weight* of the oriented edge $[v](s)$.
- Given a flow f on X and a weighting ϕ, the *weighted sum of the edge-flows of* f is $\sum_{\alpha \in A} \phi(a) f(a)$, where A is any subset of $G \times S$, such that for each $v \in G$ and $s \in S$, the set A contains either (v, s) or (vs, s^{-1}), but not both. It is independent of the choice of the set A.

Remark. In later sections of the paper, it will sometimes be necessary to define a particular weighting of X. For convenience, whenever we specify that some oriented edge $[v](s)$ has a certain weight w, it is implicitly understood that the opposite oriented edge $[vs](s^{-1})$ has weight $-w$.

Notation 2.7. Suppose $v \in G$, Y is a subgraph of X, and $f \in \mathcal{F}$. We use $[v]Y$ to denote the *translate of Y by v*, and $[v]f$ to denote the *translate of f by v*. Namely:

- $[v]Y$ is the subgraph of X defined by:
  - the vertices of $[v]Y$ are the elements of G of the form vy with $y \in Y$, and
  - there is an edge from vy_1 to vy_2 in $[v]Y$ if and only if there is an edge from y_1 to y_2 in Y.
- the edge-flow of $[v]f$ on an oriented edge $[vw](s)$ is defined to be the same as the edge-flow of f on the oriented edge $[w](s)$, for $w \in G$ and $s \in S$.

Definition 2.8. For any graphs X and Y, the *Cartesian product $X \,\square\, Y$ of X and Y* is the graph defined as follows:

- the vertices of $X \,\square\, Y$ are the ordered pairs (x, y) with $x \in X$ and $y \in Y$, and

- there is an edge from (x_1, y_1) to (x_2, y_2) if and only if either
  - $x_1 = x_2$, and there is an edge from y_1 to y_2 in Y, or
  - $y_1 = y_2$, and there is an edge from x_1 to x_2 in Y.

We use X^p to denote the Cartesian product $X \mathbin{\square} X \mathbin{\square} \cdots \mathbin{\square} X$ of p copies of X.

Observation 2.9. If S_i is a symmetric generating set for G_i, for $i = 1, 2$, then

$$\mathrm{Cay}(G_1; S_1) \mathbin{\square} \mathrm{Cay}(G_2; S_2) \cong \mathrm{Cay}\Big(G_1 \times G_2; \big(S_1 \times \{e\}\big) \cup \big(\{e\} \times S_2\big)\Big).$$

Definition 2.10.

- A *basic* 4*-cycle* in X is any 4-cycle of the form $[v](s, t, s^{-1}, t^{-1})$ with $v \in G$ and $s, t \in S$.
- An element s of G is an *involution* if s is of order 2; that is, if $s^2 = e$ and $s \neq e$.
- An element s of S is a *redundant* generator if $\langle S' \rangle = G$.
- The generating set S is *irredundant* if none of its elements are redundant.
- For a fixed element s of S, an edge of X is an s*-edge* if it is of the form $[v](s)$ or $[v](s^{-1})$, for some $v \in G$.

Notation 2.11.

- $\mathbf{Z}_n$ denotes the additive group of integers modulo n.
- K_n denotes the complete graph on n vertices.
- C_n denotes the cycle of length n. (With the expectation that it will not cause confusion, C_1, C_2 and C_3 are used to denote certain more general cycles in the proof of Corollary 10.7.)
- $|g|$ denotes the order of the element g of G.
- $|H|$ denotes the order of the subgroup H of G (that is, the number of elements of H).
- $\langle g \rangle$ denotes the subgroup of G generated by the element g.
- $\langle A \rangle$ denotes the subgroup of G generated by the subset A of G.

Definition 2.12.

- For any graph Y, we call $K_2 \mathbin{\square} Y$ the *prism over* Y.
- A *Möbius ladder* is a graph isomorphic to $\mathrm{Cay}\big(\mathbf{Z}_{2n}; \{\pm 1, n\}\big)$, for some natural number n.
- X is *the square of an even cycle* if there exist s and t in S, such that
  - $S = \{s^{\pm 1}, t^{\pm 1}\}$,
  - $t = s^2$, and
  - $t^2 \neq e$.

  (The final condition is a convention: we do not consider the cubic graph K_4 to be the square of an even cycle.) It is not difficult to show that if X is isomorphic to the square of an even cycle, then X itself is the square of an even cycle.

Warning. If we write $S = \{s^{\pm 1}, t^{\pm 1}\}$, then it is obvious that $|S| \leq 4$. However, it need not be the case that $|S|$ is exactly 4, unless additional restrictions are explicitly imposed. For example, it could be the case that $s = s^{-1}$, or that $s = t$.

Warning. We use p and q to denote arbitrary integers; they are **not** assumed to be prime numbers.

3 Remarks on involutions in S

Observation 3.1. If S is an irredundant generating set, or, more generally, if no involution in S is a redundant generator, then we may assume that S contains no more than one involution. To see this, let

- S_1 be the set of involutions in S,
- G_1 be the subgroup generated by S_1, and
- G_2 be the subgroup generated by $S \smallsetminus S_1$.

Then $G_1 \cap G_2$ is trivial (because none of the elements of S_1 are redundant), so $G = G_1 \times G_2$. Hence

$$X \cong \mathrm{Cay}(G_1; S_1) \,\square\, \mathrm{Cay}(G_2; S \smallsetminus S_1).$$

Now, the desired conclusion follows by noting that $\mathrm{Cay}(G_1; S_1)$ is isomorphic to either $(C_4)^p$ (if $|S_1| = 2p$) or $K_2 \,\square\, (C_4)^p$ (if $|S_1| = 2p + 1$), for some natural number p.

Observation 3.2. If $|S| = 4$, and X is not the prism over a Möbius ladder, then we may assume that S does not contain any involutions. Specifically:

(1) If X is isomorphic to the cartesian product of K_2 with a prism over a cycle C_n, then (3.1) applies.
(2) If X is obtained from the prism over a cycle of length $2n$ by adding the diagonals, that is, if

$$X \cong \mathrm{Cay}\Big(\mathbf{Z}_2 \oplus \mathbf{Z}_{2n}; \{(1,0), (1,n), \pm(0,1)\}\Big),$$

then

$$X \cong \begin{cases} \mathrm{Cay}\Big(\mathbf{Z}_2 \oplus \mathbf{Z}_{2n}; \{\pm(1,n/2), \pm(0,1)\}\Big) & \text{if } n \text{ is even,} \\ \mathrm{Cay}\Big(\mathbf{Z}_4 \oplus \mathbf{Z}_n; \{\pm(1,0), \pm(2,1)\}\Big) & \text{if } n \text{ is odd.} \end{cases}$$

It is not difficult to see that these cases are exhaustive, given the list of cubic graphs in Observation 1.3.

Lemma 3.3. *If there exists $s \in S$, such that*

- *s is a redundant involution in S,*
- *$|S| \geq 3$, and*
- *S' is irredundant,*

then there is a generating set T' for an abelian group H, and an involution t in H, such that

(1) $X \cong \operatorname{Cay}\Big(H; T' \cup \{t\}\Big)$,
(2) $X' \cong \operatorname{Cay}(H; T')$,
(3) t is an involution, and
(4) not every element of T' is an involution.

Proof. We may assume that every element of S' is an involution. (Otherwise, take $H = G$, $T' = S'$, and $t = s$.) Write $s = t_1 + t_2 + \cdots + t_p$, where $t_1, \ldots, t_p$ are distinct elements of S'. Let

$$\begin{aligned} H &= \mathbf{Z}_4 \times \Big\langle S' \smallsetminus \{t_1, t_2\} \Big\rangle \subseteq \mathbf{Z}_4 \times G', \\ T' &= \{(\pm 1, e)\} \cup \bigg(\{0\} \times \Big(S' \smallsetminus \{t_1, t_2\} \Big) \bigg) \subseteq H, \\ t &= (2, t_3 + \cdots + t_p) \in H. \end{aligned}$$

Then it is not difficult to verify the desired conclusions. □

4 Graphs that are cubic or of odd order

Let us recall the observations of [LW] that describe exactly which flows are in $\mathcal{H}'$, for the cases where Theorem 1.2 or 1.4 gives an imprecise answer. For completeness, we include all of these results, even though the proofs in later sections require only (4.2), (4.3), and (4.4).

Lemma 4.1 ([LW, pf. of Prop. 3.1])**.** *Suppose*

- *$G' \cong \mathbf{Z}_3 \times \mathbf{Z}_3$, and*
- *$S' = \{s^{\pm 1}, t^{\pm 1}\}$.*

Give

- *weight 1 to the oriented s-edge $[v](s)$, for each $v \in \langle s \rangle$, and*
- *weight 0 to each of the other edges of X'.*

Then a flow is in $\mathcal{H}'$ if and only if the weighted sum of its edge-flows is divisible by 3*.*

In the situation of Lemma 4.1, it is easy to see that the weighted sum of the edge-flows of any basic 4-cycle is nonzero, so the following observation is an easy consequence (cf. proof of (8.6)).

Corollary 4.2. *If*

- $G' \cong \mathbf{Z}_3 \times \mathbf{Z}_3$,
- $|S'| = 4$, *and*
- $\mathcal{H}$ *contains some basic* 4*-cycle* C *of* X',

then $\mathcal{F}' \subseteq \mathcal{H} + \mathcal{H}'$*.*

Lemma 4.3 ([LW, pf. of Prop. 3.3(1a)])**.** *Suppose X' is a Möbius ladder, and X' is bipartite, so we may write*

- $S' = \{t^{\pm 1}, u\}$,
- $G' = \langle t \rangle$,
- $|t| = |G'| = 2n$, *where n is odd, and*
- $u = t^n$ *is an involution.*

Give

- *weight $(-1)^i$ to the oriented u-edge $[t^i](u)$, for each i, and*
- *weight* 0 *to each t-edge.*

Then a flow is in $\mathcal{H}'$ if and only if the weighted sum of its edge-flows is divisible by n.

Lemma 4.4 ([LW, pf. of Prop. 3.3(2a)])**.** *Suppose X' is the prism over a cycle, and X' is bipartite, so we may write*

- $S' = \{t^{\pm 1}, u\}$,
- $G' = \langle t \rangle \times \langle u \rangle$, *and*
- $|G'| = 2|t| = 2n$, *where n is even.*

Give

- *weight $(-1)^j$ to the oriented t-edge $[t^i u^j](t)$, for each i and j, and*
- *weight* 0 *to each u-edge.*

Then a flow is in $\mathcal{H}'$ if and only if the weighted sum of its edge-flows is divisible by $n - 1$.

Lemma 4.5 ([LW, pf. of Prop. 3.3(2b)])**.** *Suppose X' is the prism over a* *cycle, and X' is* ***not*** *bipartite, so we may write*

- $S' = \{t^{\pm 1}, u\}$,
- $G' = \langle t \rangle \times \langle u \rangle$, *and*
- $|G'| = 2|t| = 2n$, *where n is odd.*

Give

- *weight* 1 *to the oriented t-edge* $[v](t)$, *for each* $v \in \langle t \rangle$, *and*
- *weight* 0 *to all of the other edges of* X'.

Then a flow is in $\mathcal{H}'$ if and only if

(1) the weighted sum of its edge-flows is divisible by $n-1$, and
(2) the flow on the oriented edge (t) is the negative of the flow on the oriented edge $[u](t)$.

5 Relations between $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{H}$, $\mathcal{H}'$, and $2\mathcal{F}$

In most cases, our goal in this paper is to show $\mathcal{E} \subseteq \mathcal{H}$, and our proof proceeds by induction on $|S|$. Thus, we usually know that $\mathcal{E}' \subseteq \mathcal{H}'$, and we wish to show that $\mathcal{E} \subseteq \mathcal{H}$. This section presents some of our main tools to accomplish this. They are of three general types:

(1) It suffices to show $2\mathcal{F} \subseteq \mathcal{H}$: (5.1).
(2) Results that show $\mathcal{E} \subseteq \mathcal{H} + \mathcal{E}'$: (5.3), (5.5), (5.6).
(3) Results that show $\mathcal{H}' \subseteq \mathcal{H}$ (or, in some cases, show only that $2\mathcal{H}' \subseteq \mathcal{H}$): (5.7), (5.8), (5.9), (5.10).

Note that if $\mathcal{E}' \subseteq \mathcal{H}'$, then combining (2) with the strong form of (3) yields the desired conclusion $\mathcal{E} \subseteq \mathcal{H}$.

Most of the results in this section assume that $\mathcal{H}$ contains certain basic 4-cycles; results in Sections 6 and 10 show that $\mathcal{H}$ often contains every basic 4-cycle.

Theorem 5.1 (Alspach-Locke-Witte [ALW, Thm. 2.1])**.** *Let X be a connected Cayley graph on a finite abelian group. If X is not a prism over an odd cycle, then $\mathcal{E} \subseteq \mathcal{H} + 2\mathcal{F}$.*

We state the following result for X', rather than X, because it applies to all groups, including those of odd order.

Theorem 5.2 (Chen-Quimpo [CQ])**.** *Suppose $|S'| \geq 3$, and let v and w be* *any two distinct vertices of X'.*

(1) If X' is not bipartite, then there is a hamiltonian path from v to w.
(2) If X' is bipartite, then either
   - *there is a hamiltonian path from v to w, or*
   - *there is a path of even length from v to w.*

Corollary 5.3 (cf. pf. of [ALW, Cor. 3.2])**.** *Suppose $s \in S$, such that*

- *$G' = G$,*
- *$|S'| \geq 3$, and*
- *either X is bipartite or X' is not bipartite.*

Then $\mathcal{E} \subseteq \mathcal{H} + \mathcal{E}'$.

Lemma 5.4. *Let $s \in S$, and assume*

- *$|S| \geq 4$,*
- *$G' \neq G$, and*
- *$(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$, for every $t \in S'$.*

Give

- *weight 1 to the oriented s-edge $[v](s)$ if $v \in s^{-1}G'$, and*
- *weight 0 to all of the other edges of X.*

If k is the weighted sum of the edge-flows of some element H of $\mathcal{H}$, then $k\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$.

Proof. Let $f \in \mathcal{F}$. We wish to show $kf \in \mathcal{H} + \mathcal{F}'$. By adding an appropriate multiple of H to kf, we obtain a flow f_1, such that the weighted edge-sum of f_1 is 0.

For each i, with $-1 \leq i \leq |G/G'| - 2$, let

$$E_i = \Big\{ [v](s) \mid v \in s^i G' \Big\}.$$

Then, for $0 \leq i \leq |G/G'| - 2$, the union $-E_{-1} \cup E_i$ is the set of oriented edges that start in $\cup_{j=0}^{i} s^j G'$, and end in the complement. So the net flow of f_1 through the edges in E_i must equal the net flow through the edges in E_{-1}, which is 0.

Therefore, by adding appropriate multiples of basic 4-cycles of the form

$$[s^m](s, t, s, t^{-1}),$$

with $t \in S'$, to f_1, we obtain a flow that does not use any edges of E_m.

Repeating this for all m (including $m = -1$), we obtain a flow f_2 that does not use any s-edges.

So f_2 is a sum of flows on various cosets of G'. The following claim shows that $f_2 \in \mathcal{H} + \mathcal{F}'$, so we conclude that $kf \in \mathcal{H} + f_2 \subseteq \mathcal{H} + \mathcal{F}'$, as desired.

Claim. For any $f' \in \mathcal{F}'$, and any $v \in G$, we have $[v]f' \in \mathcal{H} + \mathcal{F}'$. We may assume

- f' is a cycle $[w](t_1, \ldots, t_m)$, with each $t_j \in S'$, and $w \in G'$, and
- $v = s^r$, for some $r > 0$.

Then

$$[v]f' - f' = \sum_{i=1}^{r} \sum_{j=1}^{m} [s^{i-1} w t_1 t_2 \ldots t_{j-1}](s, t_j, s^{-1}, t_j^{-1}),$$

is a sum of basic 4-cycles, so $[v]f' - f' \in \mathcal{H}$. □

Corollary 5.5 (cf. [ALW, Lem. 3.8])**.** *Suppose $s \in S$, and we have*

- *$|S| \geq 4$,*
- *$G' \neq G$,*
- *$(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$, for every $t \in S'$, and*
- *either*
  - *(a) X is bipartite, or*
  - *(b) X' is not bipartite.*

Then $\mathcal{E} \subseteq \mathcal{H} + 2\mathcal{F}'$.

Proof. It suffices to show $\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$, for then multiplying by 2 yields $2\mathcal{F} \subseteq \mathcal{H} + 2\mathcal{F}'$, and then the desired conclusion follows from Theorem 5.1.

Let $m = |G/G'|$. Note that, because $|S| \geq 4$, we have $|G'| \geq 3$.

Case 1. Assume X is bipartite. Let $H' = (t_1, t_2, \ldots, t_n)$ be a hamiltonian cycle in X'. There is some r with $s^{-m} = t_1 t_2 \cdots t_r$. (Note that, because X is bipartite, we know that $r + m$ is even.) If m is even, define

$$H = \left(\left(s^{m-1}, t_{2i-1}, s^{-(m-1)}, t_{2i} \right)_{i=1}^{r/2}, \left((t_{r+i})_{i=1}^{n-r-1}, s, (t_{n-i}^{-1})_{i=1}^{n-r-1}, s \right)^{m/2} \right); \quad \text{(E1)}$$

whereas, if m is odd, let

$$H = \Bigl(\bigl(s^{m-2}, t_{2i-1}, s^{-(m-2)}, t_{2i}\bigr)_{i=1}^{(r+1)/2},$$
$$\Bigl((t_{r+1+i})_{i=1}^{n-r-2}, s, (t_{n-i}^{-1})_{i=1}^{n-r-2}, s \Bigr)^{(m-1)/2},$$
$$(t_{r+1+i})_{i=1}^{n-r-1}, (t_i)_{i=1}^{r}, s \Bigr).$$

Then Lemma 5.4 (with $k = 1$) implies $\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$, as desired.

Case 2. Assume X' is not bipartite, and $|S'| \geq 3$. Choose nonidentity elements $g_1, g_2, \ldots, g_m$ of G', such that $g_1 g_2 \cdots g_m = s^{-m}$. Theorem 5.2 implies, for each j, that there is a hamiltonian path $(t_{i,j})_{j=1}^{n-1}$ in X' from e to g_j. Define the hamiltonian cycle $H = \bigl((t_{i,j})_{j=1}^{n-1}, s \bigr)_{i=1}^{m}$. Then Lemma 5.4 (with $k = 1$) implies $\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$, as desired.

Case 3. Assume X' is not bipartite, and $|S'| = 2$. Let $t \in S'$, so $S = \{s^{\pm 1}, t^{\pm 1}\}$. Because X' is not bipartite, we know $|t|$ is odd. We have $s^{-m} = t^r$ for some r with $0 \leq r \leq |t| - 1$. We may assume r is even, by replacing s with its inverse if necessary. Define H as in Eq. (E1), with $t_i = t$ for all i. Then Lemma 5.4 (with $k = 1$) implies $\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$, as desired. □

Corollary 5.6 (cf. [ALW, Lem. 3.8])**.** *Suppose $s \in S$, and we have*

- *$|S| \geq 4$,*
- *$(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$, for every $t \in S'$, and*
- *$G' \neq G$.*

Then $\mathcal{E} \subseteq \mathcal{H} + \mathcal{E}'$.

Proof. It suffices to show $2\mathcal{F} \subseteq \mathcal{H} + \mathcal{E}'$, for then the desired conclusion follows from Theorem 5.1. We may assume that X' is bipartite, but X is not bipartite, for otherwise Corollary 5.5 applies (and yields a stronger conclusion).

Let $m = |G/G'|$. Because $|S| \geq 4$, we have $|G'| \geq 3$. Let $(t_1, t_2, \ldots, t_n)$ be a hamiltonian cycle in X'.

We will construct a hamiltonian cycle H in X, such that H contains the oriented edges $[s^{m-1}](s)$ and $[s^{m-1} t_1](s)$, but no other oriented edges of the form $\pm[v](s)$ with $v \in s^{-1}G'$. Then Lemma 5.4 (with $k = 2$) implies $2\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}'$. Since X' is bipartite, this means $2\mathcal{F} \subseteq \mathcal{H} + \mathcal{E}'$, as desired.

There are p and q with $0 \leq p, q < n$, such that $s^m = t_1 t_2 \ldots t_p$ and $s^m t_1 = t_1 t_2 \ldots t_q$. Because X' is bipartite, but X is not bipartite, it is not difficult to see that the cycle $(t_1, t_2, \ldots, t_p, s^{-m})$ must be odd. Similarly, the cycle $(t_2, t_3, \ldots, t_q, s^{-m})$ must also be odd. Therefore

- p and m have opposite parity, and
- q and m have the same parity.

Furthermore, we may assume the hamiltonian cycle $(t_1, t_2, \ldots, t_n)$ has been chosen so that

$$p \text{ is as small as possible.} \tag{E2}$$

Note that if $p > q$ and $q \neq 0$, then replacing the hamiltonian cycle $(t_1, t_2, \ldots, t_n)$ with $(t_2, t_3, \ldots, t_n, t_1)$ replaces p with $q - 1$. Because $q - 1 < q < p$, this contradicts (E2). Hence

- either $p < q$ or $q = 0$.

If m is odd, then $q \neq 0$ (because q must be odd), so we must have $p < q$. Define

$$\begin{aligned} H = \Big(s, (t_i)_{i=1}^{p}, (t_{p+2i-1}, s^{-1}, t_{p+2i}, s)_{i=1}^{(q-p-1)/2}, \\ (t_{q-1+i})_{i=1}^{n-q-1}, \Big(s, (t_{n-1-i}^{-1})_{i=1}^{n-2}, s, (t_i)_{i=1}^{n-2}\Big)^{(m-3)/2}, \\ s, (t_{n-1-i}^{-1})_{i=1}^{n-3}, s, (t_{q+i})_{i=1}^{n-q-1}, s^{m-1}, t_n, s, (t_{p+1-i}^{-1})_{i=1}^{p} \Big). \end{aligned}$$

If m is even and $p < q$, define

$$\begin{aligned} H = \Big(s^m, \Big(s, t_{p+2-2i}^{-1}, s^{-1}, t_{p+1-2i}^{-1}\Big)_{i=1}^{(p-1)/2}, s, \\ \Big(s, (t_{i+1})_{i=1}^{n-2}, s, (t_{n-i}^{-1})_{i=1}^{n-2}\Big)^{(m-2)/2}, s, (t_{q+1-i}^{-1})_{i=1}^{q-p-1}, s, \\ (t_{p+1+i})_{i=1}^{q-p}, s^{-1}, \Big(t_{q+2i}, s, t_{q+2i+1}, s^{-1}\Big)_{i=1}^{(n-q-2)/2}, t_n \Big). \end{aligned}$$

If m is even and $p > q = 0$, then, by considering the hamiltonian cycle $(t_1^{-1}, t_n^{-1}, t_{n-1}^{-1}, \ldots, t_2^{-1})$, we see that $p = 1$. Define

$$H = \Big(s^m, \Big((t_{i+1})_{i=1}^{n-2}, s, (t_{n-i}^{-1})_{i=1}^{n-2}, s\Big)^{m/2} \Big).$$

This completes the proof. □

Observation 5.7. If $G' = G$, then X' is a spanning subgraph of X, so $\mathcal{H}' \subseteq \mathcal{H}$. (In particular, if $G' = G$ and $\mathcal{E} \subseteq \mathcal{H} + \mathcal{H}'$, then $\mathcal{E} \subseteq \mathcal{H}$.)

Lemma 5.8 (cf. [ALW, Lem. 3.6]). *Suppose $s \in S$, such that*

$$|G'| \geq 4 \textit{ is even.}$$

(1) If $(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$, for every $t \in S'$, then $\mathcal{H}' \subseteq \mathcal{H}$.

(2) If $2(s,t,s^{-1},t^{-1}) \in \mathcal{H}$, for every $t \in S'$, then $2\mathcal{H}' \subseteq \mathcal{H}$.

Proof. We prove only (1), because (2) is very similar. Let $m = |G/G'|$. Given a hamiltonian cycle $H' = (t_1, t_2, \ldots, t_n)$ in X', we have a corresponding hamiltonian cycle

$$H = \left(s^{m-1}, t_{2i-1}, s^{-(m-1)}, t_{2i}\right)_{i=1}^{n/2}$$

in X. Because

$$H - H' = \sum_{i=0}^{m-2} \sum_{j=1}^{n/2} [s^i t_1 t_2 \ldots t_{2j-2}]\left(s, t_{2j-1}, s^{-1}, t_{2j-1}^{-1}\right)$$

is a sum of basic 4-cycles, we conclude that $H' \in \mathcal{H}$. □

Lemma 5.9. *Suppose $s \in S$, such that*

- *$|s| > 2$,*
- *$|G'| \geq 3$ is odd, and*
- *$(s,t,s^{-1},t^{-1}) \in \mathcal{H}$, for every $t \in S'$.*

Then $2\mathcal{H}' \subseteq \mathcal{H}$.

Proof. Let $H' = (t_1, t_2, \ldots, t_n)$ be a hamiltonian cycle in X'. Let $m = |G/G'|$. There is some r with $s^{-m} = t_1 t_2 \cdots t_r$. Because n is odd, we may assume r is even (by replacing H' with $-H' = (t_n^{-1}, t_{n-1}^{-1}, \ldots, t_1^{-1})$ if necessary). Note that, because $|s| > 2$, we have $m > 2$ if $r = 0$. Define

$$\begin{aligned}
\epsilon &= \begin{cases} 0 & \text{if } r = 0; \\ 1 & \text{otherwise,} \end{cases} \\
H_1 &= \Big(\left(t_{n-i+1}^{-1}\right)_{i=1}^{n-r-1+2\epsilon}, \left(s^{m-1}, t_{r-2i+1}^{-1}, s^{-(m-1)}, t_{r-2i}^{-1}\right)_{i=1}^{\epsilon(r-2)/2}, \\
&\qquad s, \left(s^{m-2}\right)^{\epsilon}, t_1^{-1}, (s^{-(m-2)})^{\epsilon}, \\
&\qquad \left(t_{n-2i+2}^{-1}, s^{m-2}, t_{n-2i+1}^{-1}, s^{-(m-2)}\right)_{i=1}^{(n-r-3+2\epsilon)/2}, \\
&\qquad t_{r+3-2\epsilon}^{-1}, s^{m-2}, \left(t_2^{-1}, s^{-(m-3)}, t_1^{-1}, s^{m-3}\right)^{1-\epsilon}, s \Big)
\end{aligned}$$

and

$$\begin{aligned}
H_2 = \Big(&\left(s^{m-1}, t_{2i-1}, s^{-(m-1)}, t_{2i}\right)_{i=1}^{r/2}, \\
&\left(s^{m-2}, t_{r+2i-1}, s^{-(m-2)}, t_{r+2i}\right)_{i=1}^{(n-r-1)/2}, \\
&s^{m-1}, (t_{n-i}^{-1})_{i=1}^{n-r-1}, s \Big).
\end{aligned}$$

Then $H_1 - H_2 + 2H'$ is a sum of basic 4-cycles, so $2H'$ belongs to $\mathcal{H}$.

Perhaps we should elaborate further. Let Y be the spanning subgraph of X with

$$E(Y) = \left\{ [s^i t_1 t_2 \cdots t_j](t_{j+1}) \;\middle|\; \begin{array}{c} 0 \le i \le m-1, \\ 0 \le j \le n-2 \end{array} \right\} \cup \left\{ [s^i t_1 t_2 \cdots t_j](s) \;\middle|\; \begin{array}{c} 0 \le i \le m-2, \\ 0 \le j \le n-1 \end{array} \right\}.$$

Then

$$H_1 - H_2 + H' + [s]H' \in \mathcal{F}(Y),$$

and Y is naturally isomorphic to the cartesian product of the two paths $(s)^{m-1}$ and $(t_1, t_2, \ldots, t_{n-1})$, so any flow on Y is a sum of basic 4-cycles. Furthermore,

$$[s]H' - H' = \sum_{i=1}^{n} [t_1 t_2 \cdots t_{i-1}](s, t_i, s^{-1}, t_i^{-1})$$

is a sum of basic 4-cycles. Therefore

$$H_1 - H_2 + 2H' = \Big(H_1 - H_2 + H' + [s]H'\Big) - \Big([s]H' - H'\Big)$$

is a sum of basic 4-cycles, as claimed. □

Lemma 5.10. *Suppose $S = \{s, t^{\pm 1}, u^{\pm 1}\}$, such that*

- $|S| = 5$,
- $|G'|$ *is odd,*
- $s^2 = e$,
- $G' \neq \langle t \rangle$,
- $G' \neq \langle u \rangle$, *and*
- *every basic 4-cycle is in $\mathcal{H}$.*

Then $2\mathcal{H}' \subseteq \mathcal{H}$.

Proof. Let $p = |G'|/|t|$ and $n = |t|$. Note that p and n must be odd, since $|G'|$ is odd. There is some r with $u^{-p} = t^r$. Because n is odd, we may assume r is even (by replacing t with t^{-1} if necessary).

Define

$$\begin{aligned} H_1 = \Big(& t^{n-1}, u^{p-2}, t^{-(n-3)}, u, t^{n-1}, s, t^{n-1}, u^{-(p-1)}, t, \\ & \big(t^{n-2}, u, t^{-(n-2)}, u\big)^{(p-1)/2} \sharp, s, t^{-1}, \\ & \big(u^{-1}, t^{p-2}, u^{-1}, t^{-(p-2)}\big)^{(p-3)/2}, u^{-1} \Big). \end{aligned}$$

Then $H_1 - 2(t^n)$ is a sum of basic 4-cycles. (For example, this follows from the observation that $H_1 - 2[u^{n-1}](t^n)$ belongs to $\mathcal{F}(Y)$, where Y is a spanning subgraph of X that is naturally isomorphic to the cartesian product of the paths (s), (t^{m-1}) and (u^{p-1}), cf. pf. of (5.9)). Therefore

$$2(t^n) \in \mathcal{H}. \tag{E3}$$

Now define

$$\begin{aligned} H_2 = \Big(& s, u^{-1}, (t^{-r}, u^{-1}, t^r, u^{-1})^{(p-1)/2} \sharp, \\ & \left(t, u^{p-2}, t, u^{2-p}\right)^{(n-r-1)/2}, u^{-1}, t^{2-n}, s, \\ & \left(t^{n-2}, u, t^{2-n}, u\right)^{(p-1)/2}, t^{n-1}, u^{1-p} \Big). \end{aligned}$$

By adding certain basic 4-cycles involving s to H_2, we can obtain an element of $\mathcal{H}$ that is in X' and uses only one edge of the form $[t^i]u$. We can now apply the the proof of Lemma 5.4 to X', with u taking the role of s, to obtain the conclusion that $\mathcal{F}' \subseteq \mathcal{H} + (t^n)$, since the basic 4-cycles are in $\mathcal{H}$ and not just in $\mathcal{H}'$. In particular, this tells us that $2\mathcal{H}' \subseteq \mathcal{H} + 2(t^n)$. Combining this with (E3), we see that $2\mathcal{H}' \subseteq \mathcal{H}$, as desired. $\square$

6 Some basic 4-cycles in $\mathcal{H}$

In this section, we show, for $s, t \in S$, that $\mathcal{H}$ often contains the flow $2(s, t, s^{-1}, t^{-1})$ (see 6.2) and, if $|G|$ is divisible by 4, the basic 4-cycle (s, t, s^{-1}, t^{-1}) (see 6.3). Our main tool is the construction described in Lemma 6.1, which was already used in [ALW,LW] (and goes back to [M]).

Lemma 6.1. *Suppose*

- $x, y, z \in S$,
- $v, w \in G$, *and*
- H_+ *and* H_- *are oriented hamiltonian cycles in* X.

Then:

(1) If H_+ *contains both the oriented path* $[v](x, y, x^{-1})$ *and the oriented edge* $[vxz](y)$, *then*

$$(x, y, x^{-1}, y^{-1}) + [x](z, y, z^{-1}, y^{-1}) \in \mathcal{H}.$$

(2) If H_- contains both the oriented path $[w](x, y, x^{-1})$ and the oriented edge $[wxyz](y^{-1})$, then

$$(x, y, x^{-1}, y^{-1}) - [x](z, y, z^{-1}, y^{-1}) \in \mathcal{H}.$$

(3) If both (1) and (2) apply, then $2(x, y, x^{-1}, y^{-1}) \in \mathcal{H}$.

(4) If

- *$(x, y, x^{-1}, y^{-1}) \in \mathcal{H}$, and*
- *either (1) or (2) applies,*

then $(z, y, z^{-1}, y^{-1}) \in \mathcal{H}$.

Proof. We may assume $v = w = e$.

(1) Construct a hamiltonian cycle H'_+ by replacing

- the path (x, y, x^{-1}) with the edge (y) and
- the edge $[xz](y)$ with the path $[xz](z^{-1}, y, z)$.

Then $H_+ - H'_+$ is the sum of the two given 4-cycles.

(2) Construct a hamiltonian cycle H'_- by replacing

- the path (x, y, x^{-1}) with the edge (y) and
- the edge $[xyz](y^{-1})$ with the path $[xyz](z^{-1}, y^{-1}, z)$.

Then $H_- - H'_-$ is the difference of the two given 4-cycles.

(3) Adding the flows from (1) and (2) results in $2(x, y, x^{-1}, y^{-1})$.

(4) The difference of (x, y, x^{-1}, y^{-1}) and the flow that results from either of (2) and (1) is $\pm[x](z, y, z^{-1}, y^{-1})$. □

Proposition 6.2. *If $s, t \in S$, such that*

- *$|S| = 4$,*
- *$|s| \geq 3$,*
- *$t \notin \langle s \rangle$, and*
- *$t^2 \notin \{e, s^{\pm 1}\}$,*

then, letting $Q = (s, t, s^{-1}, t^{-1})$, we have

$$Q \equiv -Q \equiv [v]Q \pmod{\mathcal{H}},$$

for all $v \in G$. In particular, $2Q \in \mathcal{H}$.

Proof. It suffices to show

(a) $Q - [s]Q \in \mathcal{H}$,
(b) $Q - [t]Q \in \mathcal{H}$, and
(c) either $Q + [s]Q \in \mathcal{H}$, or $Q + [t]Q \in \mathcal{H}$.

Let $n = |G|/|s|$, and write $t^n = s^r$, with $0 \le r \le |s|$. (If $t^n = e$, then we have a choice, which will be made later, whether to use $r = 0$ or $r = |s|$.) Because $t \notin \langle s \rangle$, we have $G \ne \langle s \rangle$, so $n \ge 2$. Note that $n|s| = |G|$ is even, so n and $|s|$ cannot both be odd. If $|s|$ is even and $n \ge 3$, define, for future reference, the hamiltonian cycle

$$H_* = \Big(t^{n-1}, s^{|s|-1}, t^{-1}, s^{-(|s|-2)}, t^{-(n-2)}, \big(s, t^{n-3}, s, t^{-(n-3)}\big)^{(|s|-2)/2}, s\Big). \qquad \text{(E4)}$$

Now let us begin by establishing (a). Let

$$H_a = \begin{cases} \Big(t^{n-1}, s, \big(s^{|s|-2}, t^{-1}, s^{-(|s|-2)}, t^{-1}\big)^{n/2} \sharp, s^{-1}\Big) & \text{if } |s| \text{ is odd,} \\ \big(t, s^{|s|-1}, t^{-1}, s^{-(|s|-1)}\big) & \text{if } n = 2, \\ H_* \text{ as in Eq. (E4)} & \text{otherwise.} \end{cases}$$

Then H_a contains both the oriented path $[s^{-2}t^{n-1}](s, t^{-1}, s^{-1})$ and the oriented edge $[t^{n-2}](t)$, so Lemma 6.1(2) (with $x = s$, $y = t^{-1}$, $z = s$, and $w = s^{-2}t^{n-1}$) establishes (a).

All that remains is to establish (b) and (c).

Case 1. Assume $n \ge 3$.

Subcase 1.1. Assume $|s|$ is odd. Note that n must be even (so $n \ge 4$), because $n|s| = |G|$ is even. We may assume $t^n \ne e$ (so $r \notin \{0, |s|\}$), for otherwise, by interchanging s and t, we may transfer to one of the cases where $|s|$ is even and n is odd. We may assume r is odd, by replacing s with its inverse if necessary. Define hamiltonian cycles

$$H_- = \Big(\big(t^{n-3}, s, t^{-(n-3)}, s\big)^{(|s|-r)/2}, s^{r-1}, t, \\ \big(t^{n-2}, s^{-1}, t^{-(n-2)}s^{-1}\big)^{(r-1)/2}, t^{n-3}, s^{-(|s|-r)}, t, s^{|s|-r}, t\Big)$$

and

$$H_+ = \Big(\big(t^{n-1}, s, t^{-(n-1)}, s\big)^{(|s|-r)/2}, \big(s^{r-1}, t, s^{-(r-1)}, t\big)^{n/2}\Big).$$

Then

- H_- contains both the oriented path $[t^{n-4}](t, s, t^{-1})$ and the oriented edge $[t^{n-2}s](s^{-1})$, so Lemma 6.1(2) (with $x = t$, $y = s$, $z = t$, and $v = t^{n-4}$) establishes (b)

- H_+ contains the oriented path $[t^{n-2}](t, s, t^{-1})$ and the oriented edge $[t^n](s)$ (because r is odd), so Lemma 6.1(1) (with $x = t$, $y = s$, $z = t$, and $v = t^{n-2}$) establishes (c).

Subcase 1.2. Assume $|s|$ is even and $n \geq 4$. We may assume $t^n \neq s$, by replacing s with its inverse if necessary. Define hamiltonian cycles

$$H_- = \Big(\big(s, t^{n-2}, s, t^{-(n-2)}\big)^{(|s|-2)/2}, s, t^{n-1}, s^{-(|s|-1)}, t^{-(n-1)} \Big) \tag{E5}$$

and

$$H_+ = \begin{cases} H_* \text{ as in Eq. (E4)} & \text{if } r \neq 2 \\ \Big(t^{n-2}, s^{-(|s|-2)}, t, s^{|s|-1}, t^{-(n-1)}, \\ \qquad \big(s, t^{n-3}, s, t^{-(n-3)}\big)^{(|s|-2)/2}, s \Big) & \text{if } r = 2. \end{cases}$$

Then:

- H_- contains both the oriented path $[t](t^{-1}, s, t)$ and the oriented edge $[t^{-1}s](s^{-1})$, so Lemma 6.1(2) (with $x = t^{-1}$, $y = s$, $z = t^{-1}$, and $v = t$) establishes (b).
- H_+ contains both the oriented path $[st](t^{-1}, s, t)$ and the oriented edge $[st^{-1}](s)$, so Lemma 6.1(2) (with $x = t^{-1}$, $y = s$, $z = t^{-1}$, and $v = st$) establishes (c).

Subcase 1.3. Assume $|s|$ is even and $n = 3$. We may assume $0 \leq r \leq |s|/2$, by replacing t with t^{-1} if necessary. Define hamiltonian cycles

$$H_- \text{ as in Eq. (E5)},$$

and

$$H_+ = \big(t^2, s^{|s|-2-r}, t, s^{-(|s|-3)}, t, s^{|s|-2}, t, s^{-(|s|-r)}, t, s\big).$$

Then:

- H_- establishes (b), exactly as in the previous subcase.
- H_+ contains both the oriented path $[s^2](s^{-1}, t, s)$ and the oriented edge (t), so Lemma 6.1(2) (with $x = s^{-1}$, $y = t$, $z = s^{-1}$, and $v = s^2$) establishes (c).

Case 2. Assume $n = 2$. We have $2 \leq r \leq |s| - 2$, because $t^2 \notin \{e, s^{\pm 1}\}$. Notice that this implies $|s| \geq 4$.

Subcase 2.1. Assume $|s|$ is even, and r is odd. We may assume $r \leq |s|/2$, by replacing s with s^{-1} if necessary.

If $r = |s|/2$, then $|t| = 4$, so $|G|/|t| = |s|/2 = r \geq 3$ (because r is odd). Thus, an earlier subcase applies, after interchanging s and t.

We may now assume $r < |s|/2$. Define

$$H_+ = \Big(\big(t, s, t^{-1}, s\big)^{(|s|-2r)/2}, \big(s^{r-1}, t, s^{-(r-1)}, t\big)^2 \Big).$$

Then H_+ contains both the oriented path $[s^{-2}](s, t, s^{-1})$ and the oriented edge (t), so Lemma 6.1(1) (with $x = s$, $y = t$, $z = s$, and $v = s^{-2}$) implies that $Q + [s]Q \in \mathcal{H}$. Therefore, because

$$Q = \big(t, s^{|s|-1}, t^{-1}, s^{-(|s|-1)}\big) + \sum_{i=1}^{(n-2)/2} [s^{2i}]\big(Q + [s]Q\big),$$

we conclude that $Q \in \mathcal{H}$, which makes (a), (b), and (c) trivial.

Subcase 2.2. Assume that either

- $|s|$ *is odd or*
- $|s|$ *and* r *are even.*

We may assume $|s| - r$ is even, by replacing t with its inverse if necessary. Define

$$H_- = \big(t, s, t^{-2}, s^{r-2}, t^{-1}, s^{-(|s|-3)}, t, s^{|s|-r-2}, t\big).$$

Then H_- contains both the oriented path (t, s, t^{-1}) and the oriented edge $[st^2](s^{-1})$, so Lemma 6.1(2) (with $v = e$, $x = t$, $y = s$ and $z = t$) tells us that $(-Q) - [t](-Q) \in \mathcal{H}$. This establishes (b).

Define the hamiltonian cycle

$$H_+ = \begin{cases} \Big(t, s, t^{-1}, s^{|s|-2}, t, s^{-(|s|-3)}, t\Big) & \text{if } r = |s| - 2 \\ \Big(t, s, t^{-1}, s^{r+1}, t^{-1}, s^{r}, \big(s, t^{-1}, s, t\big)^{(|s|-r-2)/2} \sharp\Big) & \text{if } r < |s| - 2. \end{cases}$$

Then H_+ contains both the oriented path (t, s, t^{-1}) and the oriented edge $[t^2](s)$, so Lemma 6.1(1) (with $v = e$, $x = t$, $y = s$ and $z = t$) tells us that $(-Q) + [t](-Q) \in \mathcal{H}$. This establishes (c). $\square$

Corollary 6.3. *Suppose $s, t \in S$, such that*

- $|G|$ *is divisible by* 4,
- $|s| \geq 3$,
- $t \notin \langle s \rangle$, *and*
- *either*
  (a) $|G|/|s| \geq 3$, *or*

(b) $t^2 \notin \{e, s^{\pm 1}\}$.

Then $(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$.

Proof. Case 1. Assume $\langle s, t\rangle = G$ *and* $t^2 \notin \{e, s^{\pm 1}\}$.

- If $|s|$ is even, let $x = s$, $y = t$, $m = |s|$, and $n = |G|/|s|$.
- If $|s|$ is odd, let $x = t$, $y = s$, $m = |G|/|s|$, and $n = |s|$.

In either case,

$$H = \Big(x^{m-1}, y, \big(y^{n-2}, x^{-1}, y^{-(n-2)}, x^{-1}\big)^{m/2}\sharp, y^{-1}\Big)$$

is a hamiltonian cycle in X. Letting

$$Q = (x, y, x^{-1}, y^{-1}) = \pm(s, t, s^{-1}, t^{-1}),$$

we have

$$H = \left(\sum_{i=1}^{m-1} [x^{i-1}]Q\right) + \left(\sum_{i=1}^{m/2} \sum_{j=1}^{n-2} [x^{2i-2}y^j]Q\right),$$

so H is the sum of

$$(m-1) + \frac{m}{2}(n-2) = \frac{mn}{2} - 1$$

translates of Q. Because $mn = |G|$ is divisible by 4, we know that $(mn/2) - 1$ is odd. Therefore, Proposition 6.2 implies

$$H \equiv \big((mn/2) - 1\big)Q \equiv Q \pmod{\mathcal{H}}.$$

Since $H \in \mathcal{H}$, we conclude that $Q \in \mathcal{H}$.

Case 2. Assume that either $\langle s, t\rangle \neq G$ *or* $t^2 \in \{e, s^{\pm 1}\}$. We show how to reduce to the previous case.

First, let us show $|G|/|s| \geq 3$. By hypothesis, if this fails to hold, then $t^2 \notin \{e, s^{\pm 1}\}$, so we may assume that the latter holds. Hence, the assumption of this case implies $\langle s, t\rangle \neq G$. Since $t \notin \langle s\rangle$, we conclude that

$$\frac{|G|}{|s|} \geq \frac{|G|}{|\langle s, t\rangle|} \cdot \frac{|\langle s, t\rangle|}{|s|} \geq 2 \cdot 2 > 3,$$

as claimed.

Let $(t_1, \ldots, t_n)$ be a hamiltonian cycle in

$$\operatorname{Cay}\big(G/\langle s\rangle; S \smallsetminus \{s, s^{-1}\}\big),$$

with $t_1 = t$ (cf. 5.2). Define permutations σ and τ of G by

- $\sigma(x) = xs$ and
- $\tau(x) = xt_i$, where $i = i(x)$ satisfies $1 \le i \le n$ and $x \in \langle s \rangle t_1 t_2 \cdots t_{i-1}$.

Let Y be the spanning subgraph of X whose edge set is

$$E(Y) = \left\{ \bigl(x, \sigma(x)\bigr) \mid x \in V(G) \right\} \cup \left\{ \bigl(x, \tau(x)\bigr) \mid x \in V(G) \right\}.$$

Then Y is a connected, spanning subgraph of X. It is not difficult to see that σ and τ generate a transitive, abelian group Γ of automorphisms of Y (and any transitive, abelian permutation group is regular), so Y is isomorphic to the Cayley graph

$$Y^* = \mathrm{Cay}\bigl(\Gamma; \{\sigma^{\pm 1}, \tau^{\pm 1}\}\bigr).$$

Furthermore, the natural isomorphism carries the 4-cycle (s, t, s^{-1}, t^{-1}) to $(\sigma, \tau, \sigma^{-1}, \tau^{-1})$.

Note that $|\sigma| = |s|$ and $|\Gamma|/|\sigma| = |G|/|s| \ge 3$, so $\tau^2 \notin \langle \sigma \rangle$. From Case 1, we know that $(\sigma, \tau, \sigma^{-1}, \tau^{-1}) \in \mathcal{H}(Y^*)$. Hence, via the isomorphism, we see that $(s, t, s^{-1}, t^{-1}) \in \mathcal{H}(Y) \subseteq \mathcal{H}(X)$. □

Remark. The assumption that $t \notin \{e, s^{\pm 1}\}$ is necessary in Proposition 6.2 and Corollary 6.3, as is seen from Theorem 1.4(2) and Proposition 8.1. If $|S| = 4$ and X is not bipartite, then the assumption that $|G|$ is divisible by 4 is necessary in Corollary 6.3, as is seen from Proposition 8.2.

7 The graphs of degree 4 with $\mathcal{H} = \mathcal{E}$

In this section, we show that $\mathcal{H} = \mathcal{E}$ in many cases where $|S| = 4$ (see 7.1). In Section 8, we will calculate $\mathcal{H}$ in the cases not covered by this result.

Proposition 7.1. *If*

- $|S| = 4$*, and*
- X *is* ***not*** *the square of an even cycle, and*
- *either*
  - *(a)* X *is bipartite, or*
  - *(b)* $|G|$ *is divisible by* 4*,*

then $\mathcal{H} = \mathcal{E}$*.*

We preface the proof with an observation on bipartite graphs, and the treatment of a special case.

Lemma 7.2. *If*

- $|G|$ *is not divisible by* 4*, and*
- *every element of* S *has even order,*

then X *is bipartite.*

Proof. Let $H = \{ g^2 \mid g \in G \}$. Because $|G| \equiv 2 \pmod 4$, we see that $|H| = |G|/2$, and $|H|$ is odd. We know that no element of S belongs to H (because the elements of H have odd order), so the subgraph of X induced by each of the two cosets of H has no edges. Therefore, the coset decomposition $G = H \cup Hg$ is a bipartition of G, so G is bipartite. $\square$

Lemma 7.3. *If* $X \cong K_2 \,\square\, Y$*, where* Y *is a Möbius ladder, then* $\mathcal{H} = \mathcal{E}$*.*

Proof. We may assume

$$X = \operatorname{Cay}(\mathbf{Z}_2 \oplus \mathbf{Z}_{2n}; \{s, t^{\pm 1}, u\}),$$

where $s = (0, n)$, $t = (0, 1)$, and $u = (1, 0)$.

Step 1. We have $(u, t, u, t^{-1}) \in \mathcal{H}$*.* Define the hamiltonian cycle

$$H = \Big((s, t)^n \sharp, u, (s, t^{-1})^n \sharp, u \Big).$$

Then the sum $H + [t]H$ has edge-flow 0 on each s-edge, so $H + [t]H \in \mathcal{F}'$. Under the weighting of X' specified in Lemma 4.4, with u in the role of s, the weighted sum of the edge-flows of $H + [t]H$ is $4(n-1)$, which is relatively prime to $2n - 1$. Thus, we conclude from Lemma 4.4 that $H + [t]H$ generates $\mathcal{F}'/\mathcal{H}'$, so $\mathcal{F}' \subseteq \mathcal{H} + \mathcal{H}'$. Because X' is a spanning subgraph of X, we have $\mathcal{H}' \subseteq \mathcal{H}$, so this implies $\mathcal{F}' \subseteq \mathcal{H}$. Therefore $(u, t, u, t^{-1}) \in \mathcal{H}$, as desired.

Step 2. For

$$Y = \operatorname{Cay}\Big(\{0\} \oplus \mathbf{Z}_{2n}; \{s, t^{\pm 1}\} \Big) \subseteq X,$$

we have $\mathcal{E}(Y) \subseteq \mathcal{H}$*.* The hamiltonian cycle

$$\Big(u, t^{n-1}, s, t^{-(n-1)} \Big)^2$$

contains both the oriented path $[t^{-1}](t, u, t^{-1})$ and the oriented edge $[su](u^{-1})$, so Lemma 6.1(2) (with $x = t$, $y = u$, $z = s$, and $w = t^{-1}$) implies

$$(t, u, t^{-1}, u^{-1}) - [t](s, u, s, u) \in \mathcal{H}.$$

From Step 1, we know $(t, u, t^{-1}, u) \in \mathcal{H}$, so we conclude that (s, u, s, u) also belongs to $\mathcal{H}$. Then, from Lemma 5.8(1) (with u in the role of s), we see that $\mathcal{H}(Y) \subseteq \mathcal{H}$. Thus,

$$\text{it suffices to show } \mathcal{E}(Y) \subseteq \mathcal{H} + \mathcal{H}(Y).$$

We may assume n is odd, for otherwise Theorem 1.4(1b) implies $\mathcal{E}(Y) = \mathcal{H}(Y) \subseteq \mathcal{H} + \mathcal{H}(Y)$. Consider the hamiltonian cycle

$$H' = \Bigl(s, t^{n-2}, u, t^{-(2n-3)}, u, t^{n-2}, s, u, t, u\Bigr).$$

We have

$$(s, t^{-1}, s, t) = H' - (t, u, t^{-1}, u) - \sum_{i=1}^{2n-3} [t^i](t, u, t^{-1}, u) \in \mathcal{H}.$$

Under the weighting of Y specified in Lemma 4.3, the weighted sum of the edge-flows of (s, t^{-1}, s, t) is ± 2, which is relatively prime to n. Thus, Lemma 4.3 implies that (s, t^{-1}, s, t) generates $\mathcal{F}(Y)/\mathcal{H}(Y)$, so we conclude that $\mathcal{E}(Y) \subseteq \mathcal{H} + \mathcal{H}(Y)$.

Step 3. Completion of the proof. Given any even flow $f \in \mathcal{E}$, we wish to show $f \in \mathcal{H}$. Adding appropriate 4-cycles of the form $[v](u, t, u, t^{-1})$ eliminates all u-edges from f, leaving an even flow $f_1 \in \mathcal{E}\bigl(Y\bigr) \oplus \mathcal{E}\bigl([u]Y\bigr)$. From Step 2, we know that $\mathcal{E}\bigl(Y\bigr) \oplus \mathcal{E}\bigl([u]Y\bigr) \subseteq \mathcal{H}$, so we have $f_1 \in \mathcal{H}$. Hence, $f \in \mathcal{H}$, as desired. $\square$

Proof of Proposition 7.1. By Remark 3.2 and Lemma 7.3, we may assume that S has no involutions. Let $S = \{s^{\pm 1}, t^{\pm 1}\}$.

Case 1. Assume X is bipartite. In this case, we know $|t|$ is even, and $t^2 \neq s^{\pm 1}$.

Subcase 1.1. Assume $t \in \langle s \rangle$. Write $t = s^r$. We may assume $2 \le r < |s|/2$, by replacing t with its inverse if necessary. Because X is bipartite, we know that r is odd. Give

- weight 0 to each s-edge, and
- weight $(-1)^i$ to each oriented t-edge $[s^i](t)$.

Then the weighted sum of the edge-flows of the hamiltonian cycle

$$H_1 = \Bigl(\bigl(t, s, t^{-1}, s\bigr)^{(r-1)/2}, t, s^{|s|-2r+1}\Bigr)$$

is r, and the weighted sum of the edge-flows of the hamiltonian cycle

$$H_2 = \Bigl(t, \bigl(t, s^{-1}, t^{-1}, s^{-1}\bigr)^{(r-1)/2}, t^2, s^{|s|-2r-1}\Bigr)$$

is $2 - r$.

Given any flow f on X, we wish to show $f \in \mathcal{H}$. Because r is relatively prime to $2-r$, some integral linear combination of H_1 and H_2 has the same weighted edge-sum as f; thus, by subtracting this linear combination, we may assume the weighted edge-sum of f is 0. Then, by subtracting a linear combination of hamiltonian cycles of the form

$$[v]\Big(t, s^{-(r-1)}, t, s^{|s|-r-1}\Big),$$

we may assume that f does not use any t-edges. Then f is a multiple of the hamiltonian cycle $(s)^{|s|}$, so $f \in \mathcal{H}$.

Subcase 1.2. Assume $t \notin \langle s \rangle$. By Lemma 6.2, we know $2(s, t, s^{-1}, t^{-1}) \in \mathcal{H}$. We have

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + 2\mathcal{F} && \text{(see 5.1)} \\
&\subseteq \mathcal{H} + 2\mathcal{E} && \text{(X is bipartite, so $\mathcal{F} = \mathcal{E}$)} \\
&\subseteq \mathcal{H} + 2\mathcal{E}' && \text{(cf. 5.6)} \\
&\subseteq \mathcal{H} + 2\mathcal{H}' && \text{(X' is an even cycle, so $\mathcal{E}' = \mathcal{H}'$)} \\
&\subseteq \mathcal{H} && \text{(see 5.8(2))},
\end{aligned}$$

as desired.

Case 2. Assume $|G|$ is divisible by 4. Let $m = |G|/|t|$, and write $s^m = t^r$, for some r, with $0 \le r < |t|$.

Subcase 2.1. Assume $\langle t \rangle \neq G$. Because X is not the square of an even cycle, we know $s^2 \notin \{t^{\pm 1}\}$. Therefore Corollary 6.3(b) (with the roles of s and t interchanged) implies that the 4-cycle (s, t, s^{-1}, t^{-1}) is in $\mathcal{H}$.

If $|t|$ is even, then

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(X' is a cycle, so $\mathcal{F}' = \mathcal{H}'$)} \\
&\subseteq \mathcal{H} && \text{(see 5.8(1))},
\end{aligned}$$

as desired.

If $|t|$ is odd, then

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + 2\mathcal{F}' && \text{(see 5.5(b))} \\
&\subseteq \mathcal{H} + 2\mathcal{H}' && \text{(X' is a cycle, so $\mathcal{F}' = \mathcal{H}'$)} \\
&\subseteq \mathcal{H} && \text{(see 5.9)},
\end{aligned}$$

as desired.

Subcase 2.2. Assume $\langle t \rangle = G$. Since X is not bipartite (and $|t| = |G|$ is even),

r must be even. So $\langle s \rangle \subseteq \langle t^2 \rangle \neq G$. Thus, by interchanging s and t, we can move out of this subcase. □

For future reference, let us record the following special case of the proposition. (Note that no bipartite graph is the square of an even cycle.)

Corollary 7.4. *If $|S| = 4$ and X is bipartite, then $\mathcal{H} = \mathcal{E}$.*

8 The graphs of degree 4 with $\mathcal{H} \neq \mathcal{E}$

In this section, we provide an explicit description of $\mathcal{H}$ for the graphs of degree 4 that are not covered by Proposition 7.1 (see 8.1 and 8.2). We also establish two corollaries that will be used in the study of graphs of higher degree (see 8.6 and 8.7).

Proposition 8.1. *Suppose X is the square of an even cycle, so*

$$S = \{s^{\pm 1}, t^{\pm 1}\} \text{ with } t = s^2.$$

Give

- *weight 0 to each s-edge, and*
- *weight $(-1)^i$ to each t-edge $[s^i](t)$.*

Then a flow is in $\mathcal{H}$ if and only if the weighted sum of its edge-flows is divisible by $|G| - 2$.

Proof. Let $n = |G|/2$. All hamiltonian cycles are of one of the following two forms:

$$H_1 = [v]\Big(t^{n-1}, s, t^{-(n-1)}, s^{-1}\Big)$$

or

$$H_2 = \pm[v]\Big(s^{n_0}, (t, s^{-1}, t), s^{n_1}, (t, s^{-1}, t), s^{n_2}, (t, s^{-1}, t), \ldots, s^{n_{k-1}}, (t, s^{-1}, t), s^{n_k}\Big)$$

(for some $k \geq 0$ and $n_0, \ldots, n_k \geq 0$ with $3k + \sum n_i = 2n$). In both cases, it is easy to see that the weighted sum of the edge-flows is divisible by $2n - 2$.

Conversely, given any flow f such that the weighted sum of its edge-flows is $m(2n - 2)$, for some integer m, we wish to show $f \in \mathcal{H}$. The weighted sum of the edge-flows of $f - mH_1$ is 0, so, by adding appropriate hamiltonian cycles of the form H_2 (with $k = 1$ and $n_0 = 0$), we obtain a flow f' that does not use any t-edges. Then f' is a multiple of the hamiltonian cycle (s^{2n}), so $f' \in \mathcal{H}$. Therefore $f \in \mathcal{H}$. □

Proposition 8.2. *Suppose*

- $|S| = 4$,
- *X is not bipartite,*
- *$|G|$ is not divisible by 4, and*
- *X is not the square of an even cycle,*

so

- $S = \{t^{\pm 1}, u^{\pm 1}\}$, *where*
- *t has odd order, and*
- $t \neq u^{\pm 2}$.

Give

- *weight $(-1)^j$ to each oriented t-edge $[t^i u^j](t)$, and*
- *weight 0 to each u-edge.*

Then a flow belongs to $\mathcal{H}$ if and only if the weighted sum of its edge-flows is divisible by 4.

This proposition is obtained by combining Lemmas 8.4 and 8.5.

Observation 8.3. In the situation of Proposition 8.2, we know that $|G|/|t|$ is even, so it is not difficult to see that $\sum_{v \in G} f\Big([v](u)\Big)$ is even, for all $f \in \mathcal{F}$. Therefore, a flow on X is even if and only if the weighted sum of its edge-flows is even.

Remark. In the situation of Proposition 8.2, some even flows (such as any basic 4-cycle) have weight 2, so the result implies that $\mathcal{H} \neq \mathcal{E}$. In fact, $\mathcal{E}/\mathcal{H} \cong \mathbf{Z}_2$.

Lemma 8.4. *If*

- *S and G are as described as in Proposition 8.2, and*
- *H is any hamiltonian cycle in X,*

then the weighted sum of the edge-flows of H is divisible by 4.

Proof. Because it is rather lengthy, this proof has been postponed to a section of its own (see Section 9). The reader can easily verify that this proof does not rely on any of the subsequent results in the present section. □

Lemma 8.5. *If*

- *S and G are as described as in Proposition 8.2, and*
- *f is a flow, such that the weighted sum of the edge-flows of f is divisible by 4,*

then $f \in \mathcal{H}$.

Proof. Let $\mathcal{Q}$ be the subgroup of $\mathcal{F}$ generated by the basic 4-cycles, and let $s = u$.

Step 1. We have $\mathcal{E} \subseteq \mathcal{H} + \mathcal{Q} + 2\mathcal{F}'$. Obviously, $\mathcal{H} + \mathcal{Q}$ contains every basic 4-cycle, so this follows from (the proof of) Corollary 5.5(b).

Step 2. We may assume $f \in \mathcal{Q}$. We have

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + \mathcal{Q} + 2\mathcal{F}' && \text{(from Step 1)} \\
&= \mathcal{H} + \mathcal{Q} + 2\mathcal{H}' && \text{(X' is a cycle, so $\mathcal{F}' = \mathcal{H}'$)} \\
&\subseteq \mathcal{H} + \mathcal{Q} && \text{(see proof of (5.9)).}
\end{aligned}$$

Thus, because $f \in \mathcal{E}$ (see 8.3), we may write $f = H + Q$, with $H \in \mathcal{H}$ and $Q \in \mathcal{Q}$. By assumption, the weighted sum of the edge-flows of f is divisible by 4. By Proposition 8.4, the weighted sum of the edge-flows of H is also divisible by 4. Therefore the weighted sum of the edge-flows of Q must also be divisible by 4. So there is no harm in replacing f with Q.

Step 3. Completion of the proof. From Step 2, we may assume that f is a sum of some number of basic 4-cycles. The weighted sum of the edge-flows of any basic 4-cycle is ± 2, so we conclude that the number of 4-cycles in the sum is even. Thus, Proposition 6.2 implies $f \in \mathcal{H}$, as desired. □

Corollary 8.6. *If $s \in S$, such that*

- *$|S'| = 4$, and*
- *$\mathcal{H}$ contains some basic 4-cycle C of X',*

then $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$.

Proof. We may assume that either

- X' is the square of an even cycle, or
- X' is not bipartite, and $|G|$ is not divisible by 4,

for otherwise Proposition 7.1 implies $\mathcal{E}' \subseteq \mathcal{H}'$.

Given any even flow $f \in \mathcal{E}'$, we wish to show that $f \in \mathcal{H} + \mathcal{H}'$. Under the weighting of X' specified in Proposition 8.1 or 8.2 (as appropriate), the weighted sum of the edge-flows of C is ± 2, and, because f is an even flow, it is not difficult to see that the weighted sum of the edge-flows of f is even. Therefore, there is an integer m, such that the weighted sum of the edge-flows of $f - mC$ is 0. Therefore, Proposition 8.1 or 8.2 (as appropriate) asserts that $f - mC \in \mathcal{H}'$. Because $mC \in \mathcal{H}$, we conclude that $f \in \mathcal{H} + \mathcal{H}'$, as desired. $\square$

Corollary 8.7. *If $s, t \in S$, such that*

- $|S'| = 4$,
- $|G'|$ *is not divisible by* 4,
- X' *is not the square of an even cycle,*
- $t \in S'$,
- $|t|$ *is odd, and*
- $\mathcal{H}$ *contains the flow* $2(t^{|t|})$,

then $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$.

Proof. Let $C = 2(t^{|t|})$. Under the weighting of X' specified in Proposition 8.2, the sum of the edge-flows of C is $2|t| \equiv 2 \pmod 4$. Thus, for any $f \in \mathcal{E}'$, the weighted sum of the edge-flows of either f or $f - C$ is is divisible by 4. Therefore, Proposition 8.2 asserts that either $f \in \mathcal{H}'$ or $f - C \in \mathcal{H}'$. Because $C \in \mathcal{H}$, we conclude that $f \in \mathcal{H} + \mathcal{H}'$, as desired. $\square$

9 The proof of Lemma 8.4

This entire section is devoted to the proof of Lemma 8.4. (None of the definitions, notation, or intermediate results are utilized in other sections of the paper.) After embedding X on the torus (see 9.3), we assign an integer (mod 4), called the "imbalance" (see 9.6(2)) to certain cycles (namely, those that are "essential" and have even length). Then we show that this geometrically-defined invariant can be used to calculate the weighted sum of the edge-flows of the cycle (see 9.25). Lemma 8.4 follows easily from this formula.

Assumption 9.1. Throughout this section, S, G, t, u, and the weighting of X are as described in Proposition 8.2.

Notation 9.2. We use

- $\mathrm{wt}(f)$ to denote the weighted sum of the edge-flows of a flow f, and
- $\mathrm{len}(P)$ to denote the length of a path P.

Definition 9.3.

(1) Define

- $m = |t|$,
- $n = |G : \langle t \rangle|$, and
- choose an *even* integer r, such that $u^n = t^r$.

(2) Embed X on the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, by identifying the vertex $t^a u^b$ of X with the point

$$\left(\frac{a}{m} + \frac{rb}{mn}, \frac{b}{n} \right)$$

of $\mathbf{T}^2$, and embedding the edges in the natural way (as line segments).

Notation 9.4. Suppose P is any path in X, and C is any cycle in X, such that neither the initial vertex nor the terminal vertex of P lies on C. Intuitively, we would like to

$$\text{define } \chi_C(P) \text{ to be the parity of}$$
$$\text{the number of times that } P \text{ crosses } C.$$

(Note that if P coincides with C on some subpath, then this is counted as a crossing if (and only if) P exits C on the opposite side from the one it entered on.) It would be possible to formalize the definition in purely combinatorial terms, but we find it convenient to use a topological approach.

We may think of P as a continuous curve on the torus, and C as a knot (or loop) on the torus. By perturbing P slightly, we can obtain a curve P' on the torus, with the same endpoints as P, such that P' is homotopic to P, and every intersection of P' with C is transverse (and is not a double point of P'). Let

$$\chi_C(P) = |P' \cap C| \pmod{2}.$$

This is well-defined (modulo 2) because P' is homotopic to P (cf. [ST, §§73–74]).

Definition 9.5. A cycle C in X is *essential* if the corresponding knot on the torus is *not* homotopic to a point.

More concretely, a cycle $[v](s_1, \ldots, s_n)$ is essential if and only if either

$$|\{\, i \mid s_i = t \,\}| \neq |\{\, i \mid s_i = t^{-1} \,\}|$$

or

$$|\{\, i \mid s_i = u \,\}| \neq |\{\, i \mid s_i = u^{-1} \,\}|.$$

Definition 9.6. Let C be any essential, even cycle in X.

(1) For two vertices v and w in $X \setminus C$, we say that v *and* w *have the same color* (*with respect to* C), if

$$\operatorname{len}(P) + \chi_C(P) \text{ is even,}$$

where P is any path in X from v to w. (This is independent of the choice of the path P (see Lemma 9.13(2) below).) This is an equivalence relation on $V(X \smallsetminus C)$, and has (no more than) two equivalence classes.

We may refer to the vertices in one equivalence class as being "black," and the vertices in the other equivalence class as being "white." This is a 2-coloring of $X \smallsetminus C$.

(2) If K and W are the number of vertices of $X \smallsetminus C$ that are black, and the number that are white, respectively, we define the *imbalance* $\operatorname{imb}(C)$ to be

$$\operatorname{imb}(C) = K - W \pmod 4.$$

Because $K + W = |G| - \operatorname{len}(C)$ is even, we know that $K - W$ is either 0 or $2 \pmod 4$; therefore, $K - W \equiv W - K \pmod 4$, so $\operatorname{imb}(C)$ is well-defined (modulo 4), independent of the choice of which equivalence class is colored black and which is colored white.

Observation 9.7. Because this concept is the foundation of this entire section, we describe an alternate approach to the 2-coloring that determines $\operatorname{imb}(C)$. The graph X has a natural double cover X_2 that is bipartite. Specifically,

$$X_2 = \operatorname{Cay}\Bigl(G_2; \{t_2, u_2\}\Bigr),$$

where

$$G_2 = \langle\, t_2, u_2 \mid\ t_2^{2m} = e,\ u_2^n = t_2^r,\ t_2 u_2 = u_2 t_2 \,\rangle.$$

The inverse image of C in X_2 consists of two disjoint cycles $\widetilde{C}_1$ and $\widetilde{C}_2$, with $\operatorname{len}(\widetilde{C}_1) = \operatorname{len}(\widetilde{C}_2) = \operatorname{len}(C)$. (This would be false if C were not an even cycle.) There is a natural embedding of X_2 on the torus $\mathbf{T}^2$, and $\mathbf{T}^2 \smallsetminus (\widetilde{C}_1 \cup \widetilde{C}_2)$ has exactly two connected components. (This would be false if C were not essential.) Choose one connected component $\widetilde{X}_2^\circ$. The vertices in $\widetilde{X}_2^\circ$ are in one-to-one correspondence with the vertices in $X \smallsetminus C$. Because X_2 is bipartite, the vertices in $\widetilde{X}_2^\circ$ have a natural 2-coloring. Under the natural correspondence with $V(X \smallsetminus C)$, this is precisely the 2-coloring defined above, up to the arbitrary choice of which equivalence class will be black and which will be white.

To see that this is the same 2-coloring, note that:

- Each vertex v of X has two inverse images $\widetilde{v}_1$ and $\widetilde{v}_2$ in X_2, one in each component of $\mathbf{T}^2 \smallsetminus (\widetilde{C}_1 \cup \widetilde{C}_2)$.
- Any path from $\widetilde{v}_1$ to $\widetilde{v}_2$ in X_2 has odd length, so $\widetilde{v}_1$ and $\widetilde{v}_2$ are of opposite color under the 2-coloring of X_2.
- A continuous curve in $\mathbf{T}^2$ crosses $\widetilde{C}_1 \cup \widetilde{C}_2$ an odd number of times if and only if its two endpoints are in different components of $\mathbf{T}^2 \smallsetminus (\widetilde{C}_1 \cup \widetilde{C}_2)$ (unless the curve has an endpoint on $\widetilde{C}_1 \cup \widetilde{C}_2$).

Therefore, two vertices $\widetilde{v}$ and $\widetilde{w}$ of $X_2 \smallsetminus (\widetilde{C}_1 \cup \widetilde{C}_2)$ have the same color if and only if $\operatorname{len}(\widetilde{P}) + \chi_{\widetilde{C}_1 \cup \widetilde{C}_2}(\widetilde{P})$ is even, where $\widetilde{P}$ is any path in X_2 from $\widetilde{v}$ to $\widetilde{w}$.

This establishes that this alternate approach is indeed consistent with Definition 9.6. Furthermore, we see that replacing X_2° with the other component of $\mathbf{T}^2 \smallsetminus (\widetilde{C}_1 \cup \widetilde{C}_2)$ reverses the color of each vertex of $X \smallsetminus C$.

Notation 9.8. For any essential, even cycle C in X, let

$$\operatorname{wli}(C) = \operatorname{wt}(C) + \operatorname{len}(C) + \operatorname{imb}(C).$$

Our main task in this section is to show that if C is any essential, even cycle in X, then $\operatorname{wli}(C) \equiv 2 \pmod 4$ (cf. 9.25). (This is accomplished by reducing to the case where C is "increasing" (see 9.17 and 9.20), and then calculating the imbalance in this special case (see 9.24).) Once this formula has been established, it will be easy to prove Lemma 8.4.

Notation 9.9. Let

$$\widetilde{t} = (1,0) \text{ and } \widetilde{u} = (0,1),$$

and define

$$\widetilde{X} = \operatorname{Cay}\Big(\mathbf{Z} \oplus \mathbf{Z}; \{\widetilde{t}^{\pm 1}, \widetilde{u}^{\pm 1}\}\Big).$$

Note that:

(1) There is a natural covering map π from $\widetilde{X}$ to X, defined by $\pi(\widetilde{t}^i \widetilde{u}^j) = t^i u^j$, and

(2) $\widetilde{X}$ has a natural embedding in the plane $\mathbf{R}^2$.

Definition 9.10. Suppose P is a path in X and $\widetilde{P}$ is a path in $\widetilde{X}$. We say that $\widetilde{P}$ is a *lift* of P if $\pi(\widetilde{P}) = P$.

Note that if P is a path in X, and $\widetilde{v}$ is any vertex of $\widetilde{X}$, such that $\pi(\widetilde{v})$ is the initial vertex of P, then there is a unique lift $\widetilde{P}$ of P, such that the initial vertex of $\widetilde{P}$ is $\widetilde{v}$. Namely, if $P = [\pi(\widetilde{v})](s_1, \ldots, s_\ell)$, then $\widetilde{P} = [\widetilde{v}](\widetilde{s_1}, \ldots, \widetilde{s_\ell})$. Notice that $\operatorname{len}(\widetilde{P}) = \operatorname{len}(P)$.

Definition 9.11. Suppose C is a cycle in X, and $\widetilde{P}$ is any lift of C to a path in $\widetilde{X}$. If (x_1, y_1) and (x_2, y_2) are the initial vertex of $\widetilde{P}$ and the terminal vertex of $\widetilde{P}$, respectively, then, because C is a cycle, there exist $p, q \in \mathbf{Z}$ with

$$(x_2, y_2) - (x_1, y_1) = p\,(m, 0) + q\,(-r, n).$$

The *knot class* $\operatorname{knot}(C)$ of C is the ordered pair (p, q). (Because we have not specified an orientation of C, this is well-defined only up to a sign; that is, we do not distinguish the knot class (p, q) from the knot class $(-p, -q)$. In our applications, it is only the parities of p and q that are relevant, and the parities are not affected by a change of sign.)

In topological terms, p is the number of times that the knot corresponding to C wraps around the torus longitudinally, and q is the number of times that the knot wraps around the torus meridionally [R, pp. 17–19].

Observation 9.12. Let C be a cycle in X. Then the following are equivalent:

(a) C is essential.
(b) $\operatorname{knot}(C) \neq (0,0)$ [R, Exer. 14 on p. 25].
(c) If $\widetilde{P}$ is a lift of C to a path in $\widetilde{X}$, then $\widetilde{P}$ is *not* a cycle in $\widetilde{X}$; that is, the terminal vertex of $\widetilde{P}$ is not equal to the initial vertex of $\widetilde{P}$.

Lemma 9.13. *Suppose C is any essential, even cycle in X.*

(1) Let $\widetilde{P}$ be a lift of C to a path in $\widetilde{X}$. If (x_1, y_1) and (x_2, y_2) are the initial vertex of $\widetilde{P}$ and the terminal vertex of $\widetilde{P}$, respectively, then $y_2 - y_1 \equiv 2 \pmod 4$.
(2) Let
- *v and w be vertices of X that do not lie on C, and*
- *P and Q be paths in X from v to w.*

Then $\operatorname{len}(P) + \chi_C(P) \equiv \operatorname{len}(Q) + \chi_C(Q) \pmod 2$.

Proof. (1) We may assume $(x_1, y_1) = (0,0)$. Let $\operatorname{knot}(C) = (p,q)$. Then

$$(x_2, y_2) = p\,(m,0) + q\,(-r,n) = (pm - rq, qn).$$

Note that, because m is odd, but r and n are even, we have

$$\operatorname{len}(C) \equiv x_2 + y_2 = pm - rq + qn \equiv p \pmod 2. \tag{E6}$$

Since C is an even cycle, we conclude that p is even.

Because (p,q) is the knot class of an essential knot (namely, C), a theorem of topology asserts that $\gcd(p,q) = 1$ [R, p. 19] (this can also be proved combinatorially). Because p is even, this implies that q is odd. Combining this with the fact that $n \equiv 2 \pmod 4$, we conclude that

$$y_2 - y_1 = y_2 = qn \equiv 2 \pmod 4,$$

as desired.

(2) It suffices to show that if C' is any cycle in X, then $\operatorname{len}(C') + \chi_C(C')$ is even. Let $\operatorname{knot}(C') = (p', q')$. A theorem of topology [R, Exer. 7 on p. 28] asserts that $\chi_C(C') \equiv pq' - p'q \pmod 2$. From the proof of (1), we know that p is even and q is odd. Thus, $\chi_C(C') \equiv p' \pmod 2$. Furthermore, Eq. (E6), with C' in the role of C, shows that $\operatorname{len}(C') \equiv p' \pmod 2$. Thus, $\operatorname{len}(C') + \chi_C(C') \equiv p' + p' \pmod 2$ is even. ∎

Definition 9.14. If C is any cycle in X that is **not** essential, then C may be lifted to a cycle $\widetilde{C}$ in $\widetilde{X}$ (see 9.12). The cycle $\widetilde{C}$ has a well-defined interior and exterior in $\mathbf{R}^2$. Let $\widetilde{I}$ be the set of vertices of $\widetilde{X}$ in the interior of $\widetilde{C}$. Then the restriction of π to $\widetilde{I}$ is one-to-one and maps onto a subset of $X \smallsetminus C$ that is independent of the choice of the lift $\widetilde{C}$. We say that the vertices in $\pi(\widetilde{I})$ are *the vertices in the interior of the region bounded by C*.

Lemma 9.15 (Pick's Theorem)**.** *If C is any cycle in X that is **not** essential, and N is the number of vertices in the region enclosed by C, then*

$$\mathrm{wt}(C) \equiv \mathrm{len}(C) + 2N - 2 \pmod 4.$$

Proof. Lift C to a cycle $\widetilde{C}$ in $\widetilde{X}$, and let A be the area of the region bounded by $\widetilde{H}$. Then C is the sum of A basic 4-cycles (and the weight of any basic 4-cycle is ± 2), so

$$\mathrm{wt}(C) \equiv 2A \pmod 4.$$

From Pick's Theorem [H, pp. 27–31], we know that

$$2A = \mathrm{len}(\widetilde{C}) + 2N - 2.$$

Because $\mathrm{len}(\widetilde{C}) = \mathrm{len}(C)$, the desired conclusion is immediate. □

It will be helpful to know that making certain changes to C does not affect $\mathrm{wli}(C) \pmod 4$.

Example 9.16. Let C be an essential, even cycle in X.

(1) Suppose C contains the subpath $[v](s,t,s^{-1})$, for some $v \in G$ and some $s,t \in S$. Let C' be the cycle obtained from C by replacing this subpath with the single edge $[v](t)$. Then:
  - $\mathrm{wt}(C) - \mathrm{wt}(C') = \mathrm{wt}\big([v](s,t,s^{-1},t^{-1})\big) = \pm 2$,
  - $\mathrm{len}(C) - \mathrm{len}(C') = 2$, and
  - $\mathrm{imb}(C) - \mathrm{imb}(C') = 0$, because $X \setminus C' = (X \setminus C) \cup \{vs, vst\}$ and the two additional vertices are of opposite color (because they are adjacent in X).

  So $\mathrm{wli}(C) - \mathrm{wli}(C') \equiv 0 \pmod 4$.

(2) Suppose C contains the subpath $[v](s,t)$, for some $v \in G$ and some $s,t \in S$. If the vertex $[v](t)$ is not on C, let C' be the cycle obtained from C by replacing the subpath $[v](s,t)$ with the subpath $[v](t,s)$. Then:
  - $\mathrm{wt}(C) - \mathrm{wt}(C') = \mathrm{wt}\big([v](s,t,s^{-1},t^{-1})\big) = \pm 2$,
  - $\mathrm{len}(C) - \mathrm{len}(C') = 0$, and
  - $\mathrm{imb}(C) - \mathrm{imb}(C') = \pm 2$, because the symmetric difference of $X \setminus C$ and $X \setminus C'$ is $\{vs, vt\}$, and the color of vt in $X \setminus C$ is the opposite of the color of vs in $X \setminus C'$ (because they are an even distance apart, but on opposite sides of the two cycles).

So $\mathrm{wli}(C) - \mathrm{wli}(C') \equiv 0 \pmod 4$.

The following result is a weak form of the assertion that changing C by a homotopy does not change $\mathrm{wli}(C)$. On an intuitive level, it can be justified by claiming that a sequence of the two types of replacements described in Example 9.16 will transform C into $C' = C - P + Q$. Our formal proof takes a different approach.

Proposition 9.17. *Suppose*

- *C is an essential, even cycle in X,*
- *P is a subpath of C,*
- *Q is a path in X,*
- *the initial vertex and terminal vertex of Q are the same as those of P,*
- *Q does not intersect C (except at the endpoints of Q), and*
- *the cycle $P - Q$ is not essential.*

Then $C - P + Q$ is an essential, even cycle, and $\mathrm{wli}(C - P + Q) \equiv \mathrm{wli}(C) \pmod 4$.

Proof. Because $P - Q$ is not essential, and $\widetilde{X}$ is bipartite, it is clear that $P - Q$ is an even cycle (cf. 9.12). So $C - P + Q$ is an even cycle. It is also easy to see that $C - P + Q$, like C, must be essential. For example, one may note that

$$\begin{aligned}
\mathrm{knot}(C - P + Q) &= \mathrm{knot}(C) - \mathrm{knot}(P - Q) \\
&= \mathrm{knot}(C) - (0, 0) \\
&\neq (0, 0).
\end{aligned}$$

Letting N be the number of vertices in the region enclosed by $P - Q$, and applying Pick's Theorem (9.15), we have

$$\begin{aligned}
\mathrm{wt}(C) - \mathrm{wt}(C - P + Q) &= \mathrm{wt}(P - Q) \\
&\equiv \mathrm{len}(P - Q) + 2N - 2 \pmod 4 \qquad \text{(E7)} \\
&= \mathrm{len}(P) + \mathrm{len}(Q) + 2N - 2.
\end{aligned}$$

Obviously,

$$\mathrm{len}(C) - \mathrm{len}(C - P + Q) = \mathrm{len}(P) - \mathrm{len}(Q). \qquad \text{(E8)}$$

Also, we know, from Lemma 9.18 below, that

$$\mathrm{imb}(C) - \mathrm{imb}(C - P + Q) \equiv 2N + 2\,\mathrm{len}(P) - 2 \pmod 4. \qquad \text{(E9)}$$

Combining (E7), (E8), and (E9), we obtain

$$\begin{aligned}
\operatorname{wli}(C)-\operatorname{wli}(C-P+Q) &= \Big(\operatorname{wt}(C)-\operatorname{wt}(C-P+Q)\Big)\\
&\qquad+\Big(\operatorname{len}(C)-\operatorname{len}(C-P+Q)\Big)\\
&\qquad\qquad+\Big(\operatorname{imb}(C)-\operatorname{imb}(C-P+Q)\Big)\\
&\equiv \Big(\operatorname{len}(P)+\operatorname{len}(Q)+2N-2\Big)\\
&\qquad+\Big(\operatorname{len}(P)-\operatorname{len}(Q)\Big)\\
&\qquad\qquad+\Big(2N+2\operatorname{len}(P)-2\Big)\\
&= 4\operatorname{len}(P)+4N-4\\
&\equiv 0 \pmod 4,
\end{aligned}$$

so $\operatorname{wli}(C)\equiv\operatorname{wli}(C-P+Q) \pmod 4$, as desired. $\square$

Lemma 9.18. *Under the assumptions of Proposition 9.17, and letting N be the number of vertices in the region enclosed by $P-Q$, we have* $\operatorname{imb}(C)-\operatorname{imb}(C-P+Q)\equiv 2N+2\operatorname{len}(P)-2 \pmod 4$.

Proof. We use the description of $\operatorname{imb}(C)$ given in Remark 9.7.

- Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be the two lifts of C to cycles in X_2.
- Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be the two lifts of P to paths in X_2, with $\widetilde{P}_1\subseteq\widetilde{C}_1$ and $\widetilde{P}_2\subseteq\widetilde{C}_2$.
- Let $\widetilde{Q}_1$ and $\widetilde{Q}_2$ be the two lifts of Q to paths in X_2, such that $\widetilde{Q}_1$ has the same endpoints as $\widetilde{P}_1$, and $\widetilde{Q}_2$ has the same endpoints as $\widetilde{P}_2$.
- For a curve R on $\mathbf{T}^2$, we use R° to denote the *interior* of R; that is $R^\circ = R\smallsetminus\{v_1,v_2\}$, where v_1 and v_2 are the endpoints of R.
- Let $\widetilde{X}_2^\circ$ be the component of $\mathbf{T}^2\smallsetminus(\widetilde{C}_1\cup\widetilde{C}_2)$ that contains $\widetilde{Q}_1^\circ$.
- Let $\widetilde{I}_1$ and $\widetilde{I}_2$ be the interior regions in $\mathbf{T}^2$ that are bounded by the cycles $\widetilde{P}_1-\widetilde{Q}_1$ and $\widetilde{P}_2-\widetilde{Q}_2$, respectively.
- For any subset A of $\mathbf{T}^2$, we let K_A and W_A, respectively, be the number of black vertices of X_2 that are contained in A, and the number of white vertices of X_2 that are contained in A. We define $\operatorname{kw}(A)=K_A-W_A$.

Note that:

- The inverse image of $C-P+Q$ in X_2 is

  $$(\widetilde{C}_1-\widetilde{P}_1+\widetilde{Q}_1)\cup(\widetilde{C}_2-\widetilde{P}_2+\widetilde{Q}_2),$$

  and one of the components of the complement of this inverse image is

  $$\Big(X_2^\circ\smallsetminus(\widetilde{I}_1\cup\widetilde{Q}_1^\circ)\Big)\cup(\widetilde{I}_2\cup\widetilde{P}_2^\circ).$$

Therefore,

$$\begin{aligned}\operatorname{imb}(C - P + Q) = \operatorname{kw}(X_2^\circ) - \operatorname{kw}(\widetilde{I}_1) - \operatorname{kw}(\widetilde{Q}_1^\circ) \\ + \operatorname{kw}(\widetilde{I}_2) + \operatorname{kw}(\widetilde{P}_2^\circ).\end{aligned}$$

- From the definition of $\operatorname{imb}(C)$, we have

$$\operatorname{kw}(X_2^\circ) = \operatorname{imb}(C).$$

- For any vertex $\widetilde{v}_1$ of $\widetilde{I}_1$, the image v of $\widetilde{v}$ in X is a vertex in the interior of the region enclosed by $P - Q$. Thus, there is a vertex $\widetilde{v}_2$ in $\widetilde{I}_2$ whose image is also v. It was noted in Remark 9.7 that the color of $\widetilde{v}_1$ must be the opposite of the color or $\widetilde{v}_2$. From this (and the same argument with $\widetilde{I}_1$ and $\widetilde{I}_2$ interchanged), we conclude that $K_{\widetilde{I}_1} = W_{\widetilde{I}_2}$ and $W_{\widetilde{I}_1} = K_{\widetilde{I}_2}$. Therefore

$$\begin{aligned}-\operatorname{kw}(\widetilde{I}_1) + \operatorname{kw}(\widetilde{I}_2) &= -(K_{\widetilde{I}_1} - W_{\widetilde{I}_1}) + (K_{\widetilde{I}_2} - W_{\widetilde{I}_2}) \\ &= -(W_{\widetilde{I}_2} - K_{\widetilde{I}_2}) + (K_{\widetilde{I}_2} - W_{\widetilde{I}_2}) \\ &= 2(K_{\widetilde{I}_2} - W_{\widetilde{I}_2}) \\ &\equiv 2(K_{\widetilde{I}_2} + W_{\widetilde{I}_2}) \pmod 4 \\ &= 2N.\end{aligned}$$

- If P and Q are of odd length, then each has an even number of vertices, half black and half white (and the same is true of their interiors), so $\operatorname{kw}(\widetilde{Q}_1^\circ) = 0$ and $\operatorname{kw}(\widetilde{P}_2^\circ) = 0$. Because $2\operatorname{len}(P) - 2$ is divisible by 4, we have

$$\begin{aligned}-\operatorname{kw}(\widetilde{Q}_1^\circ) + \operatorname{kw}(\widetilde{P}_2^\circ) &= 0 \\ &\equiv 2\operatorname{len}(P) - 2 \pmod 4.\end{aligned} \tag{E10}$$

  On the other hand, if P and Q are of even length, then $\operatorname{kw}(\widetilde{Q}_1^\circ) = \pm 1$ and $\operatorname{kw}(\widetilde{P}_2^\circ) = \pm 1$. Assume, without loss of generality, that $\operatorname{kw}(\widetilde{Q}_1^\circ) = -1$. Then the interior of $\widetilde{Q}_1$ has an extra white vertex, so the endpoints of $\widetilde{Q}_1$ must both be black. These are the same as the endpoints of $\widetilde{P}_1$, so the endpoints of $\widetilde{P}_1$ must both be black. The endpoints of $\widetilde{P}_2$ are of the opposite color (because they project to the same vertices of X); they must both be white. Hence, the interior of $\widetilde{P}_2$ has an extra black vertex, so $\operatorname{kw}(\widetilde{P}_2^\circ) = 1$. Noting that $2\operatorname{len}(P) - 2 \equiv 2 \pmod 4$ in the current case, we conclude that

$$\begin{aligned}-\operatorname{kw}(\widetilde{Q}_1^\circ) + \operatorname{kw}(\widetilde{P}_2^\circ) &= -(-1) + 1 \\ &\equiv 2\operatorname{len}(P) - 2 \pmod 4,\end{aligned}$$

  which has the same form as the conclusion of (E10).

The desired conclusion is obtained by combining these calculations. □

Definition 9.19. A path $[v](s_1, \ldots, s_n)$ in X is *increasing* if

$$\{s_1, \ldots, s_n\} \subseteq \{t, u\};$$

that is, if no s_i is equal to t^{-1} or u^{-1}.

Corollary 9.20. *If C is any essential, even cycle in X, then there is an essential, even cycle C' in X, such that*

(1) $\operatorname{wli}(C') \equiv \operatorname{wli}(C) \pmod 4$*, and*
(2) C' is increasing.

Proof. We may assume, without loss of generality, that

$$\begin{gathered}\text{there does not exist an essential, even cycle } C' \text{ in } x,\\ \text{such that } \operatorname{wli}(C') = \operatorname{wli}(C) \text{ and } \operatorname{len}(C') < \operatorname{len}(C).\end{gathered} \tag{E11}$$

Lift C to a path $\widetilde{C}$ in $\widetilde{X}$. Assume, for simplicity, that the initial vertex of $\widetilde{C}$ is $(0,0)$, and let (a,b) be the terminal vertex of $\widetilde{C}$. We may assume, by interchanging t with t^{-1} and/or u with u^{-1}, if necessary, that $a \geq 0$ and $b \geq 0$.

Suppose C is not increasing (this will lead to a contradiction). Then either t^{-1} or u^{-1} appears in C. The argument in the two cases is similar, so let us assume, for definiteness, that an edge of the form $[v](t^{-1})$ appears in C. There is no harm in assuming $v = 0$.

Because $b \geq 0$, the line $x = -1/2$ separates $(-1,0)$ from (a,b). Thus, at some point, the path $\widetilde{C}$ must cross this line from left to right. In other words, $\widetilde{C}$ contains an edge of the form $\big[(-1,y)\big](t)$. We may assume $|y|$ is minimal.

We now consider the vertical segment from $v = (0,0)$ to $(0,y)$. To simplify notation, without losing the main idea, let us assume $y > 0$. Let $d_1, d_2 \geq 0$ be maximal, such that C contains the paths $[u^y]\big(u^{-d_1}\big)$ and $[u^{d_2}]\big(u^{-d_2}\big)$.

Let P be the subpath of C from u^{d_2} to u^{y-d_1}, and let Q be the path $[u^{d_2}](u^{d_1-d_2})$.

Note that:

(1) The path Q is disjoint from C, except at its endpoints. (This follows from the minimality of $|y|$ and the maximality of d_1 and d_2.)
(2) The cycle $P - Q$ is not essential (because it lifts to a cycle in $\widetilde{X}$, namely, the subpath of $\widetilde{C}$ from $(0,0)$ to $(0,y)$, plus the vertical path from $(0,y)$ to $(0,0)$).

(3) The cycle $C-P+Q$ is strictly shorter than C. (The length of Q is strictly less than the length of P, because Q is a straight (vertical) path, but P, which includes the horizontal edge $[v](t^{-1})$, is inefficient.)

From Proposition 9.17, we see that $\mathrm{wli}(C-P+Q) = \mathrm{wli}(C)$. Because $C-P+Q$ is strictly shorter than C, this contradicts (E11). $\square$

In order to calculate the imbalance of an increasing cycle C, we formulate a more convenient description of the 2-coloring of $X \smallsetminus C$ in this case.

Notation 9.21. Let C be any increasing, essential, even cycle in X.

(1) For vertices u and v of C, we let $\mathrm{dist}_C(u, v)$ be the length of the shortest subpath of C with endpoints u and v.

(2) For any vertex $v \in X \smallsetminus C$, it is easy to see that there is a vertex $c(v)$ of C, and some $h(v) \in \mathbf{Z}^+$, such that

$$c(v)\, t^{h(v)} = v.$$

Furthermore, we assume that

$$\{c(v)t, c(v)t^2, \ldots, c(v)t^{h(v)-1}\} \text{ is disjoint from } C.$$

Then $c(v)$ and $h(v)$ are uniquely determined by v.

(3) Similarly, for any vertex $v \in X \smallsetminus C$, there is a unique vertex $c'(v)$ of C, and a unique $h'(v) \in \mathbf{Z}^+$, such that

$$c'(v)\, t^{-h'(v)} = v$$

and

$$\{vt, vt^2, \ldots, vt^{h'(v)-1}\} \text{ is disjoint from } C.$$

Lemma 9.22. *Let C be any increasing, essential, even cycle in X, and fix a vertex c_0 of C. Then the black vertices can be distinguished from the white vertices by the parity of $\mathrm{dist}_C\big(c_0, c(v)\big) + h(v)$.*

In other words, two vertices v and w of $X \smallsetminus C$ have the same color (w.r.t. C) if and only if

$$\mathrm{dist}_C\big(c_0, c(v)\big) + h(v) \equiv \mathrm{dist}_C\big(c_0, c(w)\big) + h(w) \pmod 2.$$

Proof. Let P_0 be a path of length $\mathrm{dist}_C\big(c(v), c(w)\big)$ from $c(v)$ to $c(w)$ in C, and let $P = [v](t^{-h(v)}, P_0, t^{h(w)})$. Then P is a path from v to w, such that

$$\mathrm{len}(P) = \mathrm{dist}_C\big(c(v), c(w)\big) + h(v) + h(w),$$

and it is easy to see that $\chi_C(P) = 0$. (Because C is increasing, translating P_0 by the vector
$$\left(\frac{1}{2m} - \frac{r}{2mn}, -\frac{1}{2n}\right)$$
in $\mathbf{T}^2$ results in a path that is disjoint from C, so it is easy to find a path homotopic to P that is disjoint from C.) Therefore, v and w have the same color if and only if $\operatorname{len}(P)$ is even. The desired conclusion follows. $\square$

Remark. In Lemma 9.22, one could use $c'(v)$ and $h'(v)$, in place of $c(v)$ and $h(v)$: vertices v and w of $X \smallsetminus C$ have the same color if and only if

$$\operatorname{dist}_C\Big(c_0, c'(v)\Big) + h'(v) \equiv \operatorname{dist}_C\Big(c_0, c'(w)\Big) + h'(w) \pmod 2.$$

Note that, because X is *not* bipartite, there is some $v \in X \smallsetminus C$, such that

$$\operatorname{dist}_C\Big(c_0, c(v)\Big) + h(v) \not\equiv \operatorname{dist}_C\Big(c_0, c'(v)\Big) + h'(v) \pmod 2. \tag{E12}$$

Then it is easy to see that we have inequality for *all* $v \in X \smallsetminus C$ (and all $c_0 \in C$).

Definition 9.23. A t-edge $[t^i u^j](t)$ of X is said to be

$$\begin{cases} \text{blue} & \text{if } i \text{ is even;} \\ \text{red} & \text{if } i \text{ is odd.} \end{cases}$$

Proposition 9.24. *If C is any increasing, essential, even cycle in X, then* $\operatorname{imb}(C) \equiv 2B \pmod 4$*, where B is the number of blue edges in C.*

Proof. Because C is increasing, there is a sequence $(\ell_1, \ldots, \ell_z)$ of natural numbers, and some $v_0 \in C$, such that

$$C = [v_0](u, t^{\ell_1}, u, t^{\ell_2}, \ldots, u, t^{\ell_z}). \tag{E13}$$

From Lemma 9.13(1), we know that

$$z \text{ is even.}$$

There is no harm in assuming that $v_0 \in \langle t \rangle$, so

$$B = \sum_{j=1}^{z/2} \ell_{2j}. \tag{E14}$$

For $1 \le i \le z$, let

- $c_i = v_0(ut^{\ell_1})(ut^{\ell_2}) \cdots (ut^{\ell_i})$,

- h_i be the unique positive integer, such that

$$c_i t^{h_i} \in C \text{ and } \{c_i t, c_i t^2, \ldots, c_i t^{h_i - 1}\} \text{ is disjoint from } C,$$

- $c_i' = c_i t^{h_i}$,
- $d_i = \operatorname{dist}_C\big(v_0, c_i\big)$,
- $d_i' = \operatorname{dist}_C\big(v_0, c_i'\big)$,
- K be the number of black vertices in $X \smallsetminus C$, and
- W be the number of white vertices in $X \smallsetminus C$.

Note that $X \smallsetminus C$ is the disjoint union of the paths $[c_i t](t^{h_i - 1})$, with the convention that this is empty if $h_i = 0$. Thus, we may calculate $K - W$ by determining the excess (which is negative if there is actually a deficiency) of black vertices in each of these paths, and adding up the results.

A path in $X \smallsetminus C$:

- has an excess of 1 black vertex if and only if both of its endpoints are black,
- has an excess of -1 black vertex if and only if both of its endpoints are white, and
- has an excess of 0 black vertices otherwise.

By interchanging black with white if necessary, we may assume that a vertex v of $X \smallsetminus C$ is black if and only if $\operatorname{dist}_C\big(v_0, c(v)\big) + h(v)$ is even (see 9.22).

- Thus, the initial vertex of the path $[c_i t](t^{h_i - 1})$ is black if and only if d_i is odd.
- Using (E12), we see that the terminal vertex of the path $[c_i t](t^{h_i - 1})$ is black if and only if d_i' is even.

Therefore

$$\begin{aligned}
K - W &= |\{\, i \mid d_i \text{ is odd and } d_i' \text{ is even} \,\}| \\
&\qquad - |\{\, i \mid d_i \text{ is even and } d_i' \text{ is odd} \,\}| \\
&= \Big(|\{\, i \mid d_i \text{ is odd and } d_i' \text{ is even} \,\}| \\
&\qquad\qquad + |\{\, i \mid d_i \text{ is odd and } d_i' \text{ is odd} \,\}| \Big) \\
&\qquad - \Big(|\{\, i \mid d_i \text{ is even and } d_i' \text{ is odd} \,\}| \\
&\qquad\qquad\qquad + |\{\, i \mid d_i \text{ is odd and } d_i' \text{ is odd} \,\}| \Big) \\
&= |\{\, i \mid d_i \text{ is odd} \,\}| - |\{\, i \mid d_i' \text{ is odd} \,\}|.
\end{aligned}$$

Note that a vertex w of C belongs to

$$\begin{cases} \{c'_1, \ldots, c'_b\} & \text{if and only if the edge } [w](u^{-1}) \text{ is in } C, \\ \{c_1, \ldots, c_b\} & \text{if and only if the edge } [w](u) \text{ is in } C. \end{cases}$$

This implies that $\{c'_1 u^{-1}, \ldots, c'_b u^{-1}\} = \{c_1, \ldots, c_b\}$, so

$$|\{\, i \mid d'_i \text{ is odd} \,\}| = |\{\, i \mid d_i \text{ is even} \,\}| = z - |\{\, i \mid d_i \text{ is odd} \,\}|.$$

Therefore

$$\begin{aligned} K - W &= |\{\, i \mid d_i \text{ is odd} \,\}| - \Big(z - |\{\, i \mid d_i \text{ is odd} \,\}|\Big) \\ &= 2|\{\, i \mid d_i \text{ is odd} \,\}| - z. \end{aligned} \tag{E15}$$

From (E13) (and the definition of d_i and c_i), we see that

$$d_i \equiv \sum_{j=1}^{i} (\ell_j + 1) \pmod 2. \tag{E16}$$

Combining this with the fact that

$$z - i + 1 \text{ is even whenever } i \text{ is odd}, \tag{E17}$$

we see that

$$\begin{aligned} |\{i \mid d_i \text{ is odd}\}| &\equiv \sum_{i=1}^{z} d_i \pmod 2 \\ &\equiv \sum_{i=1}^{z} (z - i + 1)(\ell_i + 1) \pmod 2 \qquad \text{(see E16)} \\ &\equiv \sum_{j=1}^{z/2} (\ell_{2j} + 1) \pmod 2 \qquad \text{(see E17).} \end{aligned} \tag{E18}$$

Hence

$$\begin{aligned} \operatorname{imb}(C) &= K - W && \text{(see 9.6)} \\ &= 2|\{i \mid d_i \text{ is odd}\}| - z && \text{(see E15)} \\ &\equiv 2\left(\sum_{j=1}^{z/2} (\ell_{2j} + 1)\right) - z \pmod 4 && \text{(see E18)} \\ &= 2\sum_{j=1}^{z/2} \ell_{2j} \\ &= 2B && \text{(see E14),} \end{aligned}$$

as desired. □

Corollary 9.25. *If C is any essential, even cycle in X, then*

$$\mathrm{wt}(C) \equiv \mathrm{len}(C) + \mathrm{imb}(C) + 2 \pmod 4.$$

Proof. It suffices to show $\mathrm{wli}(C) \equiv 2 \pmod 4$ (see Definition 9.8). From Corollary 9.20, we may assume C is increasing.

Let

- B be the number of blue t-edges in C,
- R be the number of red t-edges in C, and
- U be the number of u-edges in C.

We have:

(1) $\mathrm{wt}(C) = B - R$ (because C is increasing),
(2) $\mathrm{len}(C) = B + R + U$,
(3) $\mathrm{imb}(C) \equiv 2B \pmod 4$ (see 9.24), and
(4) $U \equiv 2 \pmod 4$ (because C is increasing, and see 9.13(1)).

Therefore, modulo 4, we have

$$\begin{aligned}
\mathrm{wli}(C) &= \mathrm{wt}(C) + \mathrm{len}(C) + \mathrm{imb}(C) \\
&\equiv (B - R) + (B + R + U) + 2B \\
&= 4B + U \\
&\equiv 0 + 2 \\
&= 2,
\end{aligned}$$

as desired. □

Proof of Lemma 8.4. Note that $\mathrm{len}(H) = |G| \equiv 2 \pmod 4$.

Case 1. Assume H is essential. Because the complement $X \setminus H$ is empty, it is obvious, from the definition, that $\mathrm{imb}(H) = 0$. Applying Corollary 9.25, we obtain

$$\mathrm{wt}(H) \equiv \mathrm{len}(H) + \mathrm{imb}(H) + 2 \equiv 2 + 0 + 2 \equiv 0 \pmod 4.$$

Case 2. Assume H is not essential. From Pick's Theorem (9.15), we know that

$$\mathrm{wt}(H) \equiv \mathrm{len}(H) + 2N - 2 \pmod 4,$$

where N is the number of lattice points in the interior of the region bounded by H. Because the complement $X \setminus H$ is empty, we have $N = 0$. Therefore

$$\mathrm{wt}(H) \equiv 2 + 2(0) - 2 = 0 \pmod 4,$$

as desired. □

10 4-cycles in graphs containing $P_2 \mathbin{\square} P_3 \mathbin{\square} P_3$

Proposition 10.1 is a major ingredient in our study of graphs of degree at least 5. The gist is that if X contains

- a subgraph Y that is isomorphic to $P_2 \mathbin{\square} P_3 \mathbin{\square} P_3$, and
- an appropriate hamiltonian cycle C in the complement $X \setminus V(Y)$,

then many hamiltonian cycles in Y can be extended to hamiltonian cycles in X. This yields enough hamiltonian cycles to show that $\mathcal{H}$ contains every basic 4-cycle.

Later sections do not appeal directly to this result, but only to Corollaries 10.4, 10.5, 10.6, and 10.7.

Proposition 10.1. *Suppose*

- $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$,
- $Y = \left\{ s^i t^j u^k \;\middle|\; \begin{array}{l} 0 \le i \le 2, \\ 0 \le j \le 2, \\ 0 \le k \le 1 \end{array} \right\}$ *consists of* 18 *distinct elements of* G, *and*
- *either*
  - *(a)* $G = Y$, *or*
  - *(b) there is a hamiltonian cycle* C *in* $X \setminus Y$, *such that* C *contains the edge* $[t^3](s)$.

Then $\mathcal{H}$ *contains every basic* 4*-cycle.*

Before proving the proposition, let us establish a lemma that describes the hamiltonian cycles we will construct.

Lemma 10.2. *Suppose*

(a) $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$,
(b) some element of $\mathcal{H}$ *is the sum of an odd number of basic* 4*-cycles, and*

(c) there exist oriented hamiltonian cycles $H_1, \ldots, H_6$ in X, that contain the following specific oriented paths:
H_1: $[v_1](t, u^{-1}, t^{-1})$ and $[v_1 t s^{-1}](u^{-1})$, for some $v_1 \in X$;
H_2: $[v_2](t, u^{-1}, t^{-1})$ and $[v_2 t^2](u^{-1})$, for some $v_2 \in X$;
H_3: $[v_3](t, u^{-1}, t^{-1})$ and $[v_3 t s](u^{-1})$, for some $v_3 \in X$;
H_4: $[v_4](u^{-1}, t^{-1}, u)$ and $[v_4 u^{-1} s](t^{-1})$, for some $v_4 \in X$;
H_5: $[v_5](t, s, t^{-1})$ and $[v_5 t s u^{-1}](s^{-1})$, for some $v_5 \in X$;
H_6: $[v_6](u, t^{-1}, u^{-1})$ and $[v_6 u t^{-1} s](t)$, for some $v_6 \in X$.

Then $\mathcal{H}$ contains every basic 4-cycle.

Proof. Let us begin by establishing that it suffices to show

(i) $2Q \in \mathcal{H}$, for some basic 4-cycle Q; and
(ii) for any two basic 4-cycles Q_1 and Q_2, we have $Q_1 \equiv \pm Q_2 \pmod{\mathcal{H}}$.

From (i) and (ii), we see that any even multiple of any basic 4-cycle belongs to $\mathcal{H}$. From (b) and (ii), we see that some odd multiple of any basic 4-cycle belongs to $\mathcal{H}$. Subtracting an appropriate even multiple, we conclude that the 4-cycle itself belongs to $\mathcal{H}$, as desired. Thus, (i) and (ii) do indeed suffice.

By applying Lemma 6.1(1) to H_1, H_2, H_3, and H_4, we see that

(1) $(t, u^{-1}, t^{-1}, u) + [t](s^{-1}, u^{-1}, s, u) \in \mathcal{H}$;
(2) $(t, u^{-1}, t^{-1}, u) + [t](t, u^{-1}, t^{-1}, u) \in \mathcal{H}$;
(3) $(t, u^{-1}, t^{-1}, u) + [t](s, u^{-1}, s^{-1}, u) \in \mathcal{H}$; and
(4) $(u^{-1}, t^{-1}, u, t) + [u^{-1}](s, t^{-1}, s^{-1}, t) \in \mathcal{H}$.

By applying Lemma 6.1(2) to H_5 and H_6, we see that

(5) $(t, s, t^{-1}, s^{-1}) - [t](u^{-1}, s, u, s^{-1}) \in \mathcal{H}$; and
(6) $(u, t^{-1}, u^{-1}, t) - [u](s, t^{-1}, s^{-1}, t) \in \mathcal{H}$.

To verify (i), note that, because

$$[u](s, t^{-1}, s^{-1}, t) = [t^{-1}u](t, s, t^{-1}, s^{-1}),$$
$$[u](u^{-1}, s, u, s^{-1}) = -[u](s, u^{-1}, s^{-1}, u),$$

and

$$[t^{-1}u](t, u^{-1}, t^{-1}, u) = -(u, t^{-1}, u^{-1}, t),$$

we have

$$
\begin{aligned}
2(u,t^{-1},u^{-1},t) = &\Big((u,t^{-1},u^{-1},t) - [u](s,t^{-1},s^{-1},t)\Big) && (6)\\
&+ \Big([t^{-1}u](t,s,t^{-1},s^{-1}) - [u](u^{-1},s,u,s^{-1})\Big) && (5)\\
&- \Big([u](s,u^{-1},s^{-1},u) + [t^{-1}u](t,u^{-1},t^{-1},u)\Big) && (3)\\
\in\ & \mathcal{H}.
\end{aligned}
$$

We now establish (ii). Let

$$Q_0 = (t,u^{-1},t^{-1},u).$$

Given any basic 4-cycle Q in X, we will show that $Q \equiv \pm Q_0 \pmod{\mathcal{H}}$.

From (1) and (4), we see that any basic 4-cycle of the form

$$[v](s,u,s^{-1},u^{-1}) \qquad \text{or} \qquad [v](s,t,s^{-1},t^{-1})$$

is congruent, modulo $\mathcal{H}$, to a translate of Q_0. Thus, we may assume $Q = [w]Q_0$, for some $w \in G$, so it suffices to show

(S) $Q_0 \equiv \pm[s]Q_0 \pmod{\mathcal{H}}$;
(T) $Q_0 \equiv \pm[t]Q_0 \pmod{\mathcal{H}}$; and
(U) $Q_0 \equiv \pm[u]Q_0 \pmod{\mathcal{H}}$.

First, note that (T) is immediate from (2).

For (S), noting that

$$[t](s,u^{-1},s^{-1},u) = -[st](s^{-1},u^{-1},s,u),$$

we have

$$
\begin{aligned}
Q_0 + [s]Q_0 = &\Big((t,u^{-1},t^{-1},u) + [t](s,u^{-1},s^{-1},u)\Big) && (3)\\
&+ \Big([s](t,u^{-1},t^{-1},u) + [st](s^{-1},u^{-1},s,u)\Big) && (1)\\
\in\ & \mathcal{H}.
\end{aligned}
$$

For (U), noting that

$$-Q_0 = [tu^{-1}](u,t^{-1},u^{-1},t)$$

and

$$[u]Q_0 = [tu](u^{-1},t^{-1},u,t),$$

we have

$$\begin{aligned}
-Q_0 + [u]Q_0 &= \Big([tu^{-1}](u, t^{-1}, u^{-1}, t) - [t](s, t^{-1}, s^{-1}, t)\Big) && (6)\\
&\qquad + \Big([tu](u^{-1}, t^{-1}, u, t) + [t](s, t^{-1}, s^{-1}, t)\Big) && (4)\\
&\in \mathcal{H}.
\end{aligned}$$

This completes the proof. □

Proof of Proposition 10.1. It suffices to verify hypotheses (b) and (c) of Lemma 10.2.

Let

$$H_o = (u, s, t, u^{-1}, t^{-1}, s, u, t, u^{-1}, t, u, s^{-1}, u^{-1}, s^{-1}, u, t^{-1}, u^{-1}, t^{-1})$$

and

$$H_e = (s^2, t^2, s^{-2}, u, t^{-1}, u^{-1}, s, u, t, s, t^{-2}, s^{-2}, u^{-1}),$$

so H_o and H_e are hamiltonian cycles in the subgraph of X induced by Y.

Let

$$C' = \begin{cases} 0 & \text{if } G = Y, \\ C + [t^2](s, t, s^{-1}, t^{-1}) & \text{if } G \neq Y, \end{cases}$$

and let

$$H_o' = H_o + C' \text{ and } H_e' = H_e + C'.$$

Because H_o and H_e each contain the oriented edge $[st^2](s^{-1})$, and C contains the oriented edge $[t^3](s)$ (if $G \neq Y$), we see that H_o' and H_e' are hamiltonian cycles in X.

(b) It is easy to verify that H_o is the sum of 9 basic 4-cycles, and that H_e is the sum of 8 basic 4-cycles. Thus, $H_o - H_e$ is the sum of 17 basic 4-cycles. Also, we have $H_o - H_e = H_o' - H_e' \in \mathcal{H}$. Therefore, $H_o - H_e$ is an element of $\mathcal{H}$ that is the sum of an odd number of basic 4-cycles.

(c) Let

$$H_1 = H_2 = H_3 = H_4 = H_o', \qquad H_5 = H_6 = H_e',$$

$$v_1 = v_2 = v_3 = su, \qquad v_4 = tu, \qquad v_5 = stu, \qquad v_6 = t^2.$$

Then, because H_o' and H_e' contain every edge of H_o or H_e, except (possibly) $[st^2](s^{-1})$, it is easy to verify that each H_i contains the oriented paths specified in 10.2(c). □

Corollary 10.3. *If*

- $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$, *and*

- *there exist $m, n \geq 3$ and $p \geq 2$, such that every element g of G can be written **uniquely** in the form $g = s^i t^j u^k$ with $0 \leq i < m$, $0 \leq j < n$, and $0 \leq k < p$,*

then $\mathcal{H}$ contains every basic 4-cycle.

Proof. By permuting s, t, and u, we may assume $n = \max\{m, n, p\}$. Because $|G|$ is even (see 2.3), we cannot have $m = n = p = 3$. Thus, either $m = n = 3$ and $p = 2$, or $n \geq 4$. In the former case, the desired conclusion is immediate from Proposition 10.1(a). Thus, we henceforth assume that $n \geq 4$. We will construct an appropriate hamiltonian cycle C as specified in Proposition 10.1(b).

For natural numbers a, b, and c, and $x, y \in \{s^{\pm 1}, t^{\pm 1}\}$, such that either

- $x \in \{s^{\pm 1}\}$, $y \in \{t^{\pm 1}\}$, $a \leq m$, and $b \leq n$, or
- $x \in \{t^{\pm 1}\}$, $y \in \{s^{\pm 1}\}$, $a \leq n$, and $b \leq m$,

let

$$A_{a,b}(x,y) = \begin{cases} (x^{a-1}, u, x^{-(a-1)}) & \text{if } b \text{ is odd,} \\ (x^{a-1}, y, x^{-(a-1)}, u, x^{a-1}, y^{-1}, x^{-(a-1)}) & \text{if } b \text{ is even,} \end{cases}$$

$$C_{a,b}(x,y) = \Big(\big(x^{a-1}, y, x^{-(a-1)}, y\big)^{\lfloor (b-1)/2 \rfloor}, A_{a,b}(x,y), \\ \big(y^{-1}, x^{a-1}, y^{-1}, x^{-(a-1)}\big)^{\lfloor (b-1)/2 \rfloor}, u^{-1} \Big)$$

and

$$X_{a,b,c}(x,y) = \left\{ x^i y^j u^k \;\middle|\; \begin{array}{l} 0 \leq i \leq a-1, \\ 0 \leq j \leq b-1, \\ 0 \leq k \leq c-1 \end{array} \right\}.$$

Then $C_{a,b}(x,y)$ is a hamiltonian cycle in the subgraph of X induced by $X_{a,b,2}(x,y)$.

Let

$$A = \begin{cases} -[s^2t^3]C_{3,n-3}(s^{-1}, t) & \text{if } m = 3, \\ -[s^2t^3]C_{3,n-3}(s^{-1}, t) + [s^2t^3](s, u, s^{-1}, u^{-1}) \\ \quad + [s^3t^3]C_{m-3,n-3}(s,t) & \text{if } m > 3, \\ \quad + [s^3t^3](t^{-1}, s, t, s^{-1}) - [s^3t^2]C_{m-3,3}(s, t^{-1}) \end{cases}$$

so A is a hamiltonian cycle in the subgraph of X induced by

$$X_{m,n,2}(s,t) \smallsetminus X_{3,3,2}(s,t),$$

and A contains the oriented edges $[t^3](s)$ and $[st^3u](s^{-1})$.

If $p > 2$, let B be any hamiltonian cycle in the subgraph of X induced by $[u^2]X_{m,n,p-2}(s,t)$, such that B contains the oriented edge $[t^3u^2](s)$. For example, if p is even, we may let

$$B = -\sum_{i=1}^{(p-2)/2} [u^{2i}]C_{m,n}(s,t) + \sum_{i=1}^{(p-4)/2} [t^3u^{2i+1}](s,u,s^{-1},u^{-1}).$$

On the other hand, if p is odd, then mn must be even, so it is easy to construct a hamiltonian cycle H in the subgraph of X induced by $X_{m,n,1}(s,t)$, such that H contains the oriented edge $[t^3](s)$. If $[v](x)$ is any other oriented edge of H, then we may let

$$\begin{aligned} B = \sum_{i=2}^{p-1}(-1)^i[u^i]H &+ \sum_{i=1}^{(p-3)/2} [vu^{2i}](u,x,u^{-1},x^{-1}) \\ &+ \sum_{i=1}^{(p-3)/2} [t^3u^{2i+1}](s,u,s^{-1},u^{-1}). \end{aligned}$$

Let

$$C = \begin{cases} A & \text{if } p = 2, \\ A + [t^3u](s,u,s^{-1},u^{-1}) + B & \text{if } p > 2. \end{cases}$$

Then C is a hamiltonian cycle in $X \smallsetminus X_{3,3,2}(s,t)$, and C contains the oriented edge $[t^3](s)$, so Proposition 10.1(b) implies that $\mathcal{H}$ contains every basic 4-cycle. □

Corollary 10.4. *If there exist $s, t \in S$, such that*

- $|S| = 5$ *or* 6,
- $G' \neq G$,
- $t \in S'$,
- $\langle t \rangle \neq G'$, *and*
- *either*
  - *(a)* $|G|$ *is* ***not*** *divisible by* 4, *or*
  - *(b) no more than one of* $|t|$, $|G'|/|t|$ *and* $|G/G'|$ *is equal to* 2,

then $\mathcal{H}$ *contains every basic* 4*-cycle.*

Proof. Because the product of the three orders in (b) is $|G|$, it is clear that (a) implies (b). Thus, we may assume (b).

Let $u \in S' \smallsetminus \{t^{\pm1}\}$; then $S = \{s^{\pm1}, t^{\pm1}, u^{\pm1}\}$. Let $m = |G/G'|$, $n = |t|$, and $p = |G'|/|t|$. Then every element of G can be written uniquely in the form

$s^i t^j u^k$, with $0 \le i \le m-1$, $0 \le j \le n-1$, $0 \le k \le p-1$. Thus, Corollary 10.3 applies (with s or t in the role of u, if $m = 2$ or $n = 2$). □

Corollary 10.5. *If*

- $|S| \ge 5$,
- *S is irredundant, and*
- *S contains no more than one involution,*

then $\mathcal{H}$ contains every basic 4-cycle (s, t, s^{-1}, t^{-1}).

Proof. We may assume $s \ne t^{\pm 1}$. (Otherwise, the 4-cycle is degenerate, so $(s, t, s^{-1}, t^{-1}) = 0 \in \mathcal{H}$.) By an argument similar to Case 2 of the proof of Corollary 6.3, we may assume $|S|$ is either 5 or 6. Also, since S contains no more than one involution, we may assume $|s| \ge 3$ (by interchanging s and t if necessary). The desired conclusion follows from either Corollary 6.3(a) (if $|G|$ is divisible by 4), or Corollary 10.4(a) (if $|G|$ is not divisible by 4). □

Corollary 10.6. *If $X \cong K_2 \,\square\, K_3 \,\square\, K_3$, then $\mathcal{H} = \mathcal{E}$.*

Proof. Corollary 10.5 implies that $\mathcal{H}$ contains every basic 4-cycle. Let s be the involution in S, so $G' \cong \mathbf{Z}_3 \times \mathbf{Z}_3$. We have

$$\begin{aligned} \mathcal{E} &\subseteq \mathcal{H} + 2\mathcal{F}' && \text{(see 5.5(b))} \\ &\subseteq \mathcal{H} + 2\mathcal{H}' && \text{(see 4.2)} \\ &\subseteq \mathcal{H} && \text{(see 5.10),} \end{aligned}$$

as desired. □

Corollary 10.7. *If*

- $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$, *with* $|S| = 6$,
- $G' \ne G$, *and*
- $|G/G'|$ *is odd,*

then $\mathcal{H}$ contains every basic 4-cycle.

Proof. We may assume $\langle t \rangle = \langle u \rangle$, for otherwise Corollary 10.4(b) applies (perhaps after interchanging t and u), because $|t| \ne 2$, $|u| \ne 2$, and $|G/G'| \ne 2$. Hence $\langle t \rangle = G'$. In particular, we know $|t| = |G'|$ is even (because $|G/G'|$ is odd).

We have $u = t^{-q}$ for some q. (To avoid confusion, let us point out the negative sign in the exponent.) We may assume $2 \le q < |t|/2$, by replacing u with its inverse if necessary (and noting that $q \neq |t|/2$, since $|u| = |t|$). We may also assume that $q > 2$ (by interchanging t and u, if necessary). Then

$$Y = \{\, s^i t^j u^k \mid 0 \leq i \leq 2, 0 \leq j \leq 2, 0 \leq k \leq 1 \,\}$$

consists of 18 distinct elements of G.

Let $m = |G/G'|$ (by assumption, m is odd), and let $C = C_1 + C_2 + C_3$, where

$$C_1 = [t^3]\Big(s, \big(s^{m-2}, t, s^{-(m-2)}, t\big)^{(|t|-2q)/2}\sharp, s^{-1}, t^{-(|t|-2q-1)}\Big),$$

$$C_2 = \begin{cases} 0 & \text{if } m = 3, \\ [s^{m-1}t^2]\Big(t, s^{-1}, t^{-1}, \\ \qquad \big(s^{-(m-5)}, t^{-1}, s^{m-5}, t^{-1}\big)^q\sharp, s, t^{2q-1}\Big) & \text{if } m \geq 5, \end{cases}$$

$$C_3 = \begin{cases} 0 & \text{if } q = 3, \\ [t^{-(2q-3)}]\Big(t^{-1}, s, t^{q-3}, s, t^{-(q-4)}, u^{-1}, t^{q-4}, \\ \qquad s^{-1}, t^{-(q-4)}, s^{-1}, t^{q-4}, u, t^{-(q-4)}\Big) & \text{if } q > 3. \end{cases}$$

Then C is a hamiltonian cycle in $X \smallsetminus Y$, and C contains the oriented edge $[t^3](s)$, so Proposition 10.1(b) applies. $\square$

11 Redundant generators in graphs of large degree

In this section, assuming that $|S| \geq 5$, and that s is a redundant generator, we provide an induction step: if $\mathcal{H}' = \mathcal{E}'$, then $\mathcal{H} = \mathcal{E}$ (see 11.3).

Lemma 11.1. *If there exists s in S, and elements H_+ and H_- of $\mathcal{H}$, such that*

- $H_+ = (s) + [t](s) + (\text{edges in } X')$,
- $H_- = (s) + [st](s^{-1}) + (\text{edges in } X')$,
- $G' = G$,
- $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$, *and*
- $|S| \geq 4$,

then $\mathcal{H} = \mathcal{E}$.

Proof. We have

$$H_+ + H_- = 2(s) + (\text{edges in } X'),$$

so adding or subtracting appropriate translates of $H_+ + H_-$ will eliminate all the s-edges from any flow in $2\mathcal{F}$, the result being a flow in $\mathcal{E}'$. Therefore

$$2\mathcal{F} \subseteq \mathcal{H} + \mathcal{E}'. \tag{E19}$$

We have

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + 2\mathcal{F} && \text{(see 5.1)} \\
&\subseteq \mathcal{H} + \mathcal{E}' && \text{(see E19)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(by assumption)} \\
&\subseteq \mathcal{H} && \text{(see 5.7),}
\end{aligned}$$

as desired. □

Lemma 11.2. *If there exist s and t in S, such that*

- $|S| \geq 5$,
- $t \in S'$,
- $G = \langle t \rangle$, *and*
- $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$,

then $\mathcal{H} = \mathcal{E}$.

Proof. Choose $u \in S \smallsetminus \{s^{\pm 1}, t^{\pm 1}\}$. Write $s = t^p$ and $u = t^q$. We may assume $2 \leq p, q \leq |t|/2$ (by replacing s and/or t with its inverse, if necessary). We may also assume X' is bipartite, but X is not bipartite, for otherwise we have

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.3)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(by assumption)} \\
&\subseteq \mathcal{H} && \text{(see 5.7),}
\end{aligned}$$

as desired. Hence, p is even and q is odd.

Let

$$H_+ = (s, t^{-(p-1)}, s, t^{|t|-p-1}),$$

so H_+ is a hamiltonian cycle in X, and the only oriented s-edges in H_+ are (s) and $[t](s)$.

Define H_- from the hamiltonian cycle

$$H_-^* = [su]\Big(\big(u^{-1}, t, u, t\big)^{(q-1)/2}, t^{|t|-2q}, u, t\Big),$$

by replacing

- the path $[su](u^{-1}, t, u)$ with the oriented edge $[su](t)$ and
- the oriented edge (t) with the path (s, t, s^{-1}).

Then the only oriented s-edges in H_- are (s) and $[st](s^{-1})$.

Therefore, Lemma 11.1 applies. $\square$

Proposition 11.3. *If there exists $s \in S$, such that*

- $|S| \geq 5$,
- $G' = G$, *and*
- $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$,

then $\mathcal{H} = \mathcal{E}$.

Proof. We may assume that X' is bipartite, but X is not bipartite, for otherwise we have

$$\begin{aligned} \mathcal{E} &= \mathcal{H} + \mathcal{E}' && \text{(see 5.3)} \\ &\subseteq \mathcal{H} + \mathcal{H}' && \text{(by assumption)} \\ &\subseteq \mathcal{H} && \text{(see 5.7),} \end{aligned}$$

as desired.

Choose some $t \in S'$. It suffices to find oriented hamiltonian cycles H_+ and H_- in X, such that

- the only s-edges in H_+ are (s) and $[t](s)$, and
- the only s-edges in H_- are (s) and $[st](s^{-1})$,

for then Lemma 11.1 applies.

We may assume that not every element of S' is an involution (see 3.1 and 3.3), so we may assume $|t| > 2$. We may also assume $\langle t \rangle \neq G$, for otherwise Lemma 11.2 applies. Note that, because X' is bipartite, $|t|$ must be even.

Let $m = |G|/|t|$, and let $(u_1, u_2, \ldots, u_{m-1})$ be a hamiltonian path in

$$\mathrm{Cay}\Big(G/\langle t \rangle; S' \smallsetminus \{t^{\pm 1}\}\Big).$$

We have $s = t^p u_1 u_2 \cdots u_q$, for some p and q. We may assume $0 \leq p \leq |t|/2$ and $0 \leq q \leq m/2$ (by replacing s and/or t with their inverses, if necessary). Because X' is bipartite, but X is not bipartite, we know that $p + q$ is even.

Case 1. Assume $m > 2$, and p and q are odd. Let

$$\begin{aligned}H_+ = \Big(s, \big((u_{q+1-i}^{-1})_{i=1}^{q}, t^{-1}, (u_i)_{i=1}^{q}, t^{-1}\big)^{(p-1)/2}, (u_{q+1-i}^{-1})_{i=1}^{q},\\ s, \big((u_{q+1-i}^{-1})_{i=1}^{q}, t, (u_i)_{i=1}^{q}, t\big)^{(|t|-p-1)/2}\sharp, u_{q+1},\\ \big((u_{q+1+i})_{i=1}^{m-q-2}, t^{-1}, (u_{m-i}^{-1})_{i=1}^{m-q-2}, t^{-1}\big)^{|t|/2}\sharp,\\ (u_{q+2-i}^{-1})_{i=1}^{q+1}\Big).\end{aligned}$$

Define H_- from the hamiltonian cycle

$$\begin{aligned}H_-^* = \Big(t, \big(t^{|t|-2}, u_{2i-1}, t^{-(|t|-2)}, u_{2i}\big)_{i=1}^{(q-1)/2}, t^{|t|-2}, u_q,\\ \big((u_{q+i})_{i=1}^{m-q-1}, t^{-1}, (u_{m-i}^{-1})_{i=1}^{m-q-1}, t^{-1}\big)^{|t|/2}\sharp,\\ (u_{q+1-i}^{-1})_{i=1}^{q}\Big),\end{aligned}$$

by replacing

- the path $[stu_{q+1}](u_{q+1}^{-1}, t^{-1}, u_{q+1})$ with the edge $[stu_{q+1}](t^{-1})$, and
- the edge (t) with the path (s, t, s^{-1}).

Case 2. Assume $m > 2$, and p and q are even. Let

$$\begin{aligned}H_+ = \Big(s, \Big(t^{-p}, \big(u_{q+2-2i}^{-1}, t^{p+1}, u_{q+1-2i}^{-1}, t^{-(p+1)}\big)_{i=1}^{q/2}\Big)\sharp,\\ s, u_{q+1}, t^{-(p+1)}, u_{q+2},\\ \big((u_{q+2+i})_{i=1}^{m-q-3}, t, (u_{m-i}^{-1})_{i=1}^{m-q-3}, t\big)^{|t|/2}\sharp,\\ \big(u_{q+4-2i}^{-1}, t^{-(|t|-p-3)}, u_{q+3-2i}^{-1}, t^{|t|-p-3}\big)_{i=1}^{(q+2)/2}, t\Big).\end{aligned}$$

If $p \neq 0$, define H_- from the hamiltonian cycle

$$\begin{aligned}H_-^* = \Big(t, \big(t^{|t|-2}, u_{2i-1}, t^{-(|t|-2)}, u_{2i}\big)_{i=1}^{q/2},\\ \big((u_{q+i})_{i=1}^{m-q-1}, t, (u_{m-i})_{i=1}^{m-q-1}, t\big)^{|t|/2}\sharp, (u_{q+1-i})_{i=1}^{q}\Big),\end{aligned}$$

by replacing

- the path $[su_{q+1}](u_{q+1}^{-1}, t, u_{q+1})$ with the oriented edge $[su_{q+1}](t)$, and
- the edge (t) with the path (s, t, s^{-1}).

If $p = 0$, let

$$H_- = \Big(s, u_{q+1}, \big((u_{q+1+i})_{i=1}^{m-q-2}, t, (u_{m-i}^{-1})_{i=1}^{m-q-2}, t\big)^{|t|/2}\sharp,$$
$$u_{q+1}^{-1}, t^{-(|t|-3)}, u_q^{-1}, \big((u_{q-i}^{-1})_{i=1}^{q-1}, t, (u_i)_{i=1}^{q-1}, t\big)^{(|t|-2)/2},$$
$$t, u_q, s^{-1}, (u_i)_{i=1}^{q-2}, t^{-1}, (u_{q-1-i}^{-1})_{i=1}^{q-2}\Big).$$

Case 3. Assume $m = 2$, and p and q are odd. Note that $q = 1$. Let

$$H_+ = \Big(s, \big(u_1^{-1}, t^{-1}, u_1, t^{-1}\big)^{(p-1)/2}, u_1^{-1}, s, \big(u_1^{-1}, t, u_1, t\big)^{(|t|-p-1)/2}, u_1^{-1}\Big)$$

and

$$H_- = \Big(s, t^{-(|t|-1)}, s^{-1}, t^{|t|-1}\Big).$$

Case 4. Assume $m = 2$, and p and q are even. Note that $q = 0$, so $s \in \langle t\rangle$. Let

$$H_+ = \Big(s, t^{-(p-1)}, s, t^{|t|-p-2}, u_1, t^{-(|t|-1)}, u_1^{-1}\Big),$$
$$H_- = \Big(s, t^{-(p-2)}, u_1, t^{|t|-3}, u_1^{-1}, t^{-(|t|-p-2)}, s^{-1}, u_1, t^{-1}, u_1^{-1}\Big).$$

□

12 Two troublesome cases

In this section, we treat two special cases (see 12.2 and 12.3), in order to deal with some graphs that are not covered by our previous results, and do not yield easily to a proof by induction.

Lemma 12.1. *If there exist $s, t \in S$, such that*

- $|G/G'| = 2$,
- $t \in S'$,
- $G' = \langle t\rangle$, *and*
- $|S'| = 4$,

then $\mathcal{H}$ contains the basic 4-cycle (s, t, s^{-1}, t^{-1}).

Proof. Let $u \in S \smallsetminus \{s^{\pm 1}, t^{\pm 1}\}$, so $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$. We may write $u = t^q$ for some q. Interchange u with u^{-1}, if necessary, to ensure that

$$q \text{ is even if } |t| \text{ is odd.} \tag{E20}$$

Define hamiltonian cycles

$$H_+ = \Big(s, u, t^{-(q-2)}, s^{-1}, t^{|t|-3}, s, t^{-(|t|-q-2)}, u^{-1}, s^{-1}, t^{-1}\Big)$$

and

$$H_- = \Big(t^{|t|-1}, s, t^{-(|t|-1)}, s^{-1}\Big),$$

and let

$$Q = (t, s, t^{-1}, s^{-1}),$$

so $-Q$ is the basic 4-cycle specified in the statement of the lemma. Then

- H_+ contains both the oriented path $[t^{-2}](t, s, t^{-1})$ and the oriented edge $[e](s)$, and
- H_- contains both the oriented path $[t^{-2}](t, s, t^{-1})$ and the oriented edge $[s](s^{-1})$,

so Lemma 6.1(1,2,3) (with $x = t$, $y = s$, $z = t$, and $v = w = t^{-2}$) implies that

$$\mathcal{H} \text{ contains } Q + [t]Q,\ Q - [t]Q, \text{ and } 2Q.$$

If $|t|$ is even, then

$$Q = H_- - \sum_{i=1}^{|t|-2} [t^i]Q = H_- - \sum_{j=1}^{(|t|-2)/2} [t^{2j-1}]\Big(Q + [t]Q\Big) \in \mathcal{H},$$

as desired.

If $|t|$ is odd, define the hamiltonian cycle

$$H = \bigg(t^{q-1}, s, t^{-(q-1)}, u, \Big(s^{-1}, t, s, t\Big)^{(|t|-q-1)/2}, s^{-1}, t\bigg). \tag{E21}$$

We have

$$H = (t^{|t|}) + [s](u, t^{-q}) - \sum_{i=0}^{(|t|-q-1)/2} [t^{q-1+2i}]Q$$

and

$$H_+ = H_- + [s](u, t^{-q}) - [st](u, t^{-q}) - 2Q - [t]Q,$$

so

$$\begin{aligned} H - [t]H &= [s](u, t^{-q}) - [st](u, t^{-q}) - \sum_{i=0}^{(|t|-q-1)/2} [t^{q-1+2i}]\Big(Q - [t]Q\Big) \\ &\equiv [s](u, t^{-q}) - [st](u, t^{-q}) \pmod{\mathcal{H}}, \end{aligned}$$

and therefore

$$\Big(H - [t]H\Big) - H_+ \equiv -H_- + 2Q + [t]Q \equiv Q \pmod{\mathcal{H}}.$$

Because $\Big(H - [t]H\Big) - H_+$ obviously belongs to $\mathcal{H}$, we conclude that $Q \in \mathcal{H}$, as desired. $\square$

Proposition 12.2. *If there exist $s, t \in S$, such that*

- *$|G/G'|$ is even,*
- *$t \in S'$,*
- *$G' = \langle t \rangle$, and*
- *$|S'| = 4$,*

then $\mathcal{H} = \mathcal{E}$.

Proof. Let

- $u \in S \smallsetminus \{s^{\pm 1}, t^{\pm 1}\}$, so $S = \{s^{\pm 1}, t^{\pm 1}, u^{\pm 1}\}$,
- $m = |G/G'|$, and
- $Q = (t, s, t^{-1}, s^{-1})$.

We may write $u = t^q$ for some q.

Let us first establish that $Q \equiv -Q \equiv [v]Q \pmod{\mathcal{H}}$, for all $v \in G$.

- If $m = 2$, then Lemma 12.1 asserts $Q \in \mathcal{H}$, which is a stronger statement.
- If $m > 2$, then $s^2 \notin \langle t \rangle$, so the desired conclusion is obtained by applying Proposition 6.2 to the spanning subgraph $\operatorname{Cay}\Big(G; \{s^{\pm 1}, t^{\pm 1}\}\Big)$, with the roles of s and t interchanged.

Now let us now show that

$$2(t^{|t|}) \in \mathcal{H}. \tag{E22}$$

We may assume, for the moment, that $q \le |t|/2$, by replacing u with its inverse if necessary. Let

$$C = \begin{cases} 0 & \text{if } m = 2 \\ [s]\Big(\Big(t, \big(s, t^{|t|-2}, s, t^{-(|t|-2)}\big)^{(m-2)/2}, t^{-1}, s^{-(m-2)}\Big) & \text{if } m > 2, \end{cases} \tag{E23}$$

and define the hamiltonian cycles

$$H_1 = \Big(t^{|t|-3}, s, t^{-(q-2)}, u, t^{|t|-q-1}, u, s^{-1}, t^2\Big) - C$$

and

$$H_2 = \Big(t^{|t|-1}, s, u^{-1}, t^{q-1}, u^{-1}, t^{-(|t|-q-2)}, s^{-1}\Big) + C.$$

Then

$$H_1 + H_2 = 2(t)^{|t|} - [t^{-1}]Q - [t^{-3}]Q \equiv 2(t^{|t|}) \pmod{\mathcal{H}},$$

so

$$2(t^{|t|}) \in \mathcal{H},$$

as claimed.

Let

$$X^* = \operatorname{Cay}\Big(G; \{s^{\pm 1}, t^{\pm 1}\}\Big).$$

Case 1. Assume $|s| > 2$. It suffices to show $\mathcal{E}(X^*) \subseteq \mathcal{H} + \mathcal{H}(X^*)$, for then Proposition 11.3 applies (with u in the role of s).

- If X^* is not the square of an even cycle, and either X^* is bipartite, or $|X^*|$ is divisible by 4, then Proposition 7.1 implies $\mathcal{E}(X^*) \subseteq \mathcal{H}(X^*) \subseteq \mathcal{H} + \mathcal{H}(X^*)$.
- If X^* is not the square of an even cycle, and X^* is not bipartite, and $|X^*|$ is not divisible by 4, then Corollary 8.7 (with u in the role of s) implies $\mathcal{E}(X^*) \subseteq \mathcal{H} + \mathcal{H}(X^*)$.
- If X^* is the square of an even cycle, then $s^2 \in \langle t \rangle = G'$, so $m = 2$. Therefore, Lemma 12.1 implies that $\mathcal{H}$ contains a basic 4-cycle of X^*. Therefore, Corollary 8.6 (with u in the role of s) implies $\mathcal{E}(X^*) \subseteq \mathcal{H} + \mathcal{H}(X^*)$.

Case 2. Assume that $|s| = 2$*, and that either* $|t|$ *or* q *is odd.* If $|t|$ is odd, we may assume that q is even (by replacing u with its inverse, if necessary). Thus, $|t|$ and q are of opposite parity. Define H as in (E21) and C as in (E23), and let $H' = H + C$. Because H' has only a single u-edge, we can eliminate all of the u-edges from any flow in $\mathcal{F}$, by adding appropriate translates of the hamiltonian cycle H', leaving us with a flow f in $\mathcal{F}(X^*)$. Therefore $\mathcal{F} \subseteq \mathcal{H} + \mathcal{F}(X^*)$, so

$$2\mathcal{F} \subseteq \mathcal{H} + 2\mathcal{F}(X^*). \tag{E24}$$

But any flow in $\mathcal{F}(X^*)$ is a sum of cycles of the form $[v]Q$ and/or $[v](t)^{|t|}$. (Here, we use the assumption that $|s| = 2$.) Therefore $2\mathcal{F}(X^*) \subseteq \mathcal{H}$. Combining this with (E24) and (5.1), we conclude that $\mathcal{E} \subseteq \mathcal{H}$, as desired.

Case 3. Assume that $|s| = 2$*, and that both* $|t|$ *and* q *are even.* Let us begin by showing that $\mathcal{H}$ contains every basic 4-cycle. (Recall that we already know $\mathcal{H}$ contains the basic 4-cycle Q.) Note that, because $|S'| = 4$, we know u is not an involution, so $q \neq |t|/2$. We may assume (by replacing u with its inverse if necessary) that $2 \leq q < |t|/2$.

- Define the hamiltonian cycle

  $$H'_+ = \Big(\big(s, t, s^{-1}, t\big)^{|t|/2}\Big).$$

  This contains both the oriented path (s, t, s^{-1}) and the oriented edge $[su](t)$ (because q is even), so Lemma 6.1(2,4) (with $x = s$, $y = t$, $z = u$, and $v = e$) implies that $\mathcal{H}$ contains the basic 4-cycle (u, t, u^{-1}, t^{-1}).

- Define the hamiltonian cycle

$$H'_- = \Big(t, s, t^{-(|t|-q-1)}, u^{-1}, t^{q-1}, s^{-1}, t^{-(q-1)}, u, t^{|t|-q-2}\Big).$$

This contains both the oriented path (t, s, t^{-1}) and the oriented edge $[tsu](s^{-1})$, so Lemma 6.1(2,4) (with $x = t$, $y = s$, $z = u$, and $w = e$) implies that $\mathcal{H}$ contains the basic 4-cycle (u, s, u^{-1}, s^{-1}).

We have

$$\begin{aligned} \mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\ &\subseteq \mathcal{H} + \mathcal{H}' && \text{(see 8.6)} \\ &\subseteq \mathcal{H} && \text{(see 5.8(1))}, \end{aligned}$$

as desired. □

Proposition 12.3. *If $|S| = 5$ and $\langle t \rangle = G$, for some $t \in S$, then $\mathcal{H} = \mathcal{E}$.*

Proof. Let

- $n = |G|/2$, and
- $u \in S \smallsetminus \{t^{\pm 1}\}$, with $|u| \neq 2$. We have $u = t^q$, for some $q \neq n$.

Note that $S = \{t^{\pm 1}, u^{\pm 1}, t^n\} = \{t^{\pm 1}, t^{\pm q}, t^n\}$.

Case 1. Assume n is even. Let $s = u$, so X' is a non-bipartite Möbius ladder. From Theorem 1.4, we have $\mathcal{H}' = \mathcal{E}'$, so Lemma 11.2 applies.

Case 2. Assume n and q are both odd. Note that X is bipartite. Let $s = t^n$ be the involution in S. We have

$$\begin{aligned} \mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.3)} \\ &\subseteq \mathcal{H} + \mathcal{H}' && \text{(see 7.4)} \\ &\subseteq \mathcal{H} && \text{(see 5.7)}, \end{aligned}$$

as desired.

Case 3. Assume n is odd and q is even. We may assume $2 \leq q \leq n-1$, by replacing u with u^{-1} if necessary. Let $s = t^n$ be the involution in S.

Define

$$Q = (t, s, t^{-1}, s), \qquad C = (t^n, s), \qquad H = (t, s)^n,$$

so Q is a basic 4-cycle, C is a cycle, and H is a hamiltonian cycle in X. Note that

$$C - [t]C = Q.$$

Define the hamiltonian cycles

$$H_+ = (s, t^{n-1}, s, t^{-(n-1)})$$

and

$$H_- = \begin{cases} (t^{n-3}, u^{-1}, t, s, t^{-1}, u, t^{-(n-3)}, s) & \text{if } r = n-1, \\ (u, t^{-(q-2)}, s, t^{n-3}, s, t^{-(n-q-2)}, u^{-1}, s, t^{-1}, s) & \text{otherwise.} \end{cases}$$

Then

- H_+ contains both the oriented path $[t^{-2}](t, s, t^{-1})$ and the oriented edge $[e](s)$, and
- H_- contains both the oriented path $[t^{-2}](t, s, t^{-1})$ and the oriented edge $[s](s^{-1})$,

so Lemma 6.1(1,2) (with $x = t$, $y = s$, $z = t$, and $v = w = t^{-2}$) implies that

$$\mathcal{H} \text{ contains } Q + [t]Q \text{ and } Q - [t]Q.$$

Therefore

$$2Q \in \mathcal{H} \qquad \text{and} \qquad [v]Q \equiv Q \pmod{\mathcal{H}}, \text{ for all } v \in G.$$

Let us show that

$$Q \in \mathcal{H}. \tag{E25}$$

We have

$$C = H + \sum_{i=1}^{(n-1)/2} [t^{2i-1}]Q \equiv \frac{n-1}{2} Q \pmod{\mathcal{H}},$$

so

$$Q = C - [t]C \equiv \frac{n-1}{2}\big(Q - [t]Q\big) \equiv 0 \pmod{\mathcal{H}},$$

as claimed.

The hamiltonian cycle H contains both the oriented path $[t](s, t, s)$ and the oriented edge $[tsu](t)$ (because q is even), so Lemma 6.1(1) (with $x = s$, $y = t$, $z = u$, and $v = t$) implies that

$$(s, t, s^{-1}, t^{-1}) + [s](u, t, u^{-1}, t^{-1}) \in \mathcal{H}.$$

Then, from Lemma 6.1(4) (and (E25)), we know that $(u, t, u^{-1}, t^{-1}) \in \mathcal{H}$. Hence, Lemma 8.6 implies $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$, so Lemma 11.2 implies $\mathcal{H} = \mathcal{E}$, as desired. □

13 Graphs of degree at least 5

In this section, we show that if $|S| \geq 5$, then $\mathcal{H} = \mathcal{E}$ (see 13.3). After establishing the special cases where $|S| = 5$ or 6 (see 13.1 and 13.2), it is very easy to complete the proof by induction on $|S|$.

Proposition 13.1. *If $|S| = 5$, then $\mathcal{H} = \mathcal{E}$.*

Proof. Case 1. Assume some involution s in S is redundant. We have $G' = G$. We may assume $\mathcal{H}' \neq \mathcal{E}'$, for otherwise Proposition 11.3 applies. Then Proposition 7.1 implies that either

- X' is the square of an even cycle, in which case, Proposition 12.3 applies, or
- X' is not bipartite, and $|X'|$ is not divisible by 4.

Thus, we may assume that X' is not bipartite, and $|X'|$ is not divisible by 4.

We have $|G| = |G'| \equiv 2 \pmod 4$ and $S' = \{t^{\pm 1}, u^{\pm 1}\}$, where $|t|$ is odd and $|u|$ is even. If $G = \langle u \rangle$, then Proposition 12.3 applies. If not, then, because $|G|$ is not divisible by 4, Corollary 10.4 (with the roles of s and t interchanged) implies that $\mathcal{H}$ contains every basic 4-cycle C. Hence,

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.3)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(see 8.6)} \\
&= \mathcal{H} && \text{(see 5.7),}
\end{aligned}$$

as desired.

Case 2. Assume every involution in S is irredundant. We may assume that S contains only one involution s (see 3.1). Then X is the prism over X' (and S' does not contain any involutions). Furthermore, we may assume

$$\langle t \rangle \neq G', \text{ for all } t \in S' \tag{E26}$$

(otherwise, Proposition 12.2 applies).

Subcase 2.1. Assume $|G'|$ is odd. Let $t \in S'$. We know $S' \not\subseteq \langle t \rangle$ (because $\langle t \rangle \neq G'$) and $|G| = 2|G'|$ is not divisible by 4, so Corollary 10.4 implies that $\mathcal{H}$ contains every basic 4-cycle. We may assume that $X' \not\cong K_3 \square K_3$ (otherwise, Corollary 10.6 applies), so Theorem 1.2 asserts that $\mathcal{H}' = \mathcal{F}'$. We have

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + 2\mathcal{F}' && \text{(see 5.5(b))} \\
&\subseteq \mathcal{H} + 2\mathcal{H}' && \text{(because } \mathcal{H}' = \mathcal{F}'\text{)} \\
&\subseteq \mathcal{H} && \text{(see 5.10),}
\end{aligned}$$

as desired.

Subcase 2.2. Assume that $|G'|$ is even, that X' is not the square of an even cycle, and that either X' is bipartite or $|G'|$ is divisible by 4. For any $t \in S'$, we have

$$\frac{|G|}{|t|} = \frac{|G|}{|G'|} \cdot \frac{|G'|}{|t|} \geq 2 \cdot 2 = 4,$$

so Corollary 6.3(a) (with the roles of s and t interchanged) implies that the basic 4-cycle (s, t, s^{-1}, t^{-1}) is in $\mathcal{H}$. Therefore

$$\begin{aligned}
\mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(see 7.1)} \\
&\subseteq \mathcal{H} && \text{(see 5.8(1)),}
\end{aligned}$$

as desired.

Subcase 2.3. Assume that either

- *X' is the square of an even cycle, or*
- *X' is not bipartite, and $|G'|$ is not divisible by* 4.

From (E26), we see that X' is not the square of an even cycle, so X' must be as described in Proposition 8.2. Write $S' = \{t^{\pm 1}, u^{\pm 1}\}$, where $|t|$ is odd and $|u|$ is even.

- We know $\langle u \rangle \neq G'$ (see E26).
- Because S' does not contain any involutions, we know $|u| \neq 2$.
- Because $|G'|/|u|$ is odd, we know $|G'|/|u| \neq 2$.

Thus, Lemma 10.4(b) (with u in the role of t) implies that $\mathcal{H}$ contains every basic 4-cycle. We have

$$\begin{aligned}
\mathcal{E} &= \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\
&\subseteq \mathcal{H} + \mathcal{H}' && \text{(see 8.6)} \\
&\subseteq \mathcal{H} && \text{(see 5.8(1)),}
\end{aligned}$$

as desired. $\square$

Proposition 13.2. *If $|S| = 6$, then $\mathcal{H} = \mathcal{E}$.*

Proof. Let us begin by establishing that we may assume

$$\text{there are no involutions in } S. \tag{E27}$$

First, note that if some involution s is redundant, then $\mathcal{H}' = \mathcal{E}'$ (by Proposition 13.1), so Proposition 11.3 implies $\mathcal{H} = \mathcal{E}$, as desired. On the other hand, if all of the involutions in S are irredundant, then Remark 3.1 asserts that X can be realized by a generating set with at most one involution. Because X has even degree, there must be no involutions.

Case 1. Assume S is irredundant. Choose $s \in S$, such that

$$|G'| \text{ is even.}$$

We know, from Corollary 10.5, that

$$\mathcal{H} \text{ contains every basic 4-cycle.} \tag{E28}$$

We have

$$\begin{aligned} \mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\ &\subseteq \mathcal{H} + \mathcal{H}' && \text{(see E28 and 8.6)} \\ &\subseteq \mathcal{H} && \text{(see 5.8(1)),} \end{aligned}$$

as desired.

Case 2. Assume there is a redundant generator s in S. It suffices to show $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$, for then Proposition 11.3 applies. Note that, from (E27), we know $|S'| = 4$.

Subcase 2.1. Assume that X' is not the square of an even cycle, and that either X' is bipartite, or $|G'|$ is divisible by 4. Proposition 7.1 (and the assumption of this subcase) implies that $\mathcal{H}' = \mathcal{E}'$, so $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$, as desired.

Subcase 2.2. Assume X' is the square of an even cycle. By choosing s to be the chord of length 2, we may move out of this subcase.

Subcase 2.3. Assume that X' is not bipartite, and that $|G'|$ is not divisible by 4. We have $|G| = |G'| \equiv 2 \pmod 4$, and $S' = \{x^{\pm 1}, y^{\pm 1}\}$, where $|x| \equiv |s| \pmod 2$ and $|y| \not\equiv |s| \pmod 2$.

It suffices to show that $\mathcal{H}$ contains some basic 4-cycle C of X', for then Corollary 8.6 yields $\mathcal{E}' \subseteq \mathcal{H} + \mathcal{H}'$, as desired.

Subsubcase 2.3.1. Assume $\langle s, x\rangle = G$. Let $X^* = \operatorname{Cay}\big(G; \{s, x\}\big)$. Now X^* is bipartite (see 7.2), so $\mathcal{H}(X^*) = \mathcal{E}(X^*)$ (see 7.4). Therefore Proposition 11.3 (with y in the role of s) implies $\mathcal{H} = \mathcal{E}$.

Subsubcase 2.3.2. Assume $\langle s, x\rangle \neq G$.

- If $\langle s\rangle \neq \langle x\rangle$, then, because $y \notin \langle s, x\rangle$, Corollary 10.4(a) applies (with y in the role of s, and one or the other of s and x in the role of t), so $\mathcal{H}$ contains every basic 4-cycle.
- If $|G/\langle s, x\rangle|$ is odd, then Corollary 10.7 (with y in the role of s) implies that $\mathcal{H}$ contains every basic 4-cycle.
- If $\langle s\rangle = \langle x\rangle$ and $|G/\langle s, x\rangle|$ is even, then Proposition 12.2 (with y in the role of s) implies $\mathcal{H} = \mathcal{E}$.

This completes the proof. □

Corollary 13.3. *If* $|S| \geq 5$*, then* $\mathcal{H} = \mathcal{E}$*.*

Proof. We may assume, by Theorem 1.2 and induction on $|S|$ (with Propositions 13.1 and 13.2 providing the base cases) that

$$\mathcal{E}' \subseteq \mathcal{H}', \text{ for every } s \in S. \tag{E29}$$

We may assume

$$S \text{ is irredundant,}$$

for otherwise Proposition 11.3 implies $\mathcal{H} = \mathcal{E}$, as desired.

Choose $s \in S$, such that

$$|G'| \text{ is even.}$$

Because S is irredundant, we know $G' \neq G$, and, by Remark 3.1, we may assume S has no more than one involution. Then, from Corollary 10.5, we know that $\mathcal{H}$ contains every basic 4-cycle. We have

$$\begin{aligned} \mathcal{E} &\subseteq \mathcal{H} + \mathcal{E}' && \text{(see 5.6)} \\ &\subseteq \mathcal{H} + \mathcal{H}' && \text{(see E29)} \\ &\subseteq \mathcal{H} && \text{(see 5.8(1)),} \end{aligned}$$

as desired. □

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