Lectures on Classical Analytic Theory of L-functions

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Preface

In May 2006, I gave four lectures on classical analytic theory of L-functions at IPM. The following pages are notes that I prepared for those lectures. As you will see these notes are brief and do not follow a textbook style treatment of the subject. My goal was to give a quick introduction to L-functions, by selecting some fundamental topics from the classical theory. The lectures are interrelated and should be read in progression. Many statements are given as exercises. These notes should be read in a slow pace and I encourage the reader to do the exercises. I hope that these notes serve as a guideline for the beginners interested in the theory of L-functions.

Let me give a brief overview of the contents. In the first lecture we set up our notation and terminology. Here we consider L-functions as complex functions that satisfy some nice analytic properties. Following Iwaniec and Kowalski, we axiomatize a class of L-functions, which is basically a class of complex functions satisfying properties similar to automorphic L-functions. Next we introduce the Rankin-Selberg convolution of two L-functions in this class. The fundamental role of these convolutions in the theory of L-functions and their many applications form the main theme of these lectures. We illustrate the importance of these convolutions by describing their relations with the problem of finding sharp estimates for some arithmetic functions. Moreover, in the second lecture we show that how the existence of these convolutions will guarantee the non-vanishing of L-functions on the line $\Re(s) = 1$, and consequently will lead to the prime number theorem type results. In the third lecture we show that for two L-functions associated to cusps forms the Rankin-Selberg convolution exists, and as a consequence of this fact we can apply the results of Lecture 2 to deduce, in the fourth lecture, the prime number theorem type estimates for the Fourier coefficients of a cusp form.

Lectures 1 and 2 are based on chapter 5 of [IK][1], [R1], [O] and [GHL]. The third lecture gives a detailed exposition of the classical paper of Rankin [R2]. The final lecture is based

[1] See the list of references at the end of notes.

on [Mo], [HL] and [R2].
We are using without proof many facts from complex analysis and Fourier analysis. [T] is a good complex analysis reference. [D], [I], [IK], and [M] are good analytic number theory references. For the basic material on modular forms the reader can consult [B], [CKM], [Iw], [K], [S], and [Sh].
[GM], [IS] and [Mi] are good survey articles and they include extensive bibliography.
I hope that these notes give a glimpse of this fascinating subject and motivate the reader for further studies of L-functions.

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Notation

For two complex-valued functions f and g defined on a set X, $f = O(g)$ (or equivalently $f \ll g$) for all $x \in X$ means that there exists a $C \geq 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in X$. We call C an implied constant. The dependence of C on other parameters is indicated by subscripts (for example $O_\epsilon()$, $\ll_f$).
$f = o(g)$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$. If $f \neq o(g)$, we say $f = \Omega(g)$. If $f = g+o(g)$, we say $f \sim g$.
$d(n)$ denotes the number of divisors of n, and $\phi(n)$ is the Euler function.
$\Gamma(s)$ denotes the gamma function, and $\zeta(s)$ is the Riemann zeta function.

Lecture 1
General Setting

1. Notation and Terminology

$L(f,s)$ denotes an *L-function*. It is a Dirichlet series defined on the half plane $\Re(s) > 1$. More precisely,

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_1(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_2(p)}{p^s}\right)^{-1} \cdots \left(1 - \frac{\alpha_d(p)}{p^s}\right)^{-1},$$

where $\lambda_f(1) = 1$, $\alpha_i(p)$, $\lambda_f(n) \in \mathbb{C}$ and $\Re(s) > 1$. The $\lambda_f(n)$'s are called *coefficients*, $\alpha_i(p)$'s , $1 \leq i \leq d$, are called the *local parameters*, and we have

$$|\alpha_i(p)| < p.$$

d is called the *degree* of $L(f,s)$. The series and the Euler product are both absolutely convergent for $\Re(s) > 1$.

Exercise 1 *Show that the existence of the above Euler product implies that $\lambda_f(n)$ is a multiplicative arithmetic function. (i.e. $\lambda_f(mn) = \lambda_f(m)\lambda_f(n)$, whenever* $\gcd(n,m) = 1$.*)*

The following exercise describe the intimate connection between analytic properties of an L-function and the size of its coefficients.

Exercise 2 *Show that since $L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$ is absolutely convergent on $\Re(s) > 1$, then for $\epsilon > 0$,*

$$\sum_{n \leq x} |\lambda_f(n)| = O(x^{1+\epsilon}).$$

From here conclude that

$$\lambda_f(n) = O(n^{1+\epsilon}).$$

Moreover, construct a sequence $\lambda_f(n)$ *such that* $\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$ *is absolutely convergent on* $\Re(s) > 1$*, however*

$$\lambda_f(n) = \Omega(n^{1-\epsilon}).$$

Complex Analysis 3 *The Gamma Function* $\Gamma(s)$ *is the meromorphic function defined on* $\mathbb{C}$ *by the product formula*

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} (1 + s/n)e^{-s/n},$$

where γ *is the Euler constant.* $\Gamma(s)$ *has simple poles at* $s = 0, -1, -2, \cdots$ *.*

Stirling's formula

$$\Gamma(s) = \sqrt{2\pi} e^{-s} s^{s-\frac{1}{2}} \left(1 + O\left(\frac{1}{|s|}\right)\right),$$

uniformly on angle $|\arg s| < \pi - \delta$ for $\delta > 0$, as $|s| \to \infty$.

Stirling's formula (Horizontal Version)

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{-\sigma} \sigma^{\sigma-\frac{1}{2}},$$

for fixed t as $\sigma \to \infty$.

Stirling's formula (Vertical Version)

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma-\frac{1}{2}} e^{\frac{-\pi}{2}|t|},$$

for fixed σ as $|t| \to \infty$.
A gamma factor is defined as

$$\gamma(f, s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma(\frac{s + \kappa_j}{2}),$$

where $\kappa_j \in \mathbb{C}$ are called the *local parameters of* $L(f, s)$ *at* ∞. Moreover, we assume that either $\kappa_j \in \mathbb{R}$ or they come as conjugate pairs and also $\Re(\kappa_j) > -1$.

Note A gamma factor is nonzero on $\mathbb{C}$ and has no pole on $\Re(s) \geq 1$.

An integer $q(f) \geq 1$ denotes the *conductor* of $L(f, s)$. This integer has the property that for $1 \leq i \leq d$, $\alpha_i(p) \neq 0$ for $p \nmid q(f)$. A prime $p \nmid q(f)$ is said to be *unramified.*

The *dual* of $L(f, s)$ is an L-function $L(\bar{f}, s)$ defined by $L(\bar{f}, s) = \overline{L(f, \bar{s})}$. The following are parameters of $L(\bar{f}, s)$.

$$\lambda_{\bar{f}}(n) = \overline{\lambda_f(n)}, \quad , (\bar{\alpha}_i(p))_{\bar{f}} = \overline{(\alpha_i(p))_f} \quad \gamma(\bar{f}, s) = \gamma(f, s), \quad q(\bar{f}) = q(f).$$

If $L(\bar{f}, s) = L(f, s)$ then $L(f, s)$ is called *self dual.*
We define the *complete L-function* $\Lambda(f, s)$ by

$$\Lambda(f, s) = q(f)^{\frac{s}{2}} \gamma(f, s) L(f, s).$$

It is clear that $\Lambda(f, s)$ is holomorphic on $\Re(s) > 1$.

Complex Analysis 4 *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Let* $M(r) = \max\{|f(s)|, \text{ where } |s| = r\}$. *By the Maximum Modulus Principle and Liouville's theorem, we know that*

$$\lim_{r \to \infty} M(r) = \infty.$$

Now if there is a $\beta \geq 0$ such that

$$f(s) = O(e^{|s|^\beta})$$

as $|s| \to \infty$, we say that f is of finite order. We set

$$\text{ord}(f) = \inf\{\beta \geq 0; \quad f(s) = O(e^{|s|^\beta}) \text{ as } |s| \to \infty\}.$$

It is clear that if f is of finite order then $\text{ord}(f) \geq 0$.

Examples Polynomials have order zero. e^s, $\sin s$, $\cos s$ and $1/\Gamma(s)$ have order 1. e^{e^s} has infinite order.

2. Iwaniec-Kowalski Class

In their book (Analytic Number Theory, AMS, 2004) Iwaniec and Kowalski consider the following class of L-functions.

Definition 5 *We say that an L-function $L(f,s)$ with the gamma factor $\gamma(f,s)$, the conductor $q(f)$, and complete L-function $\Lambda(f,s)$ is in class $\mathcal{IK}$ if it satisfies the following three conditions.*

- *Holomorphy: $\Lambda(f,s)$ admits a meromorphic continuation to the whole complex plane with at most poles at $s=0$ or $s=1$.*
- *Functional Equation: $\Lambda(f,s)$ satisfies a functional equation*

$$\Lambda(f,s) = \epsilon(f)\Lambda(\bar{f}, 1-s),$$

  where $\epsilon(f)$ is a complex number with $|\epsilon(f)| = 1$. $\epsilon(f)$ is called the root number of $L(f,s)$.
- *Growth: $(s(1-s))^r\Lambda(f,s)$ is an entire function of order 1, where r is the order of pole or zero of $\Lambda(f,s)$ at $s=1$. (If $\Lambda(f,s)$ has a pole at $s=1$, then $r>0$ and if $\Lambda(f,s)$ has a zero at $s=1$ then $r<0$, otherwise $r=0$.)*

Exercise 6 a) *Let $r_0(f)$ be the order of pole or zero of $\Lambda(f,s)$ at $s=0$. Show that $r_0(f)=r$.*
b) *Show that r is also the order of pole or zero of $L(f,s)$ at $s=1$.*

Exercise 7 *Show that $(s(1-s))^r L(f,s)$ is an entire function of finite order. What is the order of this function?*

Exercise 8 a) *Show that the class $\mathcal{IK}$ is closed under multiplication. Also show that if $L(f,s)\in\mathcal{IK}$ then $L(\bar{f},s)\in\mathcal{IK}$.*
b) *If $L(f,s)\in\mathcal{IK}$ is entire, then for any $t\in\mathbb{R}$, $L(f,s+it)L(\bar{f},s-it)\in\mathcal{IK}$.*

Exercise 9 *Show that if $L(f,s)$ is self dual then $\epsilon(f)=\pm1$. Moreover in this case if $\epsilon(f)=-1$ then $L(f,\frac{1}{2})=0$.*

Exercise 10 *Show that if $s_0\neq0$ is a pole of $\gamma(f,s)$ then $L(f,s_0)=0$. Such zero is called a* trivial zero *of $L(f,s)$.*

3. Fundamental Conjectures

We describe some fundamental conjectures regarding the size of the local parameters and the coefficients of an L-function in $\mathcal{IK}$.

Ramanujan-Petersson Conjecture For all $p \nmid q(f)$ we have $|\alpha_i(p)| = 1$ and for all $p \mid q(f)$ we have $|\alpha_i(p)| \leq 1$.

The following exercise describes the consequence of the Ramanujan-Petersson Conjecture for the coefficient $\lambda_f(n)$.

Exercise 11 *Let $\tau_d(n)$ denote the number of representations of $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ as the product of d natural numbers. Then*
(i) Show that $\tau_d(n) = \binom{a_1+d-1}{d-1}\binom{a_2+d-1}{d-1}\cdots\binom{a_r+d-1}{d-1}$.
(ii) Show that $\tau_d(n) = O_{\epsilon,d}(n^{\epsilon})$, where the implied constant depends only on ϵ and d.
(iii) Conclude that if $L(f,s)$ satisfies the Ramanujan-Petersson conjecture then

$$\lambda_f(n) = O_{\epsilon,d}(n^{\epsilon}).$$

Ramanujan-Petersson Conjecture at ∞ (or Generalized Selberg Conjecture) For any j, $\Re(\kappa_j) \geq 0$.

Equivalently the Ramanujan-Petersson Conjecture at ∞ states that $\gamma(f,s)$ has no pole for $\Re(s) > 0$.

Complex Analysis 12 (**The Phragmen-Lindelöf principal for a strip**) *Let f be an entire function of finite order. Assume that*

$$|f(a+it)| \leq M_a(|t|+1)^{\alpha}, \quad \text{and} \quad |f(b+it)| \leq M_b(|t|+1)^{\beta}$$

for $t \in \mathbb{R}$. Then

$$|f(\sigma+it)| \leq M_a^{l(\sigma)} M_b^{1-l(\sigma)} (|t|+1)^{\alpha l(\sigma)+\beta(1-l(\sigma))},$$

for all $s = \sigma + it$ in the strip $a \leq \sigma \leq b$, where l is the linear function such that $l(a) = 1$ and $l(b) = 0$.

Exercise 13 *Use the Phragmen-Lindelöf principal for a strip to show that any $L(f,s) \in \mathcal{IK}$ is polynomially bounded (equivalently $\Lambda(f,s)$ is bounded) in the vertical strip $s = \sigma + it$*

with $a \leq \sigma \leq b$, $|t| \geq 1$. Moreover, show that if $L(f,s)$ satisfies the Ramanujan-Petersson conjecture then there is $A > 0$ such that

$$L(f, \sigma + it) \ll_d \left(q(f) \prod_{j=1}^{d} (|s + \kappa_j| + 1)\right)^A,$$

for all $s = \sigma + it$ with $a \leq \sigma \leq b$, $|t| \geq 1$, where the implied constant depends only on d.

The next conjecture can be considered as a global version of the Ramanujan-Petersson Conjecture.

Lindelöf Hypothesis (Conjecture)

$$L(f, \frac{1}{2} + it) \ll_\epsilon \left(q(f) \prod_{j=1}^{d} (|it + \kappa_j| + 1)\right)^\epsilon.$$

The implied constant depends only on ϵ.

4. What are the principal arithmo-geometric L-functions?

In [Se], Selberg introduced a certain class of L-functions (Selberg class) and he made some conjectures regarding the elements of this class. Conjecturally all the principal arithmetic and geometric L-functions are in this class. Here, we introduce this class and the Selberg Orthogonality Conjecture to motivate our future discussion of Ranking-Selberg L-functions.

Selberg Class The Selberg class $\mathcal{S}$ consists of functions $L(f,s)$ of a complex variable s satisfying the following properties:

1. (Dirichlet series): For $\Re(s) > 1$, $L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$ where $\lambda_f(1) = 1$.
2. (Analytic continuation): For some integer $m \geq 0$, $(s-1)^m L(f,s)$ extends to an entire function of finite order.
3. (Functional equation): There are numbers $Q(f) > 0$, $\delta_j > 0$, $r_j \in \mathbb{C}$ with $\Re(r_i) \geq 0$ such that

$$\Lambda(f,s) = Q(f)^s \prod_{j=1}^{d} \Gamma(\delta_j s + r_j) L(f,s)$$

satisfies the functional equation

$$\Lambda(f,s) = \epsilon(f) \bar{\Lambda}(f, 1-s)$$

where $\epsilon(f)$ is a complex number with $|\epsilon(f)| = 1$ and $\bar{\Lambda}(f,s) = \overline{\Lambda(f,\bar{s})}$.
4. (Euler product): For $\Re(s) > 1$, $L(f,s) = \prod_p L_p(f,s)$, where

$$L_p(f,s) = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right)$$

and $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$, and p denotes a prime number here.
5. (Ramanujan hypothesis): For any fixed $\epsilon > 0$, $\lambda_f(n) = O_\epsilon(n^\epsilon)$.

Exercise 14 *Let $L(f,s) \in \mathcal{IK}$ be an L-function that satisfies the Ramanujan-Petersson Conjecture and the Generalized Selberg Conjecture. Moreover assume that $L(f,s)$ does not have a pole at $s = 0$. Then show that $L(f,s) \in \mathcal{S}$.*

Note that the functional equation for $L(f,s) \in \mathcal{S}$ is not unique, by virtue of Legendre's duplication formula (i.e. $\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})$). However, one can show that the sum of δ_j's is well defined. We define the *degree of* $L(f,s) \in \mathcal{S}$ by

$$\deg L(f,s) = 2\sum_{j=1}^{d} \delta_j.$$

An element $L(f,s) \in \mathcal{S}$ is called *primitive* if $L(f,s) \neq 1$ and $L(f,s) = L(f_1,s)L(f_1,s)$ implies $L(f_1,s) = 1$ or $L(f_2,s) = 1$.

Selberg Orthogonality Conjecture If $L(f,s), L(g,s) \in \mathcal{S}$ are primitive functions then

$$\sum_{p \leq x} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} = \delta_{f,g} \log\log x + O(1)$$

as $x \to \infty$, where $\delta_{f,g} = \begin{cases} 1 & \text{if} \quad L(f,s) = L(g,s) \\ 0 & \text{if} \quad L(f,s) \neq L(g,s) \end{cases}$.

5. Ranking-Selberg L-functions

Let $L(f,s)$, $L(g,s) \in \mathcal{IK}$ be L-functions of degree d and e, with local parameters $(\alpha_i(p))$ and $(\beta_j(p))$ and local parameters at ∞, κ_i and ν_j respectively. For $p \nmid q(f)q(g)$, let

$$L_p(f \otimes g, s) = \prod_{i,j}(1 - \alpha_i(p)\overline{\beta_j(p)}p^{-s})^{-1}.$$

Definition 15 *We say that f and g have a Ranking-Selberg convolution if there exists an L-function $L(f \otimes g, s)$ in $\mathcal{IK}$ that satisfies the following conditions:*

- *$L(f \otimes g, s)$ is a degree de L-function. More precisely*

$$L(f \otimes g, s) = \prod_{p \nmid q(f)q(g)} L_p(f \otimes g, s) \prod_{p \mid q(f)q(g)} H_p(p^{-s}),$$

where

$$H_p(p^{-s}) = \prod_{j=1}^{de} (1 - \gamma_j(p)p^{-s})^{-1} \quad \text{with} \quad |\gamma_j(p)| < p.$$

- *The gamma factor is written as*

$$\gamma(f \otimes g, s) = \pi^{-des/2} \prod_{i,j} \Gamma(\frac{s + \mu_{i,j}}{2}),$$

where $\Re(\mu_{i,j}) \leq \Re(\kappa_i + \nu_j)$ and $|\mu_{i,j}| \leq |\kappa_i| + |\nu_j|$.

- *$q(f \otimes g) \mid q(f)^e q(g)^d$, where $q(f \otimes g)$ is the conductor of $f \otimes g$.*

- *If $f = g$ then $L(f \otimes g, s)$ has a pole at $s = 1$.*

6. What Rankin-Selberg L-functions got to do with it ?

Next we show that the existence of Ranking-Selberg L-functions provides valuable information regarding the size of the coefficients of an L-function.

Exercise 16 *Show that for $(n, q(f)) = 1$, $\lambda_{f \otimes f}(n) \geq 0$.*

Exercise 17 *Show that if $L(f \otimes f, s)$ or $L(f \otimes \bar{f}, s)$ exists, then $|\alpha_i(p)| < \sqrt{p}$ for $p \nmid q(f)$ and $\Re(\kappa_i) > -\frac{1}{2}$.*

Definition 18 *The Ramanujan τ-function is defined by the formal generating function*

$$x \left\{ \prod_{n=1}^{\infty} (1 - x^n) \right\}^{24} = \sum_{n=1}^{\infty} \tau(n) x^n = x(1 - 24x + 252x^2 - 1472x^3 + 4830x^4 \cdots).$$

For historical background about $\tau(n)$ see Chapter 10 of [H].

Definition 19 $\tau^*(n) = \frac{\tau(n)}{n^{\frac{11}{2}}}$.

Some questions regarding $\tau^*(n)$:
1) What is the order of $\tau^*(n)$?
2) What is the order of $\sum_{n \le x} |\tau^*(n)|$?
3) What is the order of $\sum_{n \le x} \tau^*(n)$?
4) What is the order of $\sum_{\substack{p \le x \\ p,\text{prime}}} \tau^*(p)$?

Exercise 20 *a) Show that*

$$\prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{\frac{n(n+1)}{2}}.$$

b) Show that

$$\sum_{n=0}^{\infty} (2n+1) x^{\frac{n(n+1)}{2}} = O\left(\frac{1}{1-x}\right)$$

as $x \to 1$.

Proposition 21 $\tau^*(n) = O(n^{\frac{5}{2}})$.

Proof From the previous exercise we have

$$|\tau(n)| x^n \le \sum_{n=1}^{\infty} |\tau(n)| x^n \le x \left(\sum_{n=0}^{\infty} (2n+1) x^{\frac{n(n+1)}{2}} \right)^8 \le A \frac{x}{(1-x)^8}$$

for some constant $A > 0$ as $x \to 1^-$. So

$$|\tau(n)| \le \frac{A}{x^{n-1}(1-x)^8}$$

as $x \to 1^-$. Taking $x = 1 - \frac{1}{n}$ implies the result. $\square$

Exercise 22 *Show that* $L(\tau^* \times \tau^*, s) = \sum_{n=1}^{\infty} \frac{(\tau^*(n))^2}{n^s}$ *is convergent for* $\Re(s) > 6$. *Show that if* $L(\tau^* \times \tau^*, s)$ *has an analytic continuation to* $\Re(s) > 1$, *then for* $\epsilon > 0$

$$\sum_{n \le x} (\tau^*(n))^2 = O(x^{1+\epsilon}).$$

Theorem 23 (**Hardy**) $\sum_{n \le x} (\tau^*(n))^2 = O(x)$.

Proof (**Sketch**) Let $f(q) = \sum_{n=1}^{\infty} \tau(n) q^n$ for $|q| < 1$. We note that $q = e^{2\pi i z}$ gives a map from the upper half-plane $\mathcal{H} = \{x + iy : \ y > 0\}$ to the punctured open unit disc centered at the origin. So we can consider

$$\tilde{f}(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

from $\mathcal{H}$ to $\mathbb{C}$. It is clear that $\tilde{f}(z+1) = \tilde{f}(z)$. In other words for fixed y,

$$\tilde{\tilde{f}}(x) = \tilde{f}(x + iy)$$

is a periodic function on $\mathbb{R}$ so it has a Fourier expansion. Thus

$$\tilde{\tilde{f}}(x) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n x} = \sum_{n=1}^{\infty} \tau(n) e^{-2\pi n y} e^{2\pi i n x}.$$

Here $a(n) = \tau(n) e^{-2\pi n y}$ is the n-th Fourier coefficient of $\tilde{\tilde{f}}(x)$. So by Parseval identity we have

$$\sum_{n=1}^{\infty} (\tau(n) e^{-2\pi n y})^2 = \int_0^1 |\tilde{\tilde{f}}(x)|^2 dx = \int_0^1 |\tilde{f}(z)|^2 dx. \tag{1}$$

One can show that $\tilde{f}(z)$ has the following transformation property

$$\tilde{f}\left(\frac{-1}{z}\right) = z^{12} \tilde{f}(z).$$

From here one can conclude that $y^6 |\tilde{f}(z)|$ is invariant under action of the group $\Gamma = SL_2(\mathbb{Z})$ on the upper half plane. So $y^6 |\tilde{f}(z)|$ is a Γ-periodic function that vanishes at the cusp at ∞ and therefore it is bounded on $\mathcal{H}$. Thus

$$y^6 |\tilde{f}(z)| \ll 1.$$

Applying this bound in (1) implies that

$$\sum_{n=1}^{\infty} (\tau(n))^2 e^{-4\pi n y} \ll y^{-12}.$$

Now taking $y = 1/x$ implies the result. $\square$

The following is a direct consequence of the previous theorem and the Cauchy-Schwarz inequality.

Corollary 24 $\sum_{n\leq x}|\tau^*(n)| = O(x)$.

Corollary 25 $\tau^*(n) = O(n^{\frac{1}{2}})$.

Proof From Theorem 23 we have

$$(\tau^*(n))^2 = \sum_{m\leq n}(\tau^*(m))^2 - \sum_{m\leq n-1}(\tau^*(m))^2 \ll n.$$

$\square$

Exercise 26 *Let $A(x) = \sum_{n\leq x} a_n = O(x^\delta)$. Show that for $\Re(s) > \delta$,*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = s\int_1^{\infty}\frac{A(t)}{t^{s+1}}dt.$$

Hence the Dirichlet series converges for $\Re(s) > \delta$. Conclude that the Dirichlet series $L(\tau^*, s) = \sum_{n=1}^{\infty}\frac{\tau^*(n)}{n^s}$ is absolutely convergent for $\Re(s) > 1$.

Two of our goals in the remaining lectures are proving the following theorems.

Theorem 27 (**Rankin**) *$L(\tau^*, s) \neq 0$ on the line $\Re(s) = 1$.*

Proof See Theorem 35 and Lecture 3. $\square$

Theorem 28 (**Rankin**) *$\tau^*(n) = O(n^{\frac{3}{10}})$.*

Proof See Corollary 99. $\square$

Note Deligne proved that $|\tau^*(n)| \leq d(n)$. Note that $d(n) = O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$.

Lecture 2
Zeros of L-functions

In this lecture we show that upon existence of Rankin-Selberg L-functions and some other conditions, an L-function has no zero on the line $\Re(s) = 1$ and a narrow region to the left of this line. All the L-functions in this lecture are in $\mathcal{IK}$ class.

7. Zeros of $\Lambda(f, s)$

Recall that $\Lambda(f, s)$ is a complete L-function in $\mathcal{IK}$ class. We list some elementary properties of zeros of $\Lambda(f, s)$. These properties are direct corollary of definition of the class $\mathcal{IK}$ and the following complex analysis fact regarding entire functions of order 1.

Complex Analysis 29 *If the relation $f(s) = O(e^{|s|})$ does not hold for an entire function of order 1, then*

$$\sum_{\rho \neq 0} \frac{1}{|\rho|} \quad \text{is divergent,}$$

and

$$\sum_{\rho \neq 0} \frac{1}{|\rho|^{1+\epsilon}} \quad \text{is convergent,}$$

where ρ denotes zeros of the function and $\epsilon > 0$. So in this case $f(s)$ has infinitely many zeros.

Exercise 30 *(i) $\Lambda(f, s)$ has infinitely many zeros.*
(ii) $\Lambda(f, s)$ and $L(f, s) \neq 0$ on the half plane $\Re(s) > 1$.
(iii) All zeros ρ of $\Lambda(f, s)$ are in the critical strip $0 \leq \sigma \leq 1$.

(iv) If $\Re(\kappa_j) \geq 0$ for $1 \leq j \leq d$, then on the strip $0 < \Re(s) \leq 1$, zeros of $\Lambda(f, s)$ and zeros of $L(f, s)$ coincide.
(v) $\sum_{\rho \neq 0} \frac{1}{|\rho|^{1+\epsilon}} < \infty$, where ρ runs over zeros of $\Lambda(f, s)$ and $\epsilon > 0$.
(vi) If ρ is a zero of $\Lambda(f, s)$ then $1 - \bar{\rho}$ is also a zero of $\Lambda(f, s)$.

8. Non-vanishing of L-functions on the line $\Re(s) = 1$

In this section we give a proof of the classical theorem of Rankin and Ogg for L-functions in $\mathcal{IK}$.
We define

$$\Lambda_f(n) = \begin{cases} \sum_{j=1}^{d} \alpha_j(p)^k \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\Lambda_f(n)$ is different from $\Lambda(f, s)$.

Complex Analysis 31 (**Logarithm of functions**) *Let $f(s)$ be a function that is analytic and never 0 on a simply connected region A. Then there is a function $g(s)$ analytic on A and unique up to the addition of a constant multiple of $2\pi i$ such that $e^{g(s)} = f(s)$. Any $g(s)$ has formal properties similar to $\log f(s)$.*

Exercise 32 *Let $\log z$ denote the principle branch of logarithm. Then $-\log(1 - z)$ is analytic on $|z| < 1$ and it has the following Taylor expansion*

$$-\log(1 - z) = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$

Use this fact to show that for $\sigma = \Re(s) > 2$, we have

$$L(f, s) = \exp\left(\sum_{p} \sum_{k=1}^{\infty} \frac{\Lambda_f(p^k)}{k \log p \; p^{ks}} \right).$$

Explain why it is reasonable to define for $\sigma > 2$

$$\log L(f, s) = \sum_{p} \sum_{k=1}^{\infty} \frac{\Lambda_f(p^k)}{\log p^k \; p^{ks}}.$$

Conclude that for $\sigma > 2$

$$-\frac{L'}{L}(f,s) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}.$$

Moreover show that for $\sigma > 2$

$$\left| -\frac{L'}{L}(f,s) \right| \leq d\zeta'(\sigma - 1).$$

Complex Analysis 33 (Landau's lemma) *A Dirichlet series with non-negative coefficients has a singularity at its abscissa of convergence.*

Lemma 34 *Let $f(s)$ be a complex function that satisfies the following:*
(i) $f(s)$ is analytic on the half-plane $\Re(s) > \sigma_0$;
(ii) $f(s)$ has a representation in the form

$$f(s) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s}\right),$$

with $c(n) \geq 0$ on the half-plane $\Re(s) > \sigma_1$ ($\sigma_1 > \sigma_0$).
Then $f(s) \neq 0$ for $\Re(s) > \sigma_0$.

Proof Let σ_2 be the abscissa of convergence of $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$. We claim that $\sigma_2 \leq \sigma_0$. To prove this let us assume that $\sigma_0 < \sigma_2 \leq \sigma_1$. Then for $\sigma > \sigma_2$ we have

$$f(\sigma) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma}\right). \tag{2}$$

Now since $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ is divergent at σ_2 and $f(s)$ is well defined at σ_2 the equality (2) shows that $f(\sigma_2) = 0$ and $\lim_{\sigma \to \sigma_2} \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} = -\infty$. (This is true since if $f(\sigma_2) \neq 0$, then $|f(s)| \neq 0$ on a neighborhood of σ_2 and so $\log f(s)$ gives a holomorphic continuation of $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ to the left of σ_2 which is a contradiction.) But $\lim_{\sigma \to \sigma_2} \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} = -\infty$ is impossible since $c(n) \geq 0$. So $\sigma_2 \leq \sigma_0$. This shows that $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ is convergent for $\Re(s) > \sigma_0$, and

$$f(s) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s}\right)$$

for $\Re(s) > \sigma_0$. So $f(s) \neq 0$ on $\Re(s) > \sigma_0$. $\square$

Theorem 35 (**Rankin (1939), Ogg (1969)**)[2] *Let $L(f,s)$ be an entire L-function. Let $L(f \otimes f, s)$ exists and it has a simple pole at $s=1$. Then $L(f, 1+it) \neq 0$ for all real t.*

Proof Suppose that $L(f, 1+it_0) = 0$, and let

$$g(s) = \zeta(s)\ L(f, s+it_0)\ L(\bar{f}, s-it_0)\ L(f \otimes f, s).$$

It is clear that $g(s)$ is entire. Now note that for $\Re(s) > 1$,

$$g_{\mathrm{unr}}(s) = \exp\left(\sum_{(p,q(f))=1} \sum_{k=1}^{\infty} \frac{|1 + \sum_j \alpha_j^k p^{-kit_0}|^2}{kp^{ks}} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s} \right)$$

where $c(n) \geq 0$. So, $g_{\mathrm{unr}}(s)$ satisfies the conditions of Lemma 34 with $\sigma_1 = 1$, and therefore $g_{\mathrm{unr}}(s)$ and $g(s) \neq 0$ everywhere. This is a contradiction since $g(-2) = 0$. $\square$

9. More on zeros of $\Lambda(f, s)$

In this section we derive an identity (Lemma 41) regarding the zeros of $\Lambda(f,s)$. This identity plays an important role in establishing a zero free region for $L(f,s)$.

Complex Analysis 36 (**Weierstrass, Hadamard**) *Let f be an entire function of order 1. Then*

$$f(s) = s^r e^{a+bs} \prod_{\rho \neq 0} (1 - \frac{s}{\rho}) e^{s/\rho},$$

uniformly and absolutely on all compact subsets of $\mathbb{C}$, where r is the order of the zero of f at $s=0$ and ρ runs over zeros of f different from 0.

As part of Weierstrass's theory we assume the legitimacy of any formal transformation of the above product formula.

Theorem 37 *There exists constants $a = a(f)$ and $b = b(f)$ such that*

$$(s(1-s))^r \Lambda(f, s) = e^{a+bs} \prod_{\rho \neq 0,1} (1 - \frac{s}{\rho}) e^{s/\rho},$$

[2] Rankin and Ogg proved this theorem for modular L-functions. Rankin proved the case $t \neq 0$, and Ogg gave a proof for $t = 0$. Our proof here is in spirit of Ogg's proof.

where ρ ranges over all zeros of $\Lambda(f,s)$ different from 0 and 1. This expansion is uniformly and absolutely convergent on compact subsets of complex plane. Moreover, the following identity is valid on any subset of complex plane that avoids zeros of $L(f,s)$.

$$-\frac{L'}{L}(f,s) = \frac{1}{2}\log q(f) + \frac{\gamma'}{\gamma}(f,s) - b(f) + \frac{r}{s} + \frac{r}{s-1} - \sum_{\rho\neq 0,1}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \qquad (3)$$

Proof These are consequences of Complex Analysis 36 and logarithmic differentiation. □

Exercise 38 *Utilize the functional equation to show that*

$$\Re(b(f)) = -\sum_{\rho}\Re\left(\frac{1}{\rho}\right).$$

Analytic Conductor (Iwaniec-Sarnak) The *conductor of $L(f,s)$ at ∞* is defined as

$$q_\infty(f,s) = \prod_{j=1}^{d}(|s+\kappa_j|+3).$$

The analytic conductor of $L(f,s)$ is defined as

$$Q(f,s) = q(f)q_\infty(f,s) = q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3).$$

We let

$$Q(f) = Q(f,0) = q(f)\prod_{j=1}^{d}(|\kappa_j|+3),$$

and similarly $q_\infty(f) = q_\infty(f,0)$.

Lemma 39 $d \leq \log q_\infty(f) \leq \log Q(f)$.

Proof We have

$$q_\infty(f) = \prod_{j=1}^{d}(|\kappa_j|+3) \geq 3^d.$$

The result follows by taking the logarithm from both sides of this inequality. □

Complex Analysis 40

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right),$$

is valid as $|s|\to\infty$, in the angle $-\pi+\delta < \arg s < \pi-\delta$, for any fixed $\delta > 0$.

Estimation of the gamma factors By employing Complex Analysis 40 we can estimate the gamma factors as follows. For $\Re(s) > 1$ we have

$$\frac{\gamma'}{\gamma}(f,s) \ll d + \log q_\infty(f,s) + \sum_{|s+\kappa_j|<1} \frac{1}{|s+\kappa_j|}. \tag{4}$$

Lemma 41 (Main Identity) *On the half-plane $\Re(s) > 1$*

$$\sum_{\rho\neq 0,1} \Re(\frac{1}{s-\rho}) = \Re(\frac{r}{s-1}) + \Re(\frac{r}{s}) + O(\sum_{|s+\kappa_j|<1} \frac{1}{|s+\kappa_j|}) + \Re(\frac{L'}{L}(f,s)) + O(\log Q(f,s)),$$

where ρ ranges over all zeros of $\Lambda(f,s)$ different from 0 and 1.

Proposition 42 *Let $\rho = \beta + i\gamma$ denote the zeros of $\Lambda(f,s)$. Then*

$$\sum_\rho \frac{1}{1+(T-\gamma)^2} \ll \sum_\rho \Re(\frac{1}{3+iT-\rho}) \ll \log Q(f,iT).$$

The implied constant depends only on r.

Note You should not confuse the ordinate γ with the factors $\gamma(f,s)$.

Proof Let $s = 3+iT$ in the Main Identity. The result follows. □

Corollary 43 *Let $N(f,T)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(f,s)$ such that $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq T$. Then*

$$N(f,T+1) - N(f,T) = O(\log Q(f,iT)).$$

The implied constant depends only on r.

Proof Since zeros of $L(f,s)$ and $\Lambda(f,s)$ on the critical strip are basically the same (at most finitely many exceptions), from the previous proposition, we have

$$N(f,T+1) - N(f,T) = \sum_{\substack{\rho \\ T<\gamma\leq T+1}} 1 \leq \sum_\rho \frac{2}{1+(T-\gamma)^2} \ll \log Q(f,iT).$$

□

Note One can show that

$$N(f,T) = \frac{d}{2\pi} T\log T + cT + O(\log T),$$

as $T \to \infty$. Here c is a constant depends on $L(f,s)$.

10. A zero-free region

In this section following [GHL] and [IK], we show the existence of a zero free region for $L(f,s)$.

Lemma 44 $Q(f,s) \le Q(f)(|s|+3)^d$.

Proof We have

$$Q(f,s) = q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3) \le q(f)\prod_{j=1}^{d}(|k_j|+3)(|s|+3) = Q(f)(|s|+3)^d.$$

$\square$

Let $L_{\mathrm{r}}(f,s)$ (respectively $L_{\mathrm{ur}}(f,s)$) be the Euler product of $L(f,s)$ restricted to ramified (respectively unramified) primes.

Exercise 45 *Assume that for any ramified prime (i.e. p such that $(p,q(f)) \ne 1$) we have $|\alpha_j(p)| \le p/2$. Then show that for $\sigma > 1$*

$$\Re(\frac{L_{\mathrm{r}}'}{L_{\mathrm{r}}}(f,\sigma)) = O(d\log q(f)).$$

Lemma 46 (Goldfeld, Hoffstein and Lieman (1994) via de la Vallée Poussin (1899)) *Suppose that $\Re(\Lambda_f(n)) \ge 0$ for $(n,q(f)) = 1$. Suppose that $\Re(\kappa_j) > -1/2$ and at ramified primes $|\alpha_j(p)| \le p/2$. Let r be the order of $L(f,s)$ at $s=1$. Then*
(i) $L(f,1) \ne 0$. In other words r is non-negative.
(ii) There exists an effective constant $c > 0$, depending only on r, such that $L(f,s)$ has at most r real zeros in the interval

$$\sigma \ge 1 - \frac{c}{d(r+1)\log Q(f)}.$$

Proof Let $1 < \sigma < \frac{3}{2}$. Then from Lemmas 41 and 44

$$\sum_{\rho\ne 0,1}\Re(\frac{1}{\sigma-\rho}) = \frac{r}{\sigma-1} + \frac{r}{\sigma} + O(\sum_{|\sigma+\kappa_j|<1}\frac{1}{|\sigma+\kappa_j|}) + \Re(\frac{L'}{L}(f,\sigma)) + O(\log Q(f)),$$

where ρ ranges over all zeros of $\Lambda(f,s)$ different from 0 and 1. Now note that

$$\Re(\frac{1}{\sigma-\rho}) > 0, \quad \Re(\frac{L_{\mathrm{ur}}'}{L_{\mathrm{ur}}}(f,\sigma)) \le 0, \quad \text{and} \quad \Re(\frac{L_{\mathrm{r}}'}{L_{\mathrm{r}}}(f,\sigma)) = O(d\log q(f)).$$

So if β_j's are the zeros of $L(f,s)$ in the interval $[\frac{1}{2},1)$, there exists a constant c_1 (depending only on r) such that

$$\sum_j \frac{1}{\sigma-\beta_j} \leq \frac{r}{\sigma-1} + c_1 d \log Q(f).$$

(Note that since $\Re(\kappa_j) > -1/2$, β_j's are also zeros of $\Lambda(f,s)$). Now if $\sigma \to 1^+$ this inequality shows that r cannot be negative, so $r \geq 0$.

Now for $\delta, c > 0$ let $\sigma = 1+\delta/d\log Q(f)$ and m be the number of zeros of $L(f,s)$ in the interval $(1-c/d(r+1)\log Q(f), 1)$. Then from the previous inequality, we have

$$m \leq (\delta + \frac{c}{r+1})(\frac{r}{\delta}+c_1) = r + \delta c_1 + \frac{c}{\delta}(\frac{r}{r+1}) + \frac{cc_1}{r+1}.$$

Now let $\delta < c_1^{-1}$, then we can choose c small enough such that $m \leq r$. The proof now is complete. $\square$

Let $L(f,s)$ be an entire L-function of degree d with at least one non-real coefficient (i.e $L(f,s)$ is not self-dual). Suppose that the Ranking-Selberg L-functions $L(f\otimes f,s)$ and $L(f\otimes\bar{f},s)$ exist. Also suppose that $\Re(\mu_{i,j}) > -\frac{1}{2}$. (Note that this implies that $\Re(\kappa_j) > -1/4$.) Moreover we assume that the pole of $L(f\otimes f,s)$ at $s=1$ is simple and $L(f\otimes\bar{f},s)$ is entire. Finally we assume that the local parameters $\alpha_j(p)$ of $L(f,s)$, $L(f\otimes f,s)$ and $L(f\otimes\bar{f},s)$ at the ramified primes satisfy in the inequality $|\alpha_j(p)| \leq p/2$.

Exercise 47 $Q(f\otimes f) \ll Q(f)^{2d}$.

The next theorem establishes a zero free region for such L-functions.

Theorem 48 *There exists an absolute constant $c>0$ such that $L(f,s)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{d^4\log\left(Q(f)(|t|+3)\right)}.$$

Proof For $t\in\mathbb{R}$, we let

$$L(g,s) = \zeta(s)L(f,s+it)^2L(\bar{f},s-it)^2L(f\otimes\bar{f},s+2it)L(\bar{f}\otimes f,s-2it)L(f\otimes f,s)^2.$$

It is clear that $L(g,s)$ is an L-function of degree $(1+2d)^2$. By employing Exercise 47 we have

$$Q(g) \ll Q(f)^{4+8d}(|t|+3)^{6d^2}.$$

Also we have $\Lambda_g(n) \geq 0$ for any n coprime to $q(g)$ (or $q(f)$).

Now let $\rho = \beta + i\gamma$ be a zero of $L(f,s)$ with $\beta \geq 1/2$. In $L(g,s)$ let $t = \gamma$. Then $L(g,s)$ has a pole of order at most 3 at $s = 1$ and a zero of order at least 4 at $s = \beta$. By Lemma 46 it is clear that

$$\beta < 1 - \frac{c}{d^2 \log Q(g)} < 1 - \frac{c'}{d^4 \log\left(Q(f)(|t|+3)\right)},$$

for some absolute constants $c > 0$ and $c' > 0$. The proof now is complete. □

Let $L(f,s)$ be an entire L-function of degree d with real coefficients (i.e $L(f,s)$ is self-dual). Suppose that the Ranking-Selberg L-functions $L(f \otimes f, s)$ exists. Also we suppose that $\Re(\mu_{i,j}) > -\frac{1}{2}$. (Note that this implies that $\Re(\kappa_j) > -1/4$.) Moreover we assume that the pole of $L(f \otimes f, s)$ at $s = 1$ is simple. Finally we assume that the local parameters $\alpha_j(p)$ of $L(f,s)$ and $L(f \otimes f, s)$ at the ramified primes satisfy in the inequality $|\alpha_j(p)| \leq p/2$. The next theorem establishes an almost zero free region for such L-functions.

Theorem 49 *There exists an absolute constant $c > 0$ such that $L(f,s)$ has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{d^4 \log\left(Q(f)(|t|+3)\right)},$$

except possibly for one simple real zero $\beta_f < 1$.

Proof The proof is the same as the previous theorem. The only difference is that if $t = \gamma = 0$, then $L(g,s)$ has a pole of order at most 5 (in fact exactly 5) at $s = 1$ and a zero of order at least 4 at $s = \beta$. So by Lemma 46 there is an absolute constant $c > 0$ such that $L(f,s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c}{d^4 \log\left(Q(f)(|t|+3)\right)},$$

except possibly for one simple real zero β_f. Since $L(f,s)$ is non-vanishing on the line $\Re(s) = 1$, we have $\beta_f < 1$. □

Note The possible simple real zero β_f of $L(f,s)$ is called the *exceptional zero* or the *Siegel zero*.

Lecture 3
Poles of L-functions, Examples

10. Riemann Zeta Function

In this lecture we turn our attention to the problem of analytic continuation of a Dirichlet series to the whole complex plane. After reviewing analytic continuation of the Riemann zeta function, we apply a similar method to deduce analytic continuation of the Epstein zeta function, modular L-functions, and Rankin-Selberg convolution of two modular L-functions. This lecture is an exposition of [R2]. For the related historical background see [D] and chapter 10 of [H].

Fourier Analysis 50 *Let $\mathbb{S}$ (the Schwartz space) be the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ which decreases at infinity faster than any negative power function, i.e., $|x|^N f(x) \to 0$ as $x \to \pm\infty$ for all N. For any $f \in \mathbb{S}$ we define its Fourier transform $\hat{f}$ by*

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx.$$

Exercise 51 *Show that $f(x) = e^{-\pi x^2} \in \mathbb{S}$, and $\hat{f} = f$.*

Fourier Analysis 52 (**Poisson Summation Formula**) *If $f \in \mathbb{S}$, then*

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \hat{f}(m).$$

Definition 53 *The theta-function $\theta(\omega)$ is defined by*

$$\theta(\omega) = \sum_{n=-\infty}^{\infty} e^{-\pi \omega n^2},$$

for $\omega > 0$.

Exercise 54 *By employing the Poisson summation formula show that $\theta(\omega)$ satisfies the functional equation*

$$\theta\left(\frac{1}{\omega}\right) = \sqrt{\omega}\,\theta(\omega).$$

Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$. We next start with the definition of the gamma-function at point $\frac{s}{2}$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt.$$

Using the change of variable $t \mapsto \pi n^2 x$, multiplying both sides by $\pi^{-\frac{s}{2}} n^{-s}$ and taking sum over n's, for $\Re(s) > 1$, we arrive at

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty \left(x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} \right) dx. \tag{5}$$

Next by letting $\eta(x) = \dfrac{\theta(x) - 1}{2}$, and utilizing the transformation property of the theta function (Exercise 54), we derive the following integral representation for the zeta-function,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty \eta(x) \left(x^{\frac{s-2}{2}} + x^{-\frac{s+1}{2}} \right) dx + \frac{1}{s(s-1)}.$$

This representation holds for $\Re(s) > 1$. But the integral on the right converges absolutely for any s, and converges uniformly with respect to s in any bounded part of the plane since

$$\eta(x) = O(e^{-\pi x})$$

as $x \to \infty$. Hence, the integral represents an everywhere analytic function of s, and the above formula gives the analytic (meromorphic) continuation of $\zeta(s)$ to the whole plane. Since the right side of this integral representation is unchanged when s is replaced by $1 - s$, it also gives the functional equation

$$\Lambda(s) = \Lambda(1 - s) \tag{6}$$

where

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

11. Epstein Zeta Function

Definition 55 *For any $z = x + iy \in \mathcal{H} = \{x + iy : \ y > 0\}$ and for $s = \sigma + it \in \mathbb{C}$, we define the Epstein zeta function by*

$$E(z,s) = \sum_{m,n}{}' \frac{1}{|mz+n|^{2s}}$$

where the dash means that m and n run through all integer pairs except $(0,0)$.

Exercise 56 *Prove that for any $z \in \mathcal{H}$, the above double series is absolutely and uniformly convergent in the half-plane $\Re(s) > 1$, and therefore $E(z,s)$ is an analytic function of s on this half-plane.*

Our goal here is to prove that the Epstein zeta function has an analytic continuation and it satisfies a functional equation. Both of these statements are consequences of the transformation property of the following theta-function.

Definition 57 *For $\omega > 0$ and $z = x + iy \in \mathcal{H}$, the theta-function $\Theta(\omega)$ is defined by the following infinite sum*

$$\Theta(\omega) = \Theta(z,\omega) = \sum_{m,n}{}' \exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}.$$

Here dash has the same meaning as in the definition of $E(z,s)$.

The first target here is to establish the transformation property of $\Theta(\omega)$. To do this, first we recall some facts about the Fourier transform. For simplicity, we set $e(z) = e^{2\pi iz}$.

Fourier Analysis 58 *Let $f : \mathbb{R}^n \to \mathbb{C}$ be bounded, smooth (i.e., all partial derivatives exist and are continuous), and rapidly decreasing (i.e., for any N, $|\mathbf{x}|^N f(\mathbf{x})$ tends to zero when $|\mathbf{x}|$ goes to infinity). The Fourier transform of f is defined by*

$$\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} e(-\mathbf{x}^t\mathbf{y}) f(\mathbf{x}) d\mathbf{x}.$$

Here, $\mathbf{x} = (x_1, ..., x_n)^t$, $\mathbf{y} = (y_1, ..., y_n)^t$, $\mathbf{x}^t\mathbf{y} = \sum_{j=1}^{n} x_j y_j$, $|\mathbf{x}| = (\mathbf{x}^t\mathbf{x})^{\frac{1}{2}}$, $d\mathbf{x} = \prod_{j=1}^{n} dx_j$ and "t" stands for transposition.

Exercise 59 *It can be proved that for $f(\mathbf{x}) = e^{-\pi \mathbf{x}^t \mathbf{x}}$ we have $\hat{f} = f$.*

Recall that throughout this lecture ω is a positive real number.

Lemma 60 *Let A be a real symmetric matrix of size n with positive eigenvalues, and let*

$$g(\mathbf{x}) = e\left(\frac{i}{2}\omega \mathbf{x}^t A \mathbf{x}\right) = e^{-\pi\omega \mathbf{x}^t A \mathbf{x}}.$$

Then we have

$$\hat{g}(\mathbf{y}) = |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}} e\left(\frac{i}{2\omega}\mathbf{y}^t A^{-1}\mathbf{y}\right) = |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}} e^{-\frac{\pi}{\omega}\mathbf{y}^t A^{-1}\mathbf{y}}.$$

Here, $|A|$ is the determinant of A.

Proof By the principal axis theorem, there exists an orthogonal matrix U such that

$$A = U^t D U$$

where $D = \text{diag}[\lambda_1, ..., \lambda_n]$ is a diagonal matrix and λ_i's are the eigenvalues of A. Let

$$B = \text{diag}[\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}]U = (b_{ij})_{n\times n}.$$

B is invertible and $A = B^t B$. Consider the change of variable $\mathbf{u} = \omega^{\frac{1}{2}} B\mathbf{x}$, and let $\mathbf{v} = \omega^{-\frac{1}{2}}(B^t)^{-1}\mathbf{y}$. We have the following

$$\mathbf{u}^t\mathbf{u} = \omega \mathbf{x}^t A\mathbf{x}, \quad \mathbf{v}^t\mathbf{v} = \frac{1}{\omega}\mathbf{y}^t A^{-1}\mathbf{y}, \quad \mathbf{x}^t\mathbf{y} = \mathbf{u}^t\mathbf{v}.$$

Also for the Jacobian matrix $\mathcal{J}$ we have

$$\mathcal{J} = \left(\frac{\partial u_i}{\partial x_j}\right)_{n\times n} = \left(\omega^{\frac{1}{2}} b_{ij}\right)_{n\times n} = \omega^{\frac{1}{2}} B,$$

and therefore

$$d\mathbf{u} = |\mathcal{J}| d\mathbf{x} = \omega^{\frac{n}{2}}|B| d\mathbf{x} = \omega^{\frac{n}{2}}|A|^{\frac{1}{2}} d\mathbf{x}.$$

Applying this change of variable in the Fourier transform of g yields

$$\begin{aligned}\hat{g}(\mathbf{y}) &= \int_{\mathbb{R}^n} e(-\mathbf{x}^t\mathbf{y})e^{-\pi\omega\mathbf{x}^tA\mathbf{x}}d\mathbf{x} \\ &= |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n} e(-\mathbf{u}^t\mathbf{v})e^{-\pi\mathbf{u}^t\mathbf{u}}d\mathbf{u} \\ &= |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}}\hat{f}(\mathbf{v}) \\ &= |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}}e^{-\pi\mathbf{v}^t\mathbf{v}} \\ &= |A|^{-\frac{1}{2}}\left(\frac{1}{\omega}\right)^{\frac{n}{2}}e^{-\frac{\pi}{\omega}\mathbf{y}^tA^{-1}\mathbf{y}}.\end{aligned}$$

The proof is complete. □

Proposition 61 *The theta-function $\Theta(\omega)$ satisfies the following transformation property*

$$1+\Theta(\omega) = \frac{1}{\omega}\left(1+\Theta\left(\frac{1}{\omega}\right)\right).$$

Proof In Lemma 60 put

$$A = \begin{pmatrix} \frac{|z|^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y}\end{pmatrix}$$

where $z = x+iy \in \mathcal{H}$. A has positive eigenvalues and we have

$$A^{-1} = \begin{pmatrix} \frac{1}{y} & -\frac{x}{y} \\ -\frac{x}{y} & \frac{|z|^2}{y}\end{pmatrix}, \quad n=2, \quad |A|=1,$$

and so

$$g(\mathbf{x}) = e^{-\pi\omega\mathbf{x}^tA\mathbf{x}} = e^{-\frac{\pi\omega}{y}|x_1z+x_2|^2},$$

$$\hat{g}(\mathbf{y}) = \frac{1}{\omega}e^{-\frac{\pi}{\omega}\mathbf{y}^tA^{-1}\mathbf{y}} = \frac{1}{\omega}e^{-\frac{\pi}{y\omega}|y_1-y_2z|^2}.$$

By applying the *Poisson summation formula*, i.e.,

$$\sum_{\mathbf{m}\in\mathbb{Z}^2} g(\mathbf{m}) = \sum_{\mathbf{m}\in\mathbb{Z}^2}\hat{g}(\mathbf{m}),$$

we have

$$\sum_{m,n} e^{-\frac{\pi\omega}{y}|mz+n|^2} = \frac{1}{\omega}\sum_{m,n} e^{-\frac{\pi}{y\omega}|m-nz|^2}$$

or

$$1 + \Theta(\omega) = \frac{1}{\omega}\left(1 + \Theta\left(\frac{1}{\omega}\right)\right).$$

The proof is complete. $\square$

Exercise 62 *Show that for $-1 \leq Re(z) \leq 1$,*

$$\Theta(\omega) \ll \left(1 + \omega^{-1} + y^{\frac{1}{2}}\omega^{-\frac{1}{2}} + y^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\right)\left(e^{-\frac{\pi y \omega}{2}} + e^{-\frac{\pi\omega}{2y}}\right). \tag{7}$$

Now we are ready to prove the main result of this section.

Proposition 63 *(i) The Epstein zeta function can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue $\frac{\pi}{y}$.*
(ii) Put

$$\xi(z, s) = \left(\frac{\pi}{y}\right)^{-s}\Gamma(s)E(z, s).$$

We have the following integral representation for $\xi(z, s)$

$$\xi(z, s) = \int_1^\infty \Theta(\omega)(\omega^{s-1} + \omega^{-s})d\omega + \frac{1}{s(s-1)}$$

and so, $\xi(z, s)$ is analytic everywhere, except for simple poles at $s = 0, 1$ with residue 1.
(iii) $\xi(z, s)$ is unchanged under the replacing of s by $1 - s$. This means that

$$\xi(z, s) = \xi(z, 1 - s).$$

In other words, the Epstein zeta function satisfies the following functional equation

$$\left(\frac{\pi}{y}\right)^{-s}\Gamma(s)E(z, s) = \left(\frac{\pi}{y}\right)^{s-1}\Gamma(1 - s)E(z, 1 - s).$$

Proof For $\Re(s) > 0$, we have

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du.$$

We apply the change of variable $u \mapsto \frac{\pi}{y}|mz + n|^2\omega$, to get

$$\Gamma(s) = \left(\frac{\pi}{y}\right)^s |mz + n|^{2s}\int_0^\infty e^{-\frac{\pi\omega}{y}|mz+n|^2}\omega^{s-1}d\omega,$$

or

$$\left(\frac{\pi}{y}\right)^{-s}|mz+n|^{-2s}\Gamma(s)=\int_0^\infty e^{-\frac{\pi\omega}{y}|mz+n|^2}\omega^{s-1}d\omega.$$

This implies

$$\begin{aligned}\xi(z,s) &= \left(\frac{\pi}{y}\right)^{-s}\Gamma(s)E(z,s)\\ &= \left(\frac{\pi}{y}\right)^{-s}\Gamma(s){\sum_{m,n}}'\frac{1}{|mz+n|^{2s}}\\ &= {\sum_{m,n}}'\left(\frac{\pi}{y}\right)^{-s}\Gamma(s)|mz+n|^{-2s}\\ &= {\sum_{m,n}}'\int_0^\infty \exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}\omega^{s-1}d\omega.\end{aligned}$$

Now note that the inequality (7) allows us to interchange the order of summation and integration. So

$$\begin{aligned}\xi(z,s) &= \int_0^\infty {\sum_{m,n}}'\exp\left\{-\frac{\pi\omega}{y}|mz+n|^2\right\}\omega^{s-1}d\omega\\ &= \int_0^\infty \Theta(\omega)\omega^{s-1}d\omega\\ &= \int_0^1 \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \Theta(\omega)\omega^{s-1}d\omega.\end{aligned}$$

Changing variable $\omega\mapsto\frac{1}{u}$ in the first integral, and applying the transformation property of Proposition 61 yield

$$\begin{aligned}\xi(z,s) &= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_\infty^1 \Theta\left(\frac{1}{u}\right)\left(\frac{1}{u}\right)^{s-1}\left(-\frac{1}{u^2}\right)du\\ &= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \{\omega\left(1+\Theta(\omega)\right)-1\}\left(\frac{1}{\omega}\right)^{s+1}d\omega\\ &= \int_1^\infty \Theta(\omega)\omega^{s-1}d\omega+\int_1^\infty \Theta(\omega)\omega^{-s}d\omega+\int_1^\infty\left(\omega^{-s}-\omega^{-s-1}\right)d\omega\\ &= \int_1^\infty \Theta(\omega)\left(\omega^{s-1}+\omega^{-s}\right)d\omega+\frac{1}{s(s-1)}.\end{aligned}\tag{8}$$

Note that the inequality (7) also shows that

$$\int_1^\infty \left|\Theta(\omega)(\omega^{s-1}+\omega^{-s})\right|d\omega$$

$$\ll \int_1^\infty \left(1+\omega^{-1}+y^{\frac{1}{2}}\omega^{-\frac{1}{2}}+y^{-\frac{1}{2}}\omega^{-\frac{1}{2}}\right)\left(e^{-\frac{\pi y\omega}{2}}+e^{-\frac{\pi\omega}{2y}}\right)\left(\omega^{\sigma-1}+\omega^{-\sigma}\right)d\omega.$$

After expanding the right-hand side, we come to a finite sum of integrals in the form of

$$\int_1^\infty e^{-a\omega}\omega^b d\omega$$

where $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Since these integrals are convergent, the first summand on the right-hand side of (8) is an entire function of s. This proves *(ii)*.

The identity (8) also proves *(iii)*, because the right-hand side of (8) is invariant under the replacing of s with $1-s$.

To prove *(i)*, note that by *(ii)* the only possible poles for $E(z,s)$ are $s=0,1$. At $s=0$ since both $\Gamma(s)$ and $\xi(z,s)$ have simple poles with residue 1, $E(z,s)$ is analytic and $E(z,0)=1$. At $s=1$, $\Gamma(s)$ has a value of 1 and $\xi(z,s)$ has a simple pole with residue 1. Therefore $E(z,s)$ has a simple pole with residue $\dfrac{\pi}{y}$.

This completes the proof. $\square$

Selberg's Analytic Continuation of the Non-holomorphic Eisenstien Series Let

$$\tilde{E}(z,s)=\frac{1}{2}\left(\frac{y}{\pi}\right)^s\Gamma(s)E(z,s).$$

$\tilde{E}(z,s)$ is called the non-holomorphic Eisenstien Series for $SL_2(\mathbb{Z})$, where $SL_2(\mathbb{Z})$ is the multiplicative group of 2×2 matrices with integer entries and determinant 1.

Exercise 64 *Show that $\tilde{E}(z,s)$ is a meromorphic function with two simple poles at $s=0$ and $s=1$ and with the residue $\frac{1}{2}$ at $s=1$. Moreover show that*

$$\tilde{E}(\gamma z,s)=\tilde{E}(z,s)$$

for any $\gamma \in SL_2(\mathbb{Z})$.

In the sequel, following Selberg, we describe a different approach regarding the meromorphic continuation of $\tilde{E}(z,s)$. First of all note that since $\tilde{E}(z+1,s)=\tilde{E}(z,s)$, $\tilde{E}(z,s)$ has a Fourier expansion in the form

$$\sum_{n=-\infty}^{\infty} a_n(y,s)e^{2\pi inx}. \tag{9}$$

For $\Re(s) > 1$. One can explicitly calculate the Fourier coefficients and deduce that

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s},$$

and for $n \neq 0$,

$$a_n(y,s) = 2\sqrt{y}|n|^{s-1/2}\sigma_{1-2s}(|n|)K_{s-1/2}(2\pi|n|y).$$

Here

$$\sigma_{1-2s}(n) = \sum_{m|n} m^{1-2s},$$

and $K_s(y)$ is the *K-Bessel function* defined by

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{\frac{-y(t+t^{-1})}{2}} t^{s-1} dt.$$

If $y > 0$, $K_s(y)$ is well defined for all values of s. Moreover if $y > 4$, we have

$$|K_s(y)| \leq e^{-y/2}K_\sigma(2),$$

where $s = \sigma + it$.

Now note that each individual term of the series (9) has analytic continuation to the whole complex plane, except that $a_0(y,s)$ has simple poles at $s = 0$ and $s = 1$. (Each of the two terms in $a_0(y,s)$ has a pole at $s = 1/2$, but these cancel.) The convergence of the Fourier expansion follows from the rapid decay of the K-Bessel functions. Thus in this way we obtain the analytic continuation of $\tilde{E}(z,s)$.

To get the functional equation, it is enough to observe that

$$a_n(y,s) = a_n(y,1-s).$$

This is clear since $n^s\sigma_{-2s}(n) = n^{-s}\sigma_{2s}(n)$ and $K_{-s}(y) = K_s(y)$.

Exercise 65 *Show that the functional equation of $\tilde{E}(z,s)$ implies the functional equation of $\zeta(s)$.*

Exercise 66 *For even integer $k \geq 4$, let*

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k}.$$

$G_k(z)$ is called the Eisenstein series of weight k for $SL_2(\mathbb{Z})$. Show that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have

$$G_k(\gamma z) = (cz+d)^k G_k(z).$$

12. Modular L-functions

Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{x + iy : \ y > 0\}.$$

Let $GL_2^+(\mathbb{R})$ be the multiplicative group of 2×2 matrices with real entries and positive determinant. Then $GL_2^+(\mathbb{R})$ acts on $\mathcal{H}$ as a group of analytic functions

$$\gamma \ : \ z \mapsto \frac{az+b}{cz+d}, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

Let $\mathcal{H}^*$ denote the union of $\mathcal{H}$ and the rational numbers $\mathbb{Q}$ together with a symbol ∞ (or $i\infty$). The rational numbers together with ∞ are called *cusps*.

Let f be an analytic function on $\mathcal{H}$ and k a positive integer. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$$

define the *stroke operator* "$|_k$" as

$$(f|_k\gamma)(z) = (det\gamma)^{\frac{k}{2}}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Sometimes, we simply write $f|\gamma$ for $f|_k\gamma$. Note that $(f|\gamma)|\sigma = f|\gamma\sigma$.

Let $\Gamma = SL_2(\mathbb{Z})$ be the multiplicative group of 2×2 matrices with integer entries and determinant 1 and let Γ' be a subgroup of finite index of it. Suppose f is an analytic function on $\mathcal{H}$ such that $f|\gamma = f$ for all $\gamma \in \Gamma'$. Since Γ' has finite index,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^M = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma'$$

for some positive integer M. Hence $f(z+M) = f(z)$ for all $z \in \mathcal{H}$. So f can be expressed as a function of $q_M = e^{\frac{2\pi iz}{M}}$, which we will denote by $\tilde{f}$. More precisely, there is a function $\tilde{f}$ such that

$$f(z) = \tilde{f}(q_M).$$

The function $\tilde{f}$ is analytic in the punctured disc $0 < |q_M| < 1$. If $\tilde{f}$ extends to a meromorphic (resp. an analytic) function at the origin, we say, by abuse of language, that f is *meromorphic* (resp. *analytic) at infinity.* This means that $\tilde{f}$ has a Laurent expansion in the punctured unit disc. Therefore, f has a *Fourier expansion at infinity* in the form of

$$f(z) = \tilde{f}(q_{_M}) = \sum_{n=-\infty}^{\infty} \hat{a}_f(n) q_{_M}^n, \qquad q_{_M} = e^{\frac{2\pi i z}{M}}$$

where $\hat{a}_f(n) = 0$ for all $n \leq n_0$ ($n_0 \in \mathbb{Z}$) if f is meromorphic at infinity; and $\hat{a}_f(n) = 0$ for all $n < 0$ if f is analytic at infinity. We say that f *vanishes at infinity* if $\hat{a}_f(n) = 0$ for all $n \leq 0$.

Let $\sigma \in \Gamma$. Then $\sigma^{-1}\Gamma'\sigma$ also has finite index in Γ and $(f|\sigma)|\gamma = f|\sigma$ for all $\gamma \in \sigma^{-1}\Gamma'\sigma$. So $f|\sigma$ also has a Fourier expansion at infinity. We say that f is *analytic at the cusps* if $f|\sigma$ is analytic at infinity for all $\sigma \in \Gamma$. We say that f *vanishes at the cusps* if $f|\sigma$ vanishes at infinity for all $\sigma \in \Gamma$.

Now for $N \geq 1$, let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \;\; c \equiv 0 \pmod{N} \right\}.$$

Note that $\Gamma_0(N)$ is of finite index in Γ.

A *modular form of weight k and level N* is an analytic function f on $\mathcal{H}$ such that
(i) $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) f is analytic at the cusps.
Such a modular form is called a *cusp form* if it vanishes at the cusps.
The modular forms of weight k and level N form a finite dimensional vector space $M_k(N)$ and this has a subspace $S_k(N)$ consisting of cusp forms. Note that since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the same as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $\Gamma_0(N)$, *(i)* shows that $M_k(N) = \{0\}$ if k is odd. So from now on we assume that k is even.
Also, one can define an inner product called *Petersson inner product* on $S_k(N)$ by

$$\langle f, g \rangle = \iint_{D_0(N)} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $D_0(N)$ is a closed simply connected region in $\mathcal{H}$ with the following two properties:
(i) For any $z \in \mathcal{H}$ there is a $\gamma \in \Gamma_0(N)$ and a $z_1 \in D_0(N)$ such that $z = \gamma(z_1)$;
(ii) If $z_1 = \gamma(z_2)$ where $z_1, \; z_2 \in D_0(N)$ and $\gamma \in \Gamma_0(N)$, then z_1 and z_2 are on the boundary of $D_0(N)$. $D_0(N)$ is called a *fundamental domain* for $\Gamma_0(N)$.

Let $f \in S_k(N)$. Since $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(N)$, the Fourier expansion of f at infinity is in the form of

$$f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n) e(nz), \quad e(z) = e^{2\pi i z}.$$

Attached to f, we define the *L-function associated to* f by the Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

where $a_f(n) = \dfrac{\hat{a}_f(n)}{n^{\frac{k-1}{2}}}$ for $n = 1, 2, 3, \cdots$. We call $\hat{a}_f(n)$ the n-th Fourier coefficient and $a_f(n)$ the n-th coefficient of f.

Exercise 67 *(i) Show that f is a cusp form if and only if the Γ-invariant function $g(z) = y^{k/2}|f(z)|$ is bounded on $\mathbb{H}$.*
(ii) Show that $\sum_{n \leq x} (\hat{a}_f(n))^2 \ll_f x^k$.
(iii) Show that $\hat{a}_f(n) \ll_f x^{\frac{k}{2}}$.
(iv) Show that $L(f,s)$ is absolutely convergent for $\Re(s) > 1$ and so it represents an analytic function on $\Re(s) > 1$.

Let

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

This is not an element of Γ unless $N = 1$. However,

$$W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N).$$

Moreover, $f|W_N^2 = f$. W_N is called the *Fricke (or Atkin-Lehner) involution.* Note that since $f \mapsto f|W_N$ defines a self-inverse linear operator on $S_k(N)$, it decomposes the space of cusp forms $S_k(N)$ to two complementary subspaces corresponding to the eigenvalues ± 1. Set

$$S_k^+(N) = \left\{ f \in S_k(N); \quad f|W_N = (-1)^{\frac{k}{2}} f \right\},$$

$$S_k^-(N) = \left\{ f \in S_k(N); \quad f|W_N = (-1)^{\frac{k}{2}+1} f \right\},$$

and notice that $S_k(N) = S_k^+(N) \oplus S_k^-(N)$. The following Theorem of Hecke guarantees the analytic continuation of $L(f,s)$ for $f \in S_k^{\pm}(N)$.

Theorem 68 **(Hecke)** *Let $f \in S_k^{\pm}(N)$. Then $L(f,s)$ extends to an entire function and $\Lambda(f,s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s + \frac{k-1}{2}) L(f,s)$ satisfies the following functional equation*

$$\Lambda(f,s) = \pm\Lambda(f, 1-s).$$

Proof We assume that $f \in S_k^{+}(N)$, the proof of the other case is similar.
One can formally deduce that for $\Re(s) > \frac{3}{2}$

$$\Lambda(f,s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) L(f,s) = (2\pi)^{\frac{k-1}{2}} N^{\frac{s}{2}} \int_0^{\infty} f(iy) y^{s+\frac{k-1}{2}-1} dy.$$

Since $f(iy) \ll e^{-2\pi y}$ as $y \to \infty$, one can show that the above identities are in fact valid on $\Re(s) > \frac{3}{2}$.
Next by breaking the integral at $1/\sqrt{N}$ and employing change of variable $y = 1/Nt$, we have

$$\Lambda(f,s) = (2\pi)^{\frac{k-1}{2}} N^{\frac{s}{2}} \left(\int_{1/\sqrt{N}}^{0} f\left(\frac{i}{Nt}\right) \left(\frac{1}{Nt}\right)^{s+\frac{k-1}{2}-1} (-\frac{1}{Nt^2}) dt + \int_0^{1/\sqrt{N}} f(iy) y^{s+\frac{k-1}{2}-1} dy \right).$$

Now note that

$$f\left(\frac{i}{Nt}\right) = f\left(\frac{-1}{Nyi}\right) = N^{-\frac{k}{2}} (Nyi)^k (-1)^{\frac{k}{2}} f(it).$$

Replacing this identity in the above integral yields

$$\Lambda(f,s) = (2\pi)^{\frac{k-1}{2}} \int_{1/\sqrt{N}}^{\infty} f(iy) y^{\frac{k-1}{2}} \left(N^{\frac{1-s}{2}} y^{-s} + N^{\frac{s}{2}} y^{-(1-s)}\right) dy.$$

Since $f(iy)$ has exponential decay as $y \to \infty$, the above integral represents an everywhere analytic function, and so this gives an analytic continuation of $\Lambda(f,s)$ to the whole complex plane. Since the integral is invariant under transformation $s \to 1-s$ we have the functional equation. □

The *root number* of $L(f,s)$ is the sign appearing in the functional equation of $L(f,s)$.

Corollary 69 *Let $f \in S_k(N)$. Then $L(f,s)$ extends to an entire function.*

Note Our definition of $S_k^{+}(N)$ and $S_k^{-}(N)$ is slightly different from the conventional ones that denote them as subspaces corresponding to the eigenvalues $+1$ and -1 for operator W_N, so for $\frac{k}{2}$ odd, our $S_k^{\pm}(N)$ is the conventional $S_k^{\mp}(N)$. In our notation $S_k^{\pm}(N)$ is the set of cusp forms whose L-functions have root number ± 1, respectively.

Hecke Operators Let $f \in M_k(N)$. Let p and q be primes such that $p \nmid N$ and $q \mid N$.[3] The *Hecke operators* T_p and U_q are defined by

$$f \mid T_p = p^{\frac{k}{2}-1} \left[f \mid \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{e=0}^{p-1} f \mid \begin{pmatrix} 1 & e \\ 0 & p \end{pmatrix} \right],$$

$$f \mid U_q = q^{\frac{k}{2}-1} \left[\sum_{e=0}^{q-1} f \mid \begin{pmatrix} 1 & e \\ 0 & q \end{pmatrix} \right].$$

We can show that $f \mid T_p$ and $f \mid U_q$ are also modular forms of weight k and level N, and furthermore they are cusp forms if f is a cusp form.

Let $f \in S_k(N)$. We will say that f is an *eigenform* if f is an eigenvector for all the Hecke operators $\{T_p\ (p \nmid N),\ U_q\ (q \mid N)\}$. The following theorem gives the main property of eigenforms.

Theorem 70 (Hecke) *The following conditions are equivalent.*
(i) f is an eigenform and $a_f(1) = 1$.
(ii) Coefficients $a_f(n)$ satisfy the following three properties:
(a) They are multiplicative, i.e., if g.c.d.$(m, n) = 1$, *then* $a_f(mn) = a_f(m)a_f(n)$;
(b) For $q \mid N$, $a_f(q^l) = a_f(q)^l$;
(c) For $p \nmid N$, $a_f(p^l) = a_f(p)a_f(p^{l-1}) - a_f(p^{l-2})$.
(iii) $L_f(s)$ has a product of the form

$$L_f(s) = \prod_{q \mid N} \left(1 - a_f(q)q^{-s}\right)^{-1} \prod_{p \nmid N} \left(1 - a_f(p)p^{-s} + p^{-2s}\right)^{-1},$$

which converges absolutely for $\Re(s) > 1$.

We call the product given in part *(iii)* of the above theorem an *Euler product*. Also any f satisfying the above equivalent conditions is called a *normalized* eigenform. It can be proved that if f is an eigenform, then $a_f(1) \neq 0$. So we can always assume that an eigenform f is normalized.

The coefficients of modular forms satisfy some important inequalities. The following statement, known as the *Ramanujan-Petersson Conjecture*, gives the best possible bounds for the coefficients of cusp forms.

[3] Here $a \mid b$ means that a is a divisor of b and $a \nmid b$ means that a is not a divisor of b.

Theorem 71 (Deligne) *(i) If f is a normalized eigenform, then*

$$|a_f(n)| \leq d(n)$$

where $d(n)$ is the number of the divisors of n.
(ii) If f is a cusp form, then for any $\epsilon > 0$,

$$a_f(n) \ll n^{\epsilon}.$$

Now suppose f is a normalized eigenform. From the above inequality it follows that if $p \nmid N$, then $a_f(p)$ can be written in the form of

$$a_f(p) = \epsilon_p + \bar{\epsilon}_p$$

where $\epsilon_p \in \mathbb{C}$ and $|\epsilon_p| = 1$. In fact, ϵ_p and $\bar{\epsilon}_p$ are the roots of the quadratic equation $1 - a_f(p)x + x^2 = 0$.

Corollary 72 *If f is a normalized eigenform, then its L-function has the following Euler product, valid for $\Re(s) > 1$,*

$$L_f(s) = \prod_{p|N}(1 - a_f(p)p^{-s})^{-1} \prod_{p\nmid N}(1 - \epsilon_p p^{-s})^{-1}(1 - \bar{\epsilon}_p p^{-s})^{-1}.$$

Inspired by the above theorems we may think of finding a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{T_p\ (p \nmid N),\ U_q\ (q \mid N),\ W_N\}$. We can show that there exists a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{T_p\ (p \nmid N)\}$ and the operator W_N. The existence of such a basis is the consequence of the fact that $\{T_p\ (p \nmid N),\ W_N\}$ form a commuting family of Hermitian linear operators (with respect to the Petersson inner product) and therefore from a theorem of linear algebra the space of cusp forms is diagonalizable under these operators. Unfortunately the operators $\{U_q\ (q \mid N)\}$ are not Hermitian for $S_k(N)$ and we can not diagonalize $S_k(N)$ with respect to the operators $\{T_p\ (p \nmid N),\ U_q\ (q \mid N),\ W_N\}$. However, we may find such a basis for a certain subspace of $S_k(N)$.
It can be proved that the Fourier coefficient $a_f(n)$ of a normalized eigenform f is real. This is a consequence of the fact that the operators $\{T_p\ (p \nmid N)\}$ are Hermitian, and the fact that the coefficients $a_f(q)\ (q \mid N)$ are real.

Oldforms and Newforms Atkin and Lehner constructed a subspace of $S_k(N)$ that is diagonalizable under the operators $\{T_p\ (p \nmid N),\ U_q\ (q \mid N),\ W_N\}$. More precisely, they showed that there exists a subspace of $S_k(N)$ whose eigenspaces with respect to the Hecke operators $\{T_p\ (p \nmid N)\}$ are one dimensional. We call such a property, for a subspace of $S_k(N)$, "*multiplicity one*". Now since the operators $\{U_q\ (q \mid N),\ W_N\}$ commute with the operators $\{T_p\ (p \nmid N)\}$, an eigenform for the operators $\{T_p\ (p \nmid N)\}$ is an eigenform for the operators $\{U_q\ (q \mid N),\ W_N\}$ too.

Let $N' \mid N\ (N' \neq N)$ and suppose that the $\{g_i\}$ is a basis consisting of eigenforms for the operators $\{T_p\ (p \nmid N')\}$. It can be proved that if d is any divisor of $\frac{N}{N'}$ then $g_i(dz) \in S_k(N)$. Set

$$S_k^{\mathrm{old}}(N) = \mathrm{span}\left\{ g_i(dz) : \text{for any} \quad N' \mid N\ (N' \neq N), \quad d \mid \frac{N}{N'} \right\}.$$

We call $S_k^{\mathrm{old}}(N)$ the space of *oldforms*. Its orthogonal complement under the Petersson inner product is denoted by $S_k^{\mathrm{new}}(N)$ and the eigenforms in this space are called *newforms*. So we have

$$S_k(N) = S_k^{\mathrm{old}}(N) \oplus S_k^{\mathrm{new}}(N).$$

Since the space of newforms has multiplicity one, the set of normalized newforms of weight k and level N is uniquely determined. We denote it by $\mathcal{F}_N$. From the above discussion it is clear that if $f \in \mathcal{F}_N$, $L_f(s)$ satisfies a functional equation and has an Euler product on the half-plane $\Re(s) > 1$.

13. Rankin-Selberg Convolution

Let $z = x + iy$ be a point in the upper half-plane $\mathcal{H}$, and let $s = \sigma + it$ be a point in the complex plane $\mathbb{C}$. Let

$$f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n) e^{2\pi i n z}$$

and

$$g(z) = \sum_{n=1}^{\infty} \hat{a}_g(n) e^{2\pi i n z}$$

be cusp forms of weight k and level N. We set

$$\delta(f, g) = y^{k-2} f(z) \overline{g(z)}.$$

Recall that for $\Re(s) > 1$, the L-functions attached to f and g are defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

and

$$L(g,s)=\sum_{n=1}^{\infty}\frac{a_g(n)}{n^s}$$

where

$$a_f(n)=\frac{\hat{a}_f(n)}{n^{\frac{k-1}{2}}},\ a_g(n)=\frac{\hat{a}_g(n)}{n^{\frac{k-1}{2}}}$$

for $n=1,2,3,\cdots$.

Definition 73 *The Rankin-Selberg convolution of $L(f,s)$ and $L(g,s)$ is defined by*

$$L(f\times g,s)=\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The modified Rankin-Selberg convolution of $L(f,s)$ and $L(g,s)$ is defined by

$$L(f\otimes g,s)=\zeta_N(2s)L(f\times g,s)=\zeta_N(2s)\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}$$

where $\zeta_N(s)=\displaystyle\sum_{\substack{n=1\\ \text{g.c.d.}(n,N)=1}}^{\infty}\frac{1}{n^s}=\prod_{p\nmid N}\left(1-\frac{1}{p^s}\right)^{-1}$ *is the Riemann zeta function with the Euler p-factors corresponding to $p\mid N$ removed.*

The main goal of this section is to study the analytic properties of $L(f\times g,s)$. We will see that the analytic continuation and the functional equation of the Epstein zeta function $E(z,s)$ will result in the analytic continuation and the functional equation for the Rankin-Selberg convolution $L(f\times g,s)$.

In Lemma 75 we will relate the Rankin-Selberg convolution $L(f\times g,s)$ to a double integral on a certain region of the upper half-plane. To do this we need the following lemma.

Lemma 74 *For any fixed $y>0$,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}f(z)\overline{g(z)}dx=\sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}.$$

Proof We have

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}f(z)\overline{g(z)}dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\hat{a}_f(m)e^{2\pi im(x+iy)}\overline{\sum_{n=1}^{\infty}\hat{a}_g(n)e^{2\pi in(x+iy)}}\right)dx\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\hat{a}_f(m)\overline{\hat{a}_g(n)}e^{2\pi i(m-n)x}e^{-2\pi(m+n)y}\right)dx.\end{aligned}$$

Interchanging the order of summation and integration yields

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\hat{a}_f(m)\overline{\hat{a}_g(n)}e^{-2\pi(m+n)y}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(m-n)x}dx\right)\\ &= \sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}.\end{aligned}$$

The proof is complete. $\square$

Lemma 75 *For $\Re(s) > 1$ we have the following integral representation for the Rankin-Selberg convolution $L(f \times g, s)$*

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \iint_S y^{s+k-2}f(z)\overline{g(z)}dxdy\\ &= \iint_S y^s\delta(f,g)dxdy\end{aligned}$$

where S is the strip $|x| \le \frac{1}{2}$ and $y > 0$.

Proof We have

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= (4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}\\ &= \sum_{n=1}^{\infty}\left\{\frac{\hat{a}_f(n)\overline{\hat{a}_g(n)}}{n^{k-1}}\frac{(4\pi)^{-s-k+1}}{n^s}\Gamma(s+k-1)\right\}\\ &= \sum_{n=1}^{\infty}\left\{\hat{a}_f(n)\overline{\hat{a}_g(n)}(4\pi n)^{-s-k+1}\Gamma(s+k-1)\right\}.\end{aligned}$$

Note that by the change of variable $t \mapsto 4\pi ny$, $\Gamma(s+k-1)$ can be written as

$$\Gamma(s+k-1) = (4\pi n)^{s+k-1}\int_0^{\infty} e^{-4\pi ny}y^{s+k-2}dy.$$

So

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \sum_{n=1}^{\infty}\left\{\hat{a}_f(n)\overline{\hat{a}_g(n)}\int_0^{\infty} e^{-4\pi ny}y^{s+k-2}dy\right\}\\ &= \int_0^{\infty} y^{s+k-2}\left\{\sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}\right\}dy.\end{aligned}$$

Now by applying Lemma 74 we get

$$\begin{aligned}
(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \int_0^\infty y^{s+k-2}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx\right\}dy \\
&= \iint_S y^{s+k-2}f(z)\overline{g(z)}dxdy \\
&= \iint_S y^s\delta(f,g)dxdy.
\end{aligned}$$

This completes the proof. □

Our next step is to rewrite the double integral in the statement of the previous lemma as a new integral on a fundamental domain for $\Gamma_0(N)$.

Lemma 76 *We have*

$$\iint_S y^s\delta(f,g)dxdy = \iint_{D_0(N)} y^s\delta(f,g)F_N(z,s)dxdy$$

where

$$F_N(z,s) = 1 + \sum_{m=1}^{\infty}\sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}}$$

and $D_0(N)$ is a fundamental domain for $\Gamma_0(N)$.

Proof Let

$$\Gamma_\infty = \{\gamma\in\Gamma:\ \gamma\infty=\infty\} = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}:\ b\in\mathbb{Z}\right\}.$$

Γ_∞ is a subgroup of Γ and it is clear that the strip $S=\{(x,y):\ |x|\le\frac{1}{2},\ y>0\}$ is a fundamental domain for Γ_∞. For any two matrices $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and $\gamma'=\left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right)$ in $GL_2(\mathbb{Z})$, the right cosets $\Gamma_\infty\gamma$ and $\Gamma_\infty\gamma'$ are equal if and only if $(c,d)=\pm(c',d')$. So the right cosets of Γ_∞ in $\Gamma_0(N)$ are in one to one correspondence with the pairs (c,d) where $c\ge 0$. Therefore we can choose a set of representative $\mathcal{T}$ for the right cosets of Γ_∞ in $\Gamma_0(N)$ as follows:

$$\mathcal{T} = \{(0,1)\}\cup\{(c,d):\ c>0,\ N|c,\ (c,d)=1\}.$$

We claim that for any pair (c,d) in $\mathcal{T}$, there is a unique transformation

$$\gamma_{c,d}: z_1\to z = \frac{az_1+b}{cz_1+d}$$

that maps $D_0(N)$ into S. This is true for the pair $(0,1)$. For other pairs in $\mathcal{T}$, note that since $\infty \in D_0(N)$,

$$\left|\frac{a}{c}\right| = |\gamma_{c,d}(\infty)| \leq \frac{1}{2}.$$

Since $ad-bc=1$, equality holds only if $c=2$, $a=\pm 1$. We consider two cases.

If $c \neq 2$, then there is exactly one solution in $a,\ b$ of the equation $ad-bc=1$ for which $\left|\frac{a}{c}\right| < \frac{1}{2}$. Since $\gamma_{c,d}D_0(N)$ has the unique cusp $\frac{a}{c}$ in S, and this cusp is not on either of the lines $|x| = \frac{1}{2}$, the whole of $\gamma_{c,d}D_0(N)$ lies in S.

If $c=2$, then $a=\pm 1$. Suppose that, for example, $\gamma_{c,d}$ takes ∞ to the cusp $-\frac{1}{2}$ and takes $D_0(N)$ into S. Then the transformation $\gamma_{c,d}(z_1)+1$ has the same $c,\ d$ and maps $D_0(N)$ outside S (touching the line $x=\frac{1}{2}$), and therefore corresponds to the other solution. Hence exactly one of the transformations $\gamma_{c,d}(z_1)$ or $\gamma_{c,d}(z_1)+1$ has the desired property. The claim is proved.

This shows that the strip S can be written as the disjoint union of $\gamma_{c,d}D_0(N)$'s

$$S = \bigcup_{(c,d)\in\mathcal{T}} \gamma_{c,d}D_0(N).$$

Therefore, we have

$$\iint_S y^s\delta(f,g)dxdy = \sum_{(c,d)\in\mathcal{T}} \iint_{\gamma_{c,d}D_0(N)} y^s\delta(f,g)dxdy.$$

Now let $z_1 = x_1 + iy_1$. Changing variable $z_1 \mapsto z = \dfrac{az_1+b}{cz_1+d}$ yields

$$\begin{aligned}
\iint_S y^s\delta(f,g)dxdy &= \sum_{(c,d)\in\mathcal{T}} \iint_{D_0(N)} \left(\frac{y_1}{|cz_1+d|^2}\right)^s \delta(f,g)dx_1dy_1 \\
&= \iint_{D_0(N)} \left(y_1^s\delta(f,g) \sum_{(c,d)\in\mathcal{T}} \frac{1}{|cz_1+d|^{2s}}\right) dx_1dy_1.
\end{aligned}$$

By considering the definition of $\mathcal{T}$ in the last integral, we have

$$\begin{aligned}
\iint_S y^s\delta(f,g)dxdy &= \iint_{D_0(N)} y^s\delta(f,g) \left\{1 + \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{\substack{d=-\infty \\ \text{g.c.d.}(d,c)=1}}^{\infty} \frac{1}{|cz+d|^{2s}}\right\} dxdy \\
&= \iint_{D_0(N)} y^s\delta(f,g)F_N(z,s)dxdy.
\end{aligned}$$

The proof is complete. $\square$

Now we will show that $F_N(z,s)$ has a representation in terms of the Epstein zeta function. First we recall the definition of the Möbius function.
The *Möbius function* $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r,\ p_i \neq p_j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 77 *We have*

$$2\zeta_N(2s)F_N(z,s) = \sum_{d|N} \frac{\mu(d)}{d^{2s}} E(\frac{N}{d}z, s).$$

Proof The idea is to evaluate the double sum

$$S = \sum_{\substack{m,n \\ (n,N)=1}}{}' \frac{1}{|mNz+n|^{2s}}$$

in two different ways.

On one hand we have

$$\begin{aligned} S &= 2\sum_{\substack{n=1 \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{n^{2s}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,m)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}}. \end{aligned}$$

Note that since g.c.d.$(n,N)=1$, then g.c.d.$(n,m)=$ g.c.d.(n,mN). So

$$\begin{aligned} S &= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s) + 2\zeta_N(2s)\sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\ &= 2\zeta_N(2s)F_N(z,s). \end{aligned}$$

On the other hand by applying the classical identity

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{otherwise}\end{cases},$$

we have

$$\begin{aligned}S &= \sum_{m,n}{}' \left\{\frac{1}{|mNz+n|^{2s}}\sum_{d|\text{g.c.d.}(n,N)}\mu(d)\right\}\\ &= \sum_{d|N}\left\{\mu(d)\sum_{m,n}{}'\frac{1}{|mNz+n_1d|^{2s}}\right\}\end{aligned}$$

where $n_1=\frac{n}{d}$. So

$$\begin{aligned}S &= \sum_{d|N}\left\{\frac{\mu(d)}{d^{2s}}\sum_{m,n_1}{}'\frac{1}{|m\frac{N}{d}z+n_1|^{2s}}\right\}\\ &= \sum_{d|N}\left\{\frac{\mu(d)}{d^{2s}}E\left(\frac{N}{d}z,s\right)\right\}.\end{aligned}$$

This completes the proof. $\square$

We are ready to prove the main result of this lecture.

Theorem 78 (Rankin) *The Rankin-Selberg convolution $L(f\times g,s)$ has the following properties:*
(i) The series

$$L(f\times g,s)=\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}$$

is absolutely and uniformly convergent for $\Re(s)>1$.
(ii) $L(f\times g,s)$ has a meromorphic continuation to the whole complex plane.
(iii) $L(f\times g,s)$ is analytic at $s=1$ if $\langle f,g\rangle=0$. Otherwise, it has a simple pole at point $s=1$ with the residue

$$\begin{aligned}r &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})}\iint_{D_0(N)}\delta(f,g)dxdy\\ &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})}\langle f,g\rangle.\end{aligned}$$

(iv) Let

$$L(f \otimes g, s) = \zeta_N(2s) L(f \times g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the modified Rankin-Selberg convolution and for $\Re(s) > 1$, let

$$\begin{aligned}
\Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1) L(f \otimes g, s) \\
&= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s) L(f \times g, s).
\end{aligned}$$

Then both $L(f \otimes g, s)$ and $\Phi(s)$ are entire functions if $\langle f, g\rangle = 0$. Otherwise, if $N = 1$ they are analytic everywhere except that $L(f \otimes g, s)$ has a simple pole at point $s = 1$ and $\Phi(s)$ has simple poles at points $s = 0$ and 1, and if $N > 1$ they are analytic everywhere except that $L(f \otimes g, s)$ has a simple pole at point $s = 1$ and $\Phi(s)$ has a simple pole at $s = 1$.

(v) If $N = 1$, then the function $\Phi(s)$ is invariant under the replacing of s by $1 - s$, i.e.,

$$\Phi(s) = \Phi(1-s).$$

Proof *(i)* Suppose that $\sigma = \Re(s) \geq 1 + \delta > 1$. By Deligne's bound, we know that $|a_f(n)|, |a_g(n)| \ll n^{\delta/4}$. So,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \frac{a_f(n)\overline{a_g(n)}}{n^s} \right| &\ll \sum_{n=1}^{\infty} \frac{n^{\delta/2}}{n^{\sigma}} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta/2}} \\
&< +\infty.
\end{aligned}$$

This completes the proof of *(i)*.

(ii) & *(iv)* By Lemma 75 and Lemma 76, we have

$$\begin{aligned}
\Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1) L(f \otimes g, s) \\
&= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\zeta_N(2s)(4\pi)^{s+k-1} \iint_S y^s \delta(f,g)\, dx dy \\
&= (4\pi)^{k-1} \left(\frac{N}{\pi}\right)^{s} \Gamma(s)\zeta_N(2s) \iint_{D_0(N)} y^s \delta(f,g) F_N(z,s)\, dx dy.
\end{aligned}$$

Applying Lemma 77 in the previous integral yields

$$\begin{aligned}
\Phi(s) &= \frac{(4\pi)^{k-1}}{2}\left(\frac{N}{\pi}\right)^s \Gamma(s) \iint_{D_0(N)} y^s \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^{2s}} E\left(\frac{N}{d}z, s\right)\right) dxdy \\
&= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(N)} \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^s}\left(\frac{Ny}{d\pi}\right)^s \Gamma(s) E\left(\frac{N}{d}z, s\right)\right) dxdy.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\Phi(s) &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(N)} \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^s} \xi\left(\frac{N}{d}z, s\right)\right) dxdy \\
&= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \left(\delta(f,g) \int_1^\infty \Theta(\omega)(\omega^{s-1} + \omega^{-s})\, d\omega\right) dxdy \\
&+ \frac{(4\pi)^{k-1}}{2s(s-1)} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \delta(f,g) dxdy. \qquad (10)
\end{aligned}$$

Note that the integral in the first summand of the right-hand side of (10) is dominated by a finite sum of integrals of the form

$$\iint_{D_0(N)} y^\lambda \delta(f,g) \left(\int_1^\infty e^{-a\omega}\omega^b d\omega\right) dxdy$$

for $\lambda \in \mathbb{R}$. These integrals are all convergent, because f and g vanish at all the cusps of $D_0(N)$. Therefore the first summand in (10) is an entire function of s. This proves *(ii)* and *(iv)*.

(iii) If we multiply both sides of (10) by $s-1$ and then let $s \to 1^+$, we get

$$\begin{aligned}
&\lim_{s\to 1^+} (s-1)\left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s) L(f\times g, s) \\
&= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d} \iint_{D_0(N)} \delta(f,g) dxdy
\end{aligned}$$

and therefore

$$\begin{aligned}
r &= \operatorname{Res}(L(f\times g, s), 1) \\
&= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})} \iint_{D_0(N)} \delta(f,g) dxdy.
\end{aligned}$$

This completes the proof of part *(iii)*.

(v) Let $N = 1$. We can simplify (10) to

$$\begin{aligned}\Phi(s) &= \frac{(4\pi)^{k-1}}{2}\iint_{D_0(1)}\left(\delta(f,g)\int_1^\infty \Theta(\omega)(\omega^{s-1}+\omega^{-s})d\omega\right)dxdy \\ &+ \frac{(4\pi)^{k-1}}{2s(s-1)}\iint_{D_0(1)}\delta(f,g)dxdy.\end{aligned}$$

At a glance we realize that the right-hand side of this equality is invariant under the replacing of s with $1-s$. Therefore

$$\Phi(s) = \Phi(1-s).$$

In other words, $L(f \times g, s)$ satisfies the following functional equation

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f\times g,s)$$

$$= (2\pi)^{2s-2}\Gamma(1-s)\Gamma(k-s)\zeta_N(2-2s)L(f\times g,1-s).$$

The proof of the theorem is complete. □

Exercise 79 *Without appealing to Deligne's bound show that $L(f \times g, s)$ is absolutely convergent for $\Re(s) > 1$.*

Next we will study the Euler product of the Rankin-Selberg convolution of two modular L-functions. Let $f(z) = \sum_{n=1}^\infty \hat{a}_f(n)e^{2\pi inz}$ be a cusp form for $\Gamma_0(N)$, and let $L_f(s) = \sum_{n=1}^\infty a_f(n)n^{-s}$ be its associated L-function. We know that $L_f(s)$ has an Euler product if and only if $f(z)$ is an eigenform. The next proposition will establish the Euler product of the modified Rankin-Selberg convolution of the modular L-functions associated to two eigenforms f and g. To derive the desired Euler product we need the following lemma.

Lemma 80 *Let f and g be two normalized eigenforms in $\Gamma_0(N)$, and let*

$$L_f(s) = \prod_{p|N}\left(1-a_f(p)p^{-s}\right)^{-1}\prod_{p\nmid N}\left(1-\epsilon_p p^{-s}\right)^{-1}\left(1-\bar{\epsilon}_p p^{-s}\right)^{-1}$$

and

$$L_g(s) = \prod_{p|N}\left(1-a_g(p)p^{-s}\right)^{-1}\prod_{p\nmid N}\left(1-\delta_p p^{-s}\right)^{-1}\left(1-\bar{\delta}_p p^{-s}\right)^{-1}$$

be their associated L-functions, where $\epsilon_p + \bar{\epsilon}_p = a_f(p)$, $\delta_p + \bar{\delta}_p = a_g(p)$ *and* $|\epsilon_p| = |\delta_p| = 1$. *Then, for* $\Re(s) > 1$ *and* $p \nmid N$, *we have the following identity*

$$(1 - p^{-2s})^{-1} \sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}}$$

$$= \left(1 - \epsilon_p \delta_p p^{-s}\right)^{-1} \left(1 - \epsilon_p \bar{\delta}_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \delta_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \bar{\delta}_p p^{-s}\right)^{-1}.$$

Proof Let $p \nmid N$. We recall that the coefficients $a_f(n)$ and $a_g(n)$ satisfy the following:

$$a_f(p^k) = a_f(p) a_f(p^{k-1}) - a_f(p^{k-2}),$$

$$a_g(p^k) = a_g(p) a_g(p^{k-1}) - a_g(p^{k-2}).$$

Applying the above identities repeatedly yields

$$a_f(p^k) a_g(p^k) - a_f(p) a_f(p^{k-1}) a_g(p) a_g(p^{k-1}) + \left(a_f(p)^2 + a_g(p)^2 - 2\right) a_f(p^{k-2}) a_g(p^{k-2})$$

$$-a_f(p) a_f(p^{k-3}) a_g(p) a_g(p^{k-3}) + a_f(p^{k-4}) a_g(p^{k-4}) = 0. \tag{11}$$

Also by using the above relations between the coefficients $a_f(p)$, $a_g(p)$ and the complex units ϵ_p, δ_p, we have

$$\left(1 - \epsilon_p \delta_p p^{-s}\right) \left(1 - \epsilon_p \bar{\delta}_p p^{-s}\right) \left(1 - \bar{\epsilon}_p \delta_p p^{-s}\right) \left(1 - \bar{\epsilon}_p \bar{\delta}_p p^{-s}\right)$$

$$= 1 - a_f(p) a_g(p) p^{-s} + \left(a_f(p)^2 + a_g(p)^2 - 2\right) p^{-2s} - a_f(p) a_g(p) p^{-3s} + p^{-4s}. \tag{12}$$

Putting together (11) and (12), and following a tedious calculation, we arrive at

$$\left(1 - \epsilon_p \delta_p p^{-s}\right) \left(1 - \epsilon_p \bar{\delta}_p p^{-s}\right) \left(1 - \bar{\epsilon}_p \delta_p p^{-s}\right) \left(1 - \bar{\epsilon}_p \bar{\delta}_p p^{-s}\right) \sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}}$$

$$= 1 - \frac{1}{p^{2s}},$$

which is equivalent to the statement of the lemma.
This completes the proof. □

Proposition 81 *The modified Rankin-Selberg convolution of the modular L-functions associated to two normalized eigenforms f and g has the following Euler product*

$$\begin{aligned} L(f \otimes g, s) &= \prod_{p \mid N} \left(1 - a_f(p) a_g(p) p^{-s}\right)^{-1} \\ &\times \prod_{p \nmid N} \left(1 - \epsilon_p \delta_p p^{-s}\right)^{-1} \left(1 - \epsilon_p \bar{\delta}_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \delta_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p \bar{\delta}_p p^{-s}\right)^{-1}. \end{aligned}$$

Proof First of all we recall that the coefficients of eigenforms are multiplicative and real. So we have

$$L(f \otimes g, s) = \zeta_N(2s) \prod_{\text{all primes}} \left(\sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}} \right).$$

For $p \mid N$, since $a_f(p^k) = a_f(p)^k$ and $a_g(p^k) = a_g(p)^k$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}} &= \sum_{k=0}^{\infty} \frac{a_f(p)^k a_g(p)^k}{p^{ks}} \\ &= \left(1 - a_f(p) a_g(p) p^{-s}\right)^{-1}. \end{aligned}$$

Using this and applying the previous lemma, we attain the result. □

Lecture 4
Applications

In the previous lecture we proved that the Rankin-Selberg convolution of two modular L-functions exists. In this final lecture, we employ this fact together with the general theorems of Lecture 2 to deduce some results regarding the distribution and the size of Fourier coefficients of modular forms.

14. The Prime Number Theorem

Theorem 82 (**Wiener-Ikehara Tauberian Theorem**) *Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, with $a_n \geq 0$, and $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ be two Dirichlet series with $|b_n| \leq a_n$ for all n. Assume that $f(s)$ and $g(s)$ extend analytically to $\Re(s) \geq 1$ except possibly at $s = 1$ where they have simple poles with residues R and r (which may be zero) respectively. Then*

$$\sum_{n \leq x} b_n \sim rx,$$

as $x \to \infty$.

In this lecture $f(z) = \sum_{n=1}^{\infty} \hat{\lambda}_f(n) e^{2\pi i n z}$ is a normalized eigenform of weight k and level N and $\lambda_f(n) = \hat{\lambda}_f(n)/n^{\frac{k-1}{2}}$ is the n-th coefficient of f, unless otherwise stated.

Proposition 83 *Let f be a normalized eigenform of weight k and level N. Then*

$$\sum_{n \leq x} |\lambda_f(n)|^2 \sim rx,$$

where

$$r = \frac{12(4\pi)^{k-1}}{N(k-1)! \prod_{p|N}(1 + \frac{1}{p})} \langle f, f \rangle,$$

and

$$\sum_{n=1}^{\infty} b_f(n) \sim \frac{\pi^2}{6} \prod_{p|N} (1 - \frac{1}{p^2}) r x,$$

where

$$b_f(n) = \sum_{\substack{n=d^2 m \\ (d,N)=1}} |\lambda_f(m)|^2.$$

Proof This is a direct corollary of the Tauberian Theorem and calculation of the residues of $L(f \times f, s)$ and $L(f \otimes f, s)$ at $s = 1$. □

Proposition 84 *Let f be a normalized eigenform of level N and weight k. Then*

$$\sum_{p \leq x} \lambda_f(p) \log p = o(x).$$

Proof With notation of Lecture 2, we have

$$-\frac{L'}{L}(f, s) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s},$$

where $\Lambda_f(n) = (\alpha_1(p)^k + \alpha_2(p)^k) \log p$ if $n = p^k$, and $\Lambda_f(n) = 0$ otherwise. Since Ramanujan-Petersson conjecture is true in this case (Theorem 71), we have

$$|\Lambda_f(n)| \leq 2 \log n.$$

By non-vanishing result of Lecture 2, $-\frac{L'}{L}(f, s)$ is analytic at $s = 1$. Also $-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\log n}{n}$ has an analytic continuation to the whole complex plane with an exception of a simple pole at $s = 1$. So by the Tauberian Theorem

$$\sum_{n \leq x} \Lambda_f(n) = \sum_{p \leq x} \lambda_f(p) \log p + \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \Lambda_f(p^\alpha) = o(x). \tag{13}$$

Now let

$$\theta(x) = \sum_{p \leq x} \log p.$$

(This should not be confused with the theta function.) Then we have

$$|\sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \Lambda_f(p^\alpha)| \leq 2 \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log p = \theta(x^{1/2}) + \cdots + \theta(x^{1/r}),$$

where r is as large as $\log x$. Since $\theta(x) \leq x \log x$, we conclude that

$$|\sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \Lambda_f(p^\alpha)| \ll \sqrt{x} \log^2 x.$$

Now applying this bound in (13) implies the result. □

15. The Prime Number Theorem With the Remainder

Complex Analysis 85 (**Perron's formula**) *Let $x > 0$, $a > 0$, $T > 0$. Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series absolutely convergent in $\Re(s) > a - \epsilon$. Then if x is a non-integer*

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^a}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^a |\log \frac{x}{n}|} \right).$$

Exercise 86 *Show that*

$$\sum_{n=1}^{\infty} \frac{\log n}{n^a |\log \frac{x}{n}|} = O\left(\frac{1}{(a-1)^2} + x^{1-a} \log^2 x \right),$$

where $1 < a \leq 2$ and x is a half-integer (i.e. $x = \frac{2k+1}{2}$ for an integer k). The implied constant is absolute.

Exercise 87 *For $t > 1$ and $1 - \dfrac{c}{\log(Nk(|t|+3))} \leq \sigma \leq 1 + \dfrac{c}{\log(Nk(|t|+3))}$, we have*

$$\frac{L'}{L}(f, \sigma + it) \ll \log(Nk(|t|+3)).$$

Here c is the constant coming from the almost zero-free region.

Theorem 88 (**Moreno**) *Let f be a normalized eigenform of level N and weight k. Then there exists an absolute constant $c > 0$ such that $L(f, s)$ has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(Nk(|t|+3))}$$

except possibly one simple real zero $\beta < 1$. Moreover,

$$\sum_{p \leq x} \lambda_f(p) \log p = -\frac{x^\beta}{\beta} + O(\sqrt{Nk}\; x \exp(-c_1 \sqrt{\log x}))$$

for $x \geq 2$, where $c_1 > 0$ and the implied constant is absolute.

Proof Since f is a normalized newform, $L(f, s)$ is self-dual. So the first part of the theorem is a simple corollary of almost zero-free region theorem in Lecture 2 (Theorem 49).

For the proof of the asymptotic formula (upper bound), let x be a half-integer, $T \geq 3$, and $a = 1 + \dfrac{c}{\log(NkT)}$. So by Perron's formula, the bound $|\Lambda_f(n)| \leq 2\log n$ (see Lecture 2) and Exercise 86, we have

$$\sum_{n \leq x} \Lambda_f(n) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{L'}{L}(f,s)\frac{x^s}{s} ds + O\left(\frac{x^a}{T}\left(\frac{1}{(a-1)^2} + x^{1-a}\log^2 x\right)\right).$$

Next let $b = 1 - \dfrac{c}{\log(NkT)}$, where c is the constant coming from the first part of theorem. We consider a rectangle $R_T = R_1 \cup (-R_2) \cup (-R_3) \cup R_4$, where

$$\begin{aligned}
R_1 &: \; s = a + it, \quad -T \leq t \leq T,\\
R_2 &: \; s = \sigma + iT, \quad b \leq \sigma \leq a,\\
R_3 &: \; s = b + it, \quad -T \leq t \leq T,\\
R_4 &: \; s = \sigma + iT, \quad b \leq \sigma \leq a.
\end{aligned}$$

Since $\frac{L'}{L}(f,s)$ has a simple pole of residue 1 at $s = \beta$ (in case the exceptional zero exists), we have

$$\sum_{n \leq x} \Lambda_f(n) = -\frac{x^\beta}{\beta} - \frac{1}{2\pi i}\left(\int_{-R_2} + \int_{-R_3} + \int_{R_4}\right) + O\left(\frac{x^a}{T}\left(\frac{1}{(a-1)^2} + x^{1-a}\log^2 x\right)\right). \quad (14)$$

Now by employing Exercise 87 we have

$$\frac{1}{2\pi i} \int_{-R_2} -\frac{L'}{L}(f,s)\frac{x^s}{s} ds \ll \frac{x^a}{T}.$$

A similar bound holds for $\frac{1}{2\pi i}\int_{R_4}$. Also

$$\frac{1}{2\pi i} \int_{-R_3} -\frac{L'}{L}(f,s)\frac{x^s}{s} ds \ll x^b \log(NkT)\log T.$$

Applying these bounds in (14) yields

$$\sum_{n \leq x} \Lambda_f(n) = -\frac{x^\beta}{\beta} + O\left(x^b \log(NkT)\log T + \frac{x^a}{T} + \frac{x^a}{T(a-1)^2} + \frac{x\log^2 x}{T}\right).$$

Now let $T = x^{a-b} = x^{\frac{2c}{\log(NkT)}}$. We have

$$\begin{aligned}
\sum_{n \leq x} \Lambda_f(n) &= -\frac{x^\beta}{\beta} + O(x^b \log(NkT)\log T + x^b\log^2(NkT) + x^{1-a}x^b\log^2 x)\\
&= -\frac{x^\beta}{\beta} + O(x^b \log^2(NkT)\log^2 T)\\
&= -\frac{x^\beta}{\beta} + O(\sqrt{Nk}\; x\exp(-c_1\sqrt{\log x})),
\end{aligned}$$

for some $c_1 > 0$. This implies that

$$\sum_{p \le x} \lambda_f(p) \log p = -\frac{x^\beta}{\beta} + O(\sqrt{Nk} x \exp(-c_1 \sqrt{\log x})).$$

$\square$

Note The asymptotic formula in the previous theorem in fact is an upper bound. Alternately we can write the formula as

$$\sum_{p \le x} \lambda_f(p) \log p = O(\sqrt{Nk} x \exp(-c_1 \sqrt{x})),$$

where the implied constant depends on β. Since the position of the exceptional zero is not clear, the implied constant is not effectively computable.

Note It is proved by Hoffstein and Ramakrishnan that there is no exceptional zero for normalized newforms. So we can drop the term $-\frac{x^\beta}{\beta}$ in Moreno's theorem.

16. An Effective Lower Bound

Exercise 89 *Show that*

$$\frac{1}{2\pi i} \int_{(2)} \frac{x^s}{s(s+1)\cdots(s+r)} ds = \begin{cases} \frac{1}{r!}(1 - \frac{1}{x})^r, & x > 1; \\ 0, & 0 < x \le 1. \end{cases}$$

Note that $\int_{(2)}$ *is an abbreviation of* $\int_{2-i\infty}^{2+i\infty}$.

Proposition 90 (**Hoffstein and Lockhart**) *Let $L(f, s) \in \mathcal{IK}$ be an L-function with positive coefficients and a single simple pole at $s = 1$ of residue r. Suppose that $L(f, s)$ satisfies a growth condition on the line $\Re(s) = 1/2$ of the form*

$$|L(f, \frac{1}{2} + it)| \le M(|t| + 3)^B$$

for some constant B. If $L(f, s)$ has no real zeros in the range

$$1 - \frac{1}{\log M} < \sigma < 1,$$

then there exists an effective constant $c = c(B) > 0$ such that

$$\frac{1}{r} \le c \log M.$$

Proof Let $\frac{1}{2} < \beta < 1$ and let r be a fixed integer greater than B. (We should not confuse β with the exceptional zero.) Using Exercise 89 and the absolute convergence of $L(f, s+\beta)$ in the range of integration, we get

$$\frac{1}{2\pi i}\int_{(2)} \frac{L(f,s+\beta)x^s}{s(s+1)\cdots(s+r)}ds = \frac{1}{r!}\sum_{n<x}\frac{\lambda_f(n)}{n^\beta}(1-\frac{n}{x})^r.$$

Since $\lambda_f(n)$ are non-negative, and $\lambda_f(1) = 1$, we have for all $x \geq 2$,

$$1 \ll \frac{1}{2\pi i}\int_{(2)} \frac{L(f,s+\beta)x^s}{s(s+1)\cdots(s+r)}ds. \tag{15}$$

From the growth condition on the line $\Re(s) = \frac{1}{2}$ and the Phragmen-Lindelöf principle, we have

$$L(f, \sigma + it) = O(|t|^B)$$

for all $\frac{1}{2} \leq \sigma \leq 3$ and $t \geq 1$. Thus we can shift the line of integration to $\Re(s) = \frac{1}{2} - \beta < 0$, picking up residues at $s = 0, 1-\beta$. Using the bound on the line $\Re(s) = \frac{1}{2}$, the right-hand side of (15) becomes

$$\frac{rx^{1-\beta}}{(1-\beta)(2-\beta)\cdots(r+1-\beta)} + \frac{L(f,\beta)}{r!} + O(Mx^{\frac{1}{2}-\beta}).$$

Taking $x = M^C$, for C a sufficiently large constant, we get

$$1 \ll \frac{rM^{C(1-\beta)}}{1-\beta} + L(f,\beta). \tag{16}$$

Now we choose

$$\beta = 1 - \frac{1}{\log M}.$$

Since $L(f,s)$ has a simple pole at $s = 1$, and is positive for $\sigma > 1$, we must have $L(f,\beta) \leq 0$. Then (16) yields

$$\frac{1}{r} \ll \log M$$

as desired. $\square$

Note If $L(f,s) \in \mathcal{IK}$ the growth condition on the line $\Re(s) = \frac{1}{2}$ will be satisfied if we choose M as a suitable power of the conductor of $L(f,s)$ and the product of its local factor at ∞.

Definition 91 *The symmetric square L-function associated to a normalized eigenform f of weight k and level N is defined as*

$$L(\mathrm{sym}^2 f, s) = \frac{L(f \otimes f, s)}{\zeta_N(s)}.$$

From Rankin-Selberg theory it is clear that $L(\mathrm{sym}^2 f, s)$ has a meromorphic continuation to $\mathbb{C}$. In 1975 Shimura proved that the symmetric square L-function in fact has an analytic continuation to the whole complex plane.
For square-free N and newform f, the symmetric square L-function associated to f satisfies a functional equation. Let

$$L_\infty(\mathrm{sym}^2, s) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right),$$

and let

$$\Lambda(\mathrm{sym}^2 f, s) = N^s L_\infty(\mathrm{sym}^2, s) L(\mathrm{sym}^2, s).$$

Then the symmetric square L-function satisfies

$$\Lambda(\mathrm{sym}^2 f, s) = \Lambda(\mathrm{sym}^2 f, 1-s).$$

Exercise 92 *Show that $L(\mathrm{sym}^2 f, s)$ has an Euler product on $\Re(s) > 1$.*

Theorem 93 (**Goldfeld, Hoffstein and Lieman**) *Let f be a normalized newform of square-free level N and weight k. Then there exists an absolute constant $c > 0$ such that $L(\mathrm{sym}^2 f, s)$ has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(kN)}.$$

Proof Consider the L-function

$$L(g, s) = \zeta(s) L(\mathrm{sym}^2 f, s)^2 L(\mathrm{sym}^2 f \otimes \mathrm{sym}^2 f, s) = \zeta(s) L(\mathrm{sym}^2 f, s)^3 \frac{L(\mathrm{sym}^2 f \otimes \mathrm{sym}^2 f, s)}{L(\mathrm{sym}^2 f, s)}.$$

The last L-function is a special case of the symmetric square of a cusp form on $GL(3)$ and has been shown by Bump and Ginzburg to have a simple pole at $s = 1$. Hence $L(g, s)$ has a pole of order 2 at $s = 1$, whereas any real zero of $L(\mathrm{sym}^2 f, s)$ is a zero of $L(g, s)$ of order ≥ 3. By local computations one checks that $\Lambda_g(n) \geq 0$ for $(n, q(g)) = 1$, hence the result follows from Goldfeld, Hoffstein and Lieman's Lemma in Lecture 2. $\square$

Exercise 94 *For $\Re(s) > 0$ define*

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

(i) Show that $f(s) = (1 - \frac{1}{2^{s-1}})\zeta(s)$ for $\Re(s) > 1$.
(ii) From part (i) deduce a meromorphic continuation of $\zeta(s)$ into the half-plane $\Re(s) > 0$.
(iii) Show that $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.
(iv) From here conclude that $L(f \otimes f, \sigma) \neq 0$, in the region given in the previous theorem.

Corollary 95 (**Hoffstein and Lockhart**) *Let f be a normalized newform of square-free level N and weight k. Then there exists an effective constant c (depends only on k) such that*

$$\langle f, f\rangle = \|f\|^2 \geq c\frac{N}{\log N}.$$

Proof This is immediate from the previous exercise and Hoffstein and Lockhart's proposition. Note that the residue of $L(f \otimes f, s)$ at $s = 1$ is

$$\frac{\phi(N)\pi(4\pi)^k}{2N^2(k-1)!}\langle f, f\rangle.$$

□

Exercise 96 *Let f be a newform of square-free level N and weight k with Petersson norm $\|f\| = 1$. Let $\rho(1)$ be the first Fourier coefficient of f. Then prove that there exists an effective constant c (depends only on k) such that*

$$|\rho(1)|^2 \leq c\frac{\log N}{N}.$$

17. Bounds for the Fourier Coefficients of Cusp forms

Theorem 97 (**Landau [also Chandrasekharan and Narasimhan]**) *Let $L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$ be a Dirichlet series with non-negative coefficients $\lambda_f(n)$ and converging for $\Re(s)$ sufficiently large. Assume that $L(f, s)$ has a meromorphic continuation to $\mathbb{C}$ with at most poles of finite order at $s = 0, 1$; assume also that $L(f, s)$ is of finite order in the half plane $\Re(s) \geq -1$, and it satisfies a functional equation of the form*

$$q(f)^s\gamma(f, s)L(f, s) = \epsilon(f)q(f)^{1-s}\gamma(f, 1-s)L(f, 1-s)$$

for some constants $\epsilon(f)$, and $q(f) > 0$, where

$$\gamma(f,s) = \prod_{i=1}^{d} \Gamma(\alpha_i s + \beta_i),$$

for some $d \geq 1$ and $\alpha_i \geq 0$, $\beta_i \in \mathbb{C}$ for $1 \leq i \leq d$. Setting $\eta = \sum_{i=1}^{d} \alpha_i$, one has

$$\sum_{n \leq x} \lambda_f(n) = P_{r-1}(\log x)x + O(x^{\frac{2\eta-1}{2\eta+1}} \log^{r-1} n),$$

where r is the order of pole of $L(f,s)$ at $s = 1$ and P_{r-1} is a polynomial of degree $r-1$ that depends only on $L(f,s)$. The implied constant also depends only on $L(f,s)$.

Theorem 98 (**Rankin**) *Let f be a normalized eigenform of weight k and level 1. Then*

$$\sum_{n \leq x} |\lambda_f(n)|^2 = c_f x + O(x^{\frac{3}{5}}),$$

where

$$c_f = \frac{12(4\pi)^{k-1}}{(k-1)!} \langle f, f \rangle.$$

Proof Let $b_f(n)$ denote the coefficients of $L(f \otimes f, s)$. Since

$$L(f \times f, s) = \frac{1}{\zeta(2s)} L(f \otimes f, s),$$

we have

$$|\lambda_f(n)|^2 = \sum_{n = d^2 m} \mu(d) b_f(m), \tag{17}$$

where $\mu(d)$ denotes the Möbius function. Now one can check that $L(f \otimes f, s)$ satisfies the conditions of Landau's Theorem (Theorem 97) and so

$$\sum_{n \leq x} b_f(n) = \frac{\pi^2}{6} c_f x + O(x^{\frac{3}{5}}).$$

Now from Proposition 83, we have

$$c_f = \frac{12(4\pi)^{k-1}}{(k-1)!} \langle f, f \rangle.$$

Next from (17) and the above asymptotic formula we have

$$\begin{aligned}
\sum_{n\le x} |\lambda_f(n)|^2 &= \sum_{n\le x} \sum_{n=d^2m} \mu(d) b_f(m) \\
&= \sum_{d^2 m\le x} \mu(d) b_f(m) \\
&= \sum_{m\le \frac{x}{d^2}} b_f(m) \sum_{d\le\sqrt{x}} \mu(d) \\
&= \sum_{d\le\sqrt{x}} \mu(d) \left(\frac{\pi^2}{6} c_f \frac{x}{d^2} + O\left((\frac{x}{d^2})^{\frac{3}{5}} \right) \right) \\
&= c_f x \frac{\pi^2}{6} \sum_{d\le\sqrt{x}} \frac{\mu(d)}{d^2} + O(x^{\frac{3}{5}} \sum_{d\le\sqrt{x}} \frac{1}{d^{\frac{6}{5}}}) \\
&= c_f x + O(x^{\frac{3}{5}}).
\end{aligned}$$

□

The following is a direct corollary of the previous theorem. Note that $f(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$ is a normalized eigenform of weight 12 and level 1.

Corollary 99 (**Rankin**) *$\lambda_f(n) = O_f(n^{\frac{3}{10}})$ and $\tau(n) = O(n^{\frac{29}{5}})$.*

Exercise 100 *Let $L(f,s) \in \mathcal{IK}$ be an L-function of degree d. Also assume that $L(f\otimes f, s)$ exists and $\lambda_{f\otimes f}(n) \ge 0$ for $(n, q(f)) \ne 1$. Then show that for the local parameters $\alpha_i(p)$ for unramified prime p we have*

$$|\alpha_i(p)| \le p^{\frac{1}{2} - \frac{1}{d^2+1}}.$$

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