Non-Vanishing of Weight k Modular L-Functions with Large Level

Amir Akbary

Abstract

We will establish lower bounds in terms of the level for the number of holomorphic cusp forms of weight $k > 2$ whose various L-functions do not vanish at the central critical point. This work generalizes the work of W. Duke [1] which was for the case of weight 2.

1 Introduction

In this paper we study the non-vanishing of the L-function associated to a cusp form of weight k and level N. More precisely, let $\mathcal{F}_N$ be the set of all holomorphic (cuspidal) normalized newforms of weight k and level N. For $f \in \mathcal{F}_N$ and a primitive Dirichlet character mod q with $(q, N) = 1$, the twisted L-function associated to f and χ is defined by

$$L_f(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_f(n)}{n^s}.$$

The twisted L-function is given by an absolutely convergent series on the half-plane $Re(s) > \frac{k+1}{2}$ and has an Euler product valid there. Also it has an analytic continuation which satisfies a certain functional equation for which $s = \frac{k}{2}$ is the centre of the critical strip. In this context one may attempt the following problem:

Problem: *What can we say about* $\sharp\{f \in \mathcal{F}_N;\ L_f(\frac{k}{2},\chi) \neq 0\}$ *if* N *is large?*

One known result concerning this problem is given by W. Duke [1] for the case $k = 2$. By comparing mean and mean square estimate for the twisted L-function $L_f(s,\chi)$ attached to a newform f of weight 2, Duke proved that there is a positive absolute constant C and a constant C_q depending only on q such that for any prime $N > C_q$ there are at least $CN(\log N)^{-2}$ newforms $f \in \mathcal{F}_N$ for which $L_f(1,\chi) \neq 0$.

The main difficulty in the generalization of the above result to the cusp forms of weight k is the contribution coming from oldforms of weight k. In this paper, by using a special

[0]1991 *Mathematics Subject Classification.* Primary 11F67.

construction of a basis for the space of cusp forms of weight k and level N, introduced by A. Pizer [5], We show that the contribution of oldforms is negligible, and therefore we obtain a generalization of Duke's result to newforms of weight k and level N. More precisely, we prove the following result.

Theorem 1 *Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{q,k}$ (depending only on q and k) such that for prime $N > C_{q,k}$ there exist at least $C_k N(\log N)^{-2}$ newforms f of weight k and level N for which $L_f(\frac{k}{2}, \chi) \neq 0$.*

We also prove the following theorem about the non-vanishing of the product of two distinct twist of a modular L-function, which is again a generalization of a result of W. Duke ([1] Theorem 2).

Theorem 2 *Let $k > 2$ and χ_1 (mod q_1) and χ_2 (mod q_2) be fixed distinct real primitive Dirichlet characters such that $\chi_1\chi_2(-N) = 1$. Then there are positive constants C_1 and C_2 depending only on $q_1 q_2$ and k such that for prime $N > C_1$ there exist at least $C_2 N(\log N)^{-6}$ newforms f of weight k and level N for which $L_f(\frac{k}{2}, \chi_1) L_f(\frac{k}{2}, \chi_2) \neq 0$.*

The main technical tool in the proof of these results is the "semi-orthogonality" of the Fourier coefficients of an orthonormal basis of $S_k(N)$ (Proposition 1) which is a consequence of the Petersson formulae about Poincaré series. Section 2 describe this technical tool and also introduces a certain basis of the space of cusp forms which has been studied by Pizer. In Sections 3 and 4 we prove mean and mean square estimate for the twisted L-function attached to an element of the basis introduced in Section 2. Using these estimates and establishing a lower bound for the Petersson inner product in Section 5, we will be able to prove Theorem 1. Section 6 describes a proof of Theorem 2.

2 A basis for $S_k(N)$

We review some basic facts concerning modular forms. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ with trivial character. The space $S_k(N)$ is a finite dimensional complex vector space. Moreover, one can define an inner product called Petersson inner product on $S_k(N)$ by

$$< f, g >= \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $\mathcal{H}$ denotes the upper half plane. For $0 \neq f \in S_k(N)$ set

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} < f, f >}.$$

If $f \in S_k(N)$, we write the Fourier expansion of f at $i\infty$ as

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz).$$

The following proposition explains the so called "semi-orthogonality" of the Fourier coefficients of an orthogonal basis of $S_k(N)$.

Proposition 1 *If $\{f_1, ..., f_r\}$ is an orthogonal basis for $S_k(N)$, for m and n positive integers we have the inequality*

$$|\sum_i \omega_{f_i} \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} - \delta_{mn}| \leq M\boldsymbol{d}(N) N^{\frac{1}{2}-k}(m,n)^{\frac{1}{2}} \sqrt{(mn)^{k-1}}$$

where M is a constant depending only on k and $\boldsymbol{d}(N)$ is the number of divisor of N.

Proof: See [1] Lemma 1. $\square$

We are going to generalize Duke's result to cusp forms of weight k and prime level N (see [1], Theorem 1). The first difficulty that we encounter is that $\mathcal{F}_N$ is not a basis for $S_k(N)$ when k is large (more precisely if $k > 12$ and $k \neq 14$). So we must find a basis for $S_k(N)$ with good analytic properties. A theorem of Pizer guarantees the existence of such basis for $S_k(N)$.

Given a two by two real matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive determinant, define its action on a modular form f of weight k to be

$$(f|\gamma)(z) = (det\ \gamma)^{\frac{k}{2}}(cz+d)^{-k} f(\frac{az+b}{cz+d}).$$

Now let $\{T_p\ (p \not| N),\ U_q\ (q|N)\}$ be the collection of the classical Hecke operators and let $W_q\ (q|N)$ be the "W operator" of Atkin and Lehner . In 1983 A. Pizer introduced the operators C_q on $S_k(N)$ for $q|N$, such that the action of C_q on the new part of $S_k(N)$ is the same as the action of the classical U_q operators. More precisely he defined C_q as

$$C_q = U_q + W_q U_q W_q + q^{\frac{k}{2}-1} W_q \quad if \quad q\|N$$

$$C_q = U_q + W_q U_q W_q \quad if \quad q^2|N.$$

then he showed that $T_p\ (p \not| N),\ C_q\ (q|N)$ form a commuting family of Hermitian operators. Using this, he proved ([5] Theorem 3.10) the following result.

Theorem *There exists a basis $f_i(z)$ $(1 \leq i \leq dim\ S_k(N))$ of $S_k(N)$ such that each $f_i(z)$ is an eigenform for all the T_p and C_q operators with $p \not| N$ and $q|N$. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e(z)$ be an element of this basis. Then $a_f(1) \neq 0$ and assuming $f(z)$ is normalized so that $a_f(1) = 1$, we have $f|T_p = a_f(p)f$ for all $p \not| N$, $f|C_q = a_f(q)f$ for all $q|N$, and $a_f(nm) = a_f(n)a_f(m)$ whenever $(n,m) = 1$. Furthermore $f(z)$ is an eigenform for all W_q operators, $q|N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the T_p and C_q operators with $p \not| N$ and $q|N$, then $g(z) = cf_i(z)$ for some $c \in \mathbb{C}^*$ and some unique i, $1 \leq i \leq dim\ S_k(N)$.*

Now let $\mathcal{P}_N$ be the basis of $S_k(N)$ given by the above theorem. The elements of $\mathcal{P}_N$ form an orthogonal basis for $S_k(N)$ and their L-functions have analytic continuation and satisfy certain functional equations. We can show that the action of C_q on $S_k(N)^{new}$ is the same as the action of U_q (see [5] Remark 2.9). This shows that $\mathcal{F}_N \subset \mathcal{P}_N$.

In the sequel we need an estimation for the Fourier coefficient of an oldform in $\mathcal{P}_N$.

Lemma 1 *Suppose N is a prime and $f \in \mathcal{P}_N$. Then*

$$|a_f(n)| \le c_0 n^{\frac{k}{2}}$$

where c_0 is an absolute constant independent of f.

Proof: If $f \in \mathcal{F}_N$ we know that $|a_f(n)| \le \mathbf{d}(n) n^{\frac{k-1}{2}}$ (Deligne's bound) and therefore the result is clear.

If $f \in \mathcal{P}_N - \mathcal{F}_N$ Propositions 3.6 and 3.4 of [5] imply that

$$f(z) = h(z) \pm N^{\frac{k}{2}} h(Nz)$$

where h is the normalized newform of weight k and level 1 associated to f. Now if $(n, N) = 1$ then $a_f(n) = c_h(n)$ where $c_h(N)$ is the N-th Fourier coefficient of h, and therefore the Deligne bound implies the result, and if $(n, N) \neq 1$ then $n = mN$ and we can write

$$a_f(Nm) = c_h(Nm) + A c_h(m).$$

By using the Deligne bound for the Fourier coefficients of h we get

$$\begin{aligned} |a_f(Nm)| &\le \mathbf{d}(Nm)(Nm)^{\frac{k-1}{2}} + N^{\frac{k}{2}} \mathbf{d}(m) m^{\frac{k-1}{2}} \\ &= \left(\frac{\mathbf{d}(Nm)}{(Nm)^{\frac{1}{2}}} + \frac{\mathbf{d}(m)}{m^{\frac{1}{2}}} \right) (Nm)^{\frac{k}{2}}. \end{aligned}$$

The result follows from the fact that $\mathbf{d}(n) = O(n^{\frac{1}{2}})$ with an absolute constant. $\square$

3 First moments

In this section we will find an asymptotic formula for $\sum_{f \in \mathcal{P}_N} \omega_f L_f(\frac{k}{2}, \chi)$. Let $f \in \mathcal{P}_N$, then since it is an eigenform for the Atkin-Lehner involution, the twisted L-function $L_f(s, \chi)$ is known to be entire and to satisfy the functional equation

$$\left(\frac{q\sqrt{N}}{2\pi} \right)^s \Gamma(s) L_f(s, \chi) = \epsilon_\chi \left(\frac{q\sqrt{N}}{2\pi} \right)^{k-s} \Gamma(k-s) L_f(k-s, \bar{\chi})$$

where $\epsilon_\chi = \epsilon_f \chi(N) \tau(\chi)^2 q^{-1}$ with $\epsilon_f = \pm 1$ (the root number of f) which depends only on f and $\tau(\chi)$ is the Gauss sum (see [6] p. 93).

We start with giving a representation of $L_f(\frac{k}{2}, \chi)$ as a sum of two convergent series for $f \in \mathcal{P}_N$ using the functional equation.

Lemma 2 *For any $x > 0$, let*

$$\mathcal{A}(x) = \sum_{n\geq 1} \chi(n) a_f(n) n^{-\frac{k}{2}} \{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (\frac{2\pi n}{x})^j \} e^{-\frac{2\pi n}{x}}.$$

Where $f \in \mathcal{P}_N$ and χ is a fixed primitive Dirichlet character mod q with $(q, N) = 1$. Then we have

$$L_f(\frac{k}{2}, \chi) = \mathcal{A}(x) + \epsilon_\chi \bar{\mathcal{A}}(Nq^2/x)$$

where ϵ_χ is the root number of $L_f(s, \chi)$ and $\bar{\mathcal{A}}$ is the conjugate of $\mathcal{A}$.

Proof: Define the function $\mathcal{E}(x)$ by

$$\mathcal{E}(x) = \frac{1}{2\pi i} \int_{(\frac{3}{4})} (-\frac{1}{x})^s \Gamma(s + \frac{k}{2}) \frac{ds}{s}$$

then

$$\frac{1}{\Gamma(\frac{k}{2})} \mathcal{E}(-\frac{1}{x}) = (\sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (\frac{1}{x})^j) e^{-\frac{1}{x}}.$$

Now by definition of $\mathcal{E}(x)$, it is clear that

$$\mathcal{A}(x) = \frac{1}{2\pi i} \int_{(\frac{3}{4})} L_f(s + \frac{k}{2}, \chi) (\frac{x}{2\pi})^s \frac{\Gamma(s + \frac{k}{2})}{\Gamma(\frac{k}{2})} s^{-1} ds.$$

Moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, and using the functional equation for $L_f(s, \chi)$ yields

$$\mathcal{A}(x) = L_f(\frac{k}{2}, \chi) + \epsilon_\chi \int_{(-\frac{3}{4})} (\frac{2\pi x}{q^2 N})^s \frac{\Gamma(-s + \frac{k}{2})}{\Gamma(\frac{k}{2})} L_f(-s + \frac{k}{2}, \bar{\chi}) s^{-1} ds$$

Now changing variables $s \mapsto -s$ gives the result. $\square$

Proposition 2 *Let χ be a fixed primitive character modulo q. Then we have*

$$\sum_{f \in \mathcal{P}_N} \omega_f L_f(\frac{k}{2}, \chi) = 1 + O(N^{-\frac{1}{2}} (\log N)^{k-1})$$

for N prime. The implied constant depends on q and k.

Proof: Choosing $x = q^2 N \log N$ in Lemma 2 gives

$$\bar{\mathcal{A}}(\frac{Nq^2}{x}) = \sum_{n \geq 1} \overline{\chi(n)} a_f(n) n^{-\frac{k}{2}} \left\{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (2\pi n \log N)^j \right\} (N^{-2\pi})^n.$$

Using Lemma 1, we get

$$|\bar{\mathcal{A}}(\frac{Nq^2}{x})| \leq c_0 \frac{k}{2} (2\pi)^{\frac{k}{2}-1} (\log N)^{\frac{k}{2}-1} O(N^{-2\pi}).$$

Therefore from Lemma 2, we have

$$\sum_{f\in\mathcal{P}_N}\omega_f L_f(\frac{k}{2},\chi)-1=\sum_{n\geq 1}\chi(n)\left(\sum_{f\in\mathcal{P}_N}\omega_f\frac{a_f(n)}{\sqrt{n^{k-1}}}-\delta_{1n}\right)\left\{\sum_{j=0}^{\frac{k}{2}-1}\frac{1}{j!}(\frac{2\pi n}{q^2N\log N})^j\right\}\frac{1}{\sqrt{n}}e^{-\frac{2\pi n}{q^2N\log N}}$$

$$+(\sum_{j=0}^{\frac{k}{2}-1}\frac{1}{j!}(\frac{2\pi}{q^2N\log N})^j)e^{-\frac{2\pi}{q^2N\log N}}-1+(\sum_{f\in\mathcal{P}_N}\omega_f)O(N^{-6}(\log N)^{\frac{k}{2}-1}).$$

Proposition 1, with $m=n=1$ implies

$$\sum_{f\in\mathcal{P}_N}\omega_f=1+O(N^{\frac{1}{2}-k}).$$

Now by applying $m=1$ in Proposition 1 and using the above identity, we have

$$|\sum_{f\in\mathcal{P}_N}\omega_f L_f(\frac{k}{2},\chi)-1|\leq M_1N^{\frac{1}{2}-k}\sum_{n\geq 1}n^{k-2}e^{-\frac{2\pi n}{q^2N\log N}}+\left(\sum_{j=\frac{k}{2}}^{\infty}\frac{1}{j!}(\frac{2\pi}{q^2N\log N})^j\right)e^{-\frac{2\pi}{q^2N\log N}}$$

$$+M_2N^{-6}(\log N)^{\frac{k}{2}-1}\leq M_3N^{-\frac{1}{2}}(\log N)^{k-1}+M_4(N\log N)^{-\frac{k}{2}}+M_2N^{-6}(\log N)^{\frac{k}{2}-1}$$

where M_1, M_2, M_3, M_4 are constants. This completes the proof. □

4 Second moments

In this section we are going to find an asymptotic relation for the average values of $|L_f(\frac{k}{2},\chi)|^2$ where f varies over $\mathcal{P}_N$. To start let $P_f(s)=L_f(s,\chi_1)L_f(s,\chi_2)$ where χ_1 and χ_2 are fixed primitive Dirichlet characters mod q_1 and q_2. Then we have $P_f(s)=\sum_{l\geq 1}b_f(l)l^{-s}$, where

$$b_f(l)=\sum_{mn=l}\chi_1(m)\chi_2(n)a_f(m)a_f(n).$$

Define for $x>0$

$$g(x)=\frac{1}{2\pi i}\int_{(\frac{3}{4})}(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}x^{-s}\frac{ds}{s}\tag{1}$$

and set $\mathcal{B}(x)=\sum_{l\geq 1}b_f(l)l^{-\frac{k}{2}}g(\frac{l}{x})$. Then we have

Lemma 3 *Let $f\in\mathcal{P}_N$ and suppose that χ_1 and χ_2 are primitive Dirichlet characters mod q_1, q_2 with $(q_1q_2,N)=1$. For any $x>0$, we have*

$$P_f(\frac{k}{2})=\mathcal{B}(x)+\hat{\epsilon}_{\chi_1\chi_2}\bar{\mathcal{B}}(\frac{(Nq_1q_2)^2}{x})$$

where $\hat{\epsilon}_{\chi_1\chi_2}=\chi_1\chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2(q_1q_2)^{-1}$ is the root number of $P_f(s)$ and $\bar{\mathcal{B}}$ is the conjugate of $\mathcal{B}$.

Proof: It is similar to the proof of Lemma 2, by writing $\mathcal{B}(x)$ as a line integral, moving the line of integration to the left of zero and applying the functional equation of $P_f(s)$, we get the desired result. □

We come now to the following proposition.

Proposition 3 *Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(\frac{k}{2},\chi)|^2 = \sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = \prod_{p|q}(1-p^{-1})\log N + c + O(N^{-\frac{1}{2}}\log N)$$

for N prime with $(q,N)=1$, where c and the implied constant depend on q and k.

Proof: In Lemma 3, set $\chi_1=\chi$, $\chi_2=\bar{\chi}$, we have $\mathcal{B}=\bar{\mathcal{B}}$ and $\hat{\epsilon}_{\chi\bar{\chi}}=1$. In Lemma 3 let $x=Nq^2$, then

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2\sum_{m,n\geq 1}\chi(m)\bar{\chi}(n)g(\frac{mn}{Nq^2})\frac{1}{(mn)^{\frac{1}{2}}}\sum_{f\in\mathcal{P}_N}\omega_f\frac{a_f(m)}{\sqrt{m^{k-1}}}\frac{a_f(n)}{\sqrt{n^{k-1}}}.$$

By Proposition 1, it is clear that

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2\sum_{n\geq 1}|\chi(n)|^2 g(\frac{n^2}{Nq^2})n^{-1} + R \tag{2}$$

where

$$R \ll N^{\frac{1}{2}-k}\sum_{m,n\geq 1} g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1}. \tag{3}$$

The first term on the right hand side of (2) is evaluated using the definition of g as

$$\frac{1}{\pi i}\int_{(\frac{3}{4})} L(2s+1,\chi_0)(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}(Nq^2)^s\frac{ds}{s}$$

where χ_0 is the principal character mod q and $L(s,\chi_0)=\zeta(s)\prod_{p|q}(1-\frac{1}{p^s})$. Since the integrand has a double pole at $s=0$, by moving the line of integration from $\frac{3}{4}$ to $-\frac{1}{2}$, we see that the above integral is equal to

$$\prod_{p|q}(1-p^{-1})\log N + c + O(N^{-\frac{1}{2}}). \tag{4}$$

Now in (3) we calculate $\sum_{m,n\geq 1} g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1}$. It is

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2})}(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}\left(\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)}\right)(Nq^2)^s\frac{ds}{s}$$

because the integrand does not have any poles in the strip $\frac{3}{4} < Re(s) < \frac{k+1}{2}$ and

$$\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)}$$

is absolutely convergent. Next we use the following identity

$$\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)}=\frac{\zeta(2s-k+\frac{3}{2})\zeta(s-\frac{k}{2}+1)^2}{\zeta(2s-k+2)}$$

(See [1] Lemma 4). By moving the line of integration from $\frac{k+1}{2}$ to $\frac{k}{2}-\epsilon$ ($\epsilon>0$) we get

$$\sum_{m,n\geq 1} g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1}\sim c_1 N^{\frac{k}{2}}\log N$$

and by (3), $R \ll N^{\frac{1}{2}-\frac{k}{2}}\log N$. This and (4) prove the Proposition. $\square$

5 A lower bound for the Petersson inner product

To complete the proof of Theorem 1 we need a lower bound in terms of N for $<f,f>$ when $f\in\mathcal{P}_N$. Note that if $N_1|N_2$ then $S_k(N_1)\subset S_k(N_2)$, therefore the value of the Petersson inner product depends on N. To emphasize this dependency from now on we show the Petersson inner product by $<.,.>_N$.

Lemma 4 *If h is a normalized newform of level 1, then*

$$<h,h(Nz)>_N \quad = N^{1-k}c_h(N)<h,h>_1$$

where $c_h(N)$ is the N-th Fourier coefficient of h.

Proof: Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, and $h\in S_k(1)$, and the operator W_N is Hermitian, we have

$$<h,h(Nz)>_N \quad = N^{-\frac{k}{2}}<h|W_N,h(z)>_N.$$

Now let F be a fundamental domain of $\Gamma_0(1)\backslash\mathcal{H}$ and let the elements

$$\left\{\gamma_{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\gamma_j=\begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix},0\leq j<N\right\}$$

be coset representatives for $\Gamma=\Gamma_0(N)\backslash\Gamma_0(1)$. Then since $\Gamma_0(1)=\bigcup_{i=-1}^{N-1}\Gamma_0(N)\gamma_i$,

$$F'=\bigcup_{i=-1}^{N-1}\gamma_i F$$

is a fundamental domain of $\Gamma_0(N)\backslash\mathcal{H}$. So we have

$$<h,h(Nz)>_N = N^{-\frac{k}{2}}\sum_{j=-1}^{N-1}\int_{\gamma_j F}(h|W_N)(z)\overline{h(z)}y^k\frac{dxdy}{y^2}.$$

Using the change of variable $z = \gamma_j w$, where $w = u + iv$ we find that this is

$$= N^{-\frac{k}{2}} \sum_{j=-1}^{N-1} \int_F ((h|W_N)|\gamma_j)(w)\overline{(h|\gamma_j)(w)} v^k \frac{dudv}{v^2}.$$

Now let $Tr(h|W_N) = \sum_{j=-1}^{N-1}(h|W_N)|\gamma_j$, then since $h|\gamma_j = h$ ($h \in S_k(1)$), we have

$$< h, h(Nz) >_N = N^{-\frac{k}{2}} < Tr(h|W_N), h >_1 .$$

But we know that

$$Tr(h|W_N) = N^{1-\frac{k}{2}} c_h(N) h$$

where $c_h(N)$ is the N-th Fourier coefficient of h (see [4] P. 175, Problem 8). This completes the proof. □

Now we use the above Lemma to get a lower bound for $< f, f >_N$.

Lemma 5 *If $f \in \mathcal{P}_N - \mathcal{F}_N$ and N is a prime then*

$$< f, f >_N \;\; \geq \;\; (N - 4N^{\frac{1}{2}} + 1) < h, h >_1$$

where h is the normalized newform of weight k and level 1 associated to f.

Proof: Proposition 3.6 and 3.4 of [5] imply that $f(z) = h(z) \pm N^{\frac{k}{2}} h(Nz)$. Now by applying Lemma 4 we have

$$< f, f >_N \;\; = \;\; < h \pm N^{\frac{k}{2}} h(Nz), h \pm N^{\frac{k}{2}} h(Nz) >_N \;\; \geq \;\; \left(N + 1 \pm 2N^{\frac{k}{2}} N^{1-k} c_h(N)\right) < h, h >_1.$$

Now applying the Deligne bound ($|c_h(n)| \leq \mathbf{d}(n) n^{\frac{k-1}{2}}$) for $c_h(N)$ yields the result. □

The following proposition is the direct consequence of Lemma 5.

Proposition 4 *If $f \in \mathcal{P}_N - \mathcal{F}_N$, for N prime large enough*

$$\omega_f \ll_k \frac{1}{N}$$

with implied constant depending on k.

We are in the situation that we can prove the main theorem of this paper.

Proof of Theorem 1:

We know that $\omega_f \ll_k \frac{\log N}{N}$ if $f \in \mathcal{F}_N$ (see [3] p. 178, remark and paragraph following the Main Theorem), now by Proposition 4 and the Cauchy-Schwarz inequality, we have

$$|\sum_{f \in \mathcal{P}_N} \omega_f L_f(\frac{k}{2}, \chi)|^2 \leq \left(\sum_{f \in \mathcal{F}_N; L_f(\frac{k}{2},\chi) \neq 0} \omega_f + \sum_{f \in \mathcal{P}_N - \mathcal{F}_N; L_f(\frac{k}{2},\chi) \neq 0} \omega_f \right) \sum_{f \in \mathcal{P}_N} \omega_f |L_f(\frac{k}{2}, \chi)|^2$$

$$\ll \left(\sharp\{f \in \mathcal{F}_N;\; L_f(\frac{k}{2}, \chi) \neq 0\} \frac{\log N}{N} + 2 dim S_k(1) \frac{1}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f |L_f(\frac{k}{2}, \chi)|^2 .$$

Now theorem follows from Propositions 2 and 3. □

6 Non-vanishing of product of twisted modular L-functions

We may try to use the above method to find a lower bound for the number of newforms f for which $P_f(s) = L_f(s,\chi_1)L_f(s,\chi_2)$ is non-zero at the centre of the critical strip. Here we assume that χ_1 and χ_2 are real and distinct such that $\chi_1\chi_2(-N) = 1$. To do this we need to derive asymptotic formulae for $\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2})$ and $\sum_{f\in\mathcal{P}_N}\omega_f |P_f(\frac{k}{2})|^2$.

Proposition 5 *Let χ_1 (mod q_1) and χ_2 (mod q_2) be distinct real primitive Dirichlet characters such that $\chi_1\chi_2(-N) = 1$, then for N prime we have*

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2L(1,\chi_1\chi_2) + O(N^{-\frac{1}{2}}\log N)$$

where the implied constant depends on q_1q_2 and k.

Proof: In Lemma 3 we have $\hat{\epsilon}_{\chi_1\chi_2} = 1$. This is because $(\tau(\chi_i))^2 = \chi_i(-1)q_i$ for $i = 1,\ 2$ (see [6] p. 91). So we may repeat the proof of Proposition 3 line by line. The result follows with the observation that

$$\frac{1}{\pi i}\int_{(\frac{3}{4})} L(2s+1,\chi_1\chi_2)(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}(Nq_1q_2)^s\frac{ds}{s}$$

is equal to

$$2L(1,\chi_1\chi_2) + O(N^{-\frac{1}{2}}).\ \square$$

We recall from (1) the definition of $g(x)$ as

$$g(x) = \frac{1}{2\pi i}\int_{(\frac{3}{4})}(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}x^{-s}\frac{ds}{s}\ .$$

For $x > 0$ and a non-negative integer v, let

$$K_v(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(u+\frac{1}{u})}u^{-(v+1)}du$$

be the K_v-Bessel function.

In the next lemma we give a representation of $g(x)$ as a sum of the K-Bessel functions.

Lemma 6 $g(x) = \frac{2}{\Gamma(\frac{k}{2})}\sum_{j=0}^{\frac{k}{2}-1}\frac{1}{j!}(2\pi\sqrt{x})^{\frac{k}{2}+j}K_{\frac{k}{2}-j}(4\pi\sqrt{x})$

Proof: From the definition of $g(x)$ and Γ function we have

$$I = \Gamma(\frac{k}{2})^2 g(x) = \frac{1}{2\pi i}\int_{(\frac{3}{4})}(\int_0^\infty\int_0^\infty {t_1}^{s+\frac{k}{2}-1}{t_2}^{s+\frac{k}{2}-1}e^{-(t_1+t_2)}dt_1dt_2)(4\pi^2x)^{-s}\frac{ds}{s}.$$

By interchanging the order of integration we get

$$I = \int_0^\infty t_1^{\frac{k}{2}-1} e^{-t_1} \left(\int_{\frac{4\pi^2 x}{t_1}}^\infty e^{-t_2} t_2^{\frac{k}{2}-1} dt_2 \right) dt_1.$$

Now the result follows by applying integration by parts in I and the fact that

$$\int_0^\infty t^{\frac{k}{2}-1-j} e^{-(t+\frac{4\pi^2 x}{t})} dt = 2(4\pi^2 x)^{\frac{k}{4}-\frac{j}{2}} K_{\frac{k}{2}-j}(4\pi\sqrt{x})$$

(see [7] p. 235, Formula 9.42). □

Lemma 7 $g(x) \ll \begin{cases} 1 & for \quad x \leq 1 \\ x^{\frac{k}{2}-\frac{3}{4}} e^{-4\pi\sqrt{x}} & for \quad x > 1 \end{cases}$.

Proof: By moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, we have

$$g(x) = 1 + O(x^{\frac{3}{4}})$$

which proves the Lemma if $x \leq 1$.

If $x > 1$, we know

$$K_v(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} [1 + O(\frac{1}{x})]$$

(see [8] p. 202). Now applying this identity to Lemma 6 yields the result. □

Lemma 8 *Under the assumptions of Proposition 5, for $f \in \mathcal{P}_N$ and $X = Nq_1q_2(\log N)^2$, we have*

$$P_f(\frac{k}{2}) = \sum_{l \leq X} c_l a_f(l) + O(N^{-11})$$

where $c_l \ll \frac{\boldsymbol{d}(j)}{l^{\frac{k}{2}}} \log N$ and the implied constants depend on q_1q_2 and k.

Proof: In Lemma 3 set $x = Nq_1q_2$, then we have

$$P_f(\frac{k}{2}) = 2 \sum_{l=1}^\infty b_f(l) l^{-\frac{k}{2}} g(\frac{l}{Nq_1q_2}).$$

Now by using Lemma 7 and the fact that $b_f(l) \leq {c_0}^2 \mathbf{d}(l) l^{\frac{k}{2}}$ (see Lemma 1), we have

$$P_f(\frac{k}{2}) = 2 \sum_{l \leq X} b_f(l) l^{-\frac{k}{2}} g(\frac{l}{Nq_1q_2}) + O(N^{-11}). \tag{5}$$

In (5) the sum can be written as

$$\sum_{l \leq X} 2l^{-\frac{k}{2}} g(\frac{l}{Nq_1q_2}) \sum_{mn=l} \chi_1(m)\chi_2(n) a_f(m) a_f(n) = (*) + (\dagger) \tag{6}$$

where $(*)$ is the sum over the terms with $(m, N) = 1$, and $(\dagger)$ is the sum over the terms with $N|m$.

We know that if $(m,N)=1$ then for $f\in\mathcal{P}_N$

$$a_f(m)a_f(n)=\sum_{d|(m,n)} d^{k-1}a_f(\frac{mn}{d^2})$$

(see [4], p. 163, Proposition 39). Using this identity in (5) yields

$$(*)=\sum_{l\le X} 2l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2})\sum_{mn=l,(m,N)=1}\chi_1(m)\chi_2(n)\sum_{d|(m,n)} d^{k-1}a_f(\frac{l}{d^2}).$$

By setting $j=\frac{l}{d^2}$ and rearranging the above sum, we have

$$(*)=\sum_{j\le X}\left(\sum_{d\le\sqrt{\frac{X}{j}}}\frac{2}{j^{\frac{k}{2}}d}\,g(\frac{jd^2}{Nq_1q_2})\sum_{\substack{mn=jd^2\\ d|(m,n)}}\chi_1(m)\chi_2(n)\right)a_f(j)=\sum_{j\le X}\alpha_j a_f(j) \tag{7}$$

where $\alpha_j\ll\frac{\mathbf{d}(j)}{j^{\frac{k}{2}}}\log N$ by using Lemma 7.

Now suppose that $N|m$. Since $m\le X=Nq_1q_2(\log N)^2$, for N large enough we can assume that $m=m_0N$ where $(m_0,N)=1$. Using the multiplicative property of $a_f(n)$'s, we have

$$(\dagger)=\sum_{l\le X} 2l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2})\sum_{mn=l,m=m_0N}\chi_1(m)\chi_2(n)a_f(N)\sum_{d|(m_0,n)} d^{k-1}a_f(\frac{l}{Nd^2}).$$

Now set $\frac{l}{Nd^2}=j$. Rearranging $(\dagger)$ yields

$$(\dagger)=\sum_{j\le\frac{X}{N}}\left(\sum_{d\le\sqrt{\frac{X}{Nj}}}\frac{2N^{-\frac{k}{2}}a_f(N)}{j^{\frac{k}{2}}d}\,g(\frac{jd^2}{q_1q_2})\sum_{\substack{mn=Njd^2,m=m_0N\\ d|(m_0,n)}}\chi_1(m)\chi_2(n)\right)a_f(j)=\sum_{j\le\frac{X}{N}}\beta_j a_f(j) \tag{8}$$

where $\beta_j\ll\frac{\mathbf{d}(j)}{j^{\frac{k}{2}}}\log N$. Here again we are using Lemma 7 and the fact that $|a_f(N)|\le c_0N^{\frac{k}{2}}$ (Lemma 1).

The result follows from (6), (7) and (8). □

We now employ the following mean value result.

Lemma 9 *For N prime and complex numbers c_n we have*

$$\sum_{f\in\mathcal{P}_N}\omega_f\Big|\sum_{l\le X}c_la_f(l)\Big|^2=(1+O(N^{-1}X\log X))\sum_{l\le X}l|c_l|^2$$

with an absolute implied constant.

Proof: See [2] Theorem 1. □

Now by applying Lemma 9 to Lemma 8, we get

Proposition 6 *Under the assumption of Proposition 5 we have*

$$\sum_{f\in\mathcal{P}_N} \omega_f |P_f(\frac{k}{2})|^2 \ll (\log N)^5$$

for $k > 2$. The implied constant depends on q_1q_2 and k.

We can now state the proof of Theorem 2.

Proof of Theorem 2:

It is enough to replace $L_f(\frac{k}{2},\chi)$ with $P_f(\frac{k}{2},\chi)$ in the proof of Theorem 1 and apply propositions 5 and 6. □

Note: In the case $k = 2$ we get the lower bound $C_2N(\log N)^{-10}$ for the number of non-vanishing $P_f(\frac{k}{2})$ (see [1] Theorem 2).

Acknowledgements: I would like to thank Professor Kumar Murty for reading the manuscript and providing many helpful suggestions and discussions. Also I would like to thank Vijay Patankar for helpful discussions.

References

[1] W. Duke, The critical order of vanishing of automorphic L-functions with large level, *Invent. Math.* **119** (1995), 165-174.

[2] W. Duke, J. Friedlander, H. Iwaniec, Bounds for automorphic L-functions. II., *Invent. Math.* **115** (1994), 219-239.

[3] D. Goldfeld, J. Hoffstein, and D. Lieman, An effective zero free region, Appendix to: Coefficients of Maass forms and the Sigel zero, *Ann. Math* **140** (1994), 177-181.

[4] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1993.

[5] A. Pizer, Hecke operators for $\Gamma_0(N)$, *Journal of Algebra* **83** (1983), 39-64.

[6] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, 1971.

[7] N. M. Temme, *Special functions: an introduction to the classical functions of mathematical physics*, John Wiely, 1996.

[8] G. N. Watson, *A treatise on the theory of Bessel functions, Second ed.*, Cambridge University Press, 1948.

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montréal, Quebec, H3G 1M8, CANADA
E–mail: akbary@cicma.concordia.ca