ON NON-VANISHING OF SYMMETRIC SQUARE L-FUNCTIONS

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Abstract. We find a lower bound in terms of N for the number of newforms of weight k and level N whose symmetric square L-functions are non-vanishing at a fixed point s_0 with $\frac{1}{2} < Re(s_0) < 1$ or $s_0 = \frac{1}{2}$.

1. Introduction

Let $S_k(N)$ be the space of cusp forms of weight k and level N with trivial character. For $f \in S_k(N)$ let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$$

be the Fourier expansion of f at $i\infty$ and let $\mathcal{F}_N$ be the set of all normalized ($a_f(1) = 1$) newforms in $S_k(N)$. The symmetric square L-function associated to $f \in \mathcal{F}_N$ is defined (for $Re(s) > 1$) by

$$L(\mathrm{sym}^2 f, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.$$

Here $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p|N$ removed. Shimura [Shi75] proved that $L(\mathrm{sym}^2 f, s)$ extends to an entire function. Moreover, let

$$L_\infty(\mathrm{sym}^2, s) = \pi^{-\frac{3\pi}{2}} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right),$$

and

$$\Lambda(\mathrm{sym}^2 f, s) = N^s L_\infty(\mathrm{sym}^2, s) L(\mathrm{sym}^2 f, s).$$

Then it is known that $\Lambda(\mathrm{sym}^2 f, s)$ is entire and for square-free N it satisfies the functional equation

$$\Lambda(\mathrm{sym}^2 f, s) = \Lambda(\mathrm{sym}^2 f, 1-s). \tag{1}$$

In recent years, the problem of non-vanishing of $L(\mathrm{sym}^2 f, s)$ in the critical strip $0 < Re(s) < 1$ has drawn the attention of many authors. In [Li96], Li proved that for a given complex number $s_0 \neq \frac{1}{2}$ satisfying $0 < Re(s_0) < 1$ and $\zeta(s_0) \neq 0$, there are infinitely many level 1 newforms f of different weight such that $L(\mathrm{sym}^2 f, s_0) \neq 0$. Kohnen and Sengupta [KS00] showed that for any fixed s_0 with $0 < Re(s_0) < 1$ and $Re(s_0) \neq \frac{1}{2}$, and for all sufficiently large k, there exists a level 1 newform f of weight k such that $L(\mathrm{sym}^2 f, s_0) \neq 0$. Very recently, Lau [Lau02] proved that for any fixed s_0 with $0 < Re(s_0) < 1$, there exist infinitely many even k such that $L(\mathrm{sym}^2 f, s_0) \neq 0$ for some level 1 newform f of weight k. Furthermore, when

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$Re(s_0) \neq \frac{1}{2}$ or $s_0 = \frac{1}{2}$, there exists a constant $k_0(s_0)$ depending on s_0 such that for all $k \geq k_0(s_0)$, $L(\mathrm{sym}^2 f, s_0) \neq 0$ for some level 1 newform of weight k.

All of the above results are related to the symmetric square L-functions associated to normalized eigenforms for the full modular group and various weights k. In this paper we consider a different point of view and we prove similar non-vanishing results while weight k is fixed and level N varies, moreover we give a density estimate in terms of N for the number of newforms of weight k and level N whose symmetric square L-functions are non-vanishing at a fixed point inside the critical strip. Precisely speaking, we prove the following.

Theorem 1.1. *Let N be a prime number and let $s_0 = \sigma_0 + it_0$ with $\frac{1}{2} < \sigma_0 < 1$ or $s_0 = \frac{1}{2}$. Then for any $\epsilon > 0$, there are positive constants $C_{s_0,k,\epsilon}$ and $C'_{s_0,k,\epsilon}$ (depending only on s_0, k and ϵ), such that for any prime $N > C'_{s_0,k,\epsilon}$, there exist at least $C_{s_0,k,\epsilon} N^{1-\epsilon}$ newforms f of weight k and level N for which $L(\mathrm{sym}^2 f, s_0) \neq 0$.*

Corollary 1.2. *For any $s_0 = \sigma_0 + it_0$ with $\frac{1}{2} < \sigma_0 < 1$ or $s_0 = \frac{1}{2}$, there are infinitely many weight k newforms f such that $L(\mathrm{sym}^2 f, s_0) \neq 0$.*

Note that by the functional equation (1) similar statements are true for points s_0 with $0 < Re(s_0) < \frac{1}{2}$.

The proof is based on a comparison of mean values. In section 2, by modifying our arguments in [Akb00] we derive an asymptotic formula for $L(\mathrm{sym}^2 f, s_0)$ on average. In section 3, we establish an upper bound for the mean square values of the symmetric square L-functions at a fixed point s_0 in the critical strip. Our proof in this section closely follows the proof given in [IM01] for the case $Re(s_0) = \frac{1}{2}$. The proof of our theorem is given in section 4.

2. Mean Estimate

We start by finding a representation for $L(\mathrm{sym}^2 f, s_0)$ as a sum of two absolutely convergent series. Recall that $L_\infty(\mathrm{sym}^2, s)$ is the product of gamma-factors in the functional equation of the symmetric square L-functions.

Lemma 2.1. *For any s_0 with $0 \leq Re(s_0) \leq 1$, let*

$$W_{s_0}(y) = \frac{1}{2\pi i} \int_{(2)} \pi^{\frac{3}{2}s_0}\ L_\infty(\mathrm{sym}^2, s_0 + s)\ y^{-s} \frac{ds}{s}$$

and

$$I_f(s_0) = \sum_{\substack{d,e \\ (d,N)=1}} \frac{a_f(e^2)}{d^{2s_0} e^{s_0}} W_{s_0}\left(\frac{d^2 e}{N}\right)$$

where $f \in \mathcal{F}_N$. Then we have

$$\pi^{\frac{3}{2}s_0} L_\infty(\mathrm{sym}^2, s_0) L(\mathrm{sym}^2 f, s_0) = I_f(s_0) + (\pi^{-\frac{3}{2}} N)^{1-2s_0} I_f(1-s_0).$$

Proof. We have

$$I_f(s_0) = \frac{1}{2\pi i} \int_{(2)} \left(\pi^{\frac{3}{2}} N^{-1}\right)^{s_0} \Lambda(\mathrm{sym}^2 f, s + s_0) \frac{ds}{s}.$$

By moving the line of integration from (2) to (-2), calculating the residue at $s = 0$ and applying the functional equation (1), we get

$$I_f(s_0) = \pi^{\frac{3}{2}s_0} L_\infty(\mathrm{sym}^2, s_0) L(\mathrm{sym}^2 f, s_0) + \frac{1}{2\pi i} \int_{(-2)} \left(\pi^{\frac{3}{2}} N^{-1}\right)^{s_0} \Lambda(\mathrm{sym}^2 f, 1-s-s_0) \frac{ds}{s}.$$

Now the change of variable s to $-s$ yields the result. $\square$

By employing the Legendre duplication formula one can deduce the following expression for $W_{s_0}(y)$ (see [Akb00], Lemma 6 for details).

$$W_{s_0}(y) = \frac{\sqrt{\pi}}{2^{s_0+k-2}} \int_0^\infty {t_1}^{\frac{s_0+1}{2}-1} e^{-t_1} \left(\int_{\frac{2\pi^{\frac{3}{2}} y}{{t_1}^{\frac{1}{2}}}}^\infty {t_2}^{s_0+k-3} e^{-t_2} dt_2 \right) dt_1.$$

Note that this integral representation shows that $|W_{s_0}(y)| \leq W_{\sigma_0}(y)$.

In the next two lemmas we derive asymptotics for some series involving $W_{s_0}(y)$.

Lemma 2.2. *Let* $s_0 = \sigma_0 + it_0$ *with* $\sigma_0 > \frac{1}{2}$, *then*

$$\sum_{\substack{d \\ (d,N)=1}} \frac{1}{d^{2u}} W_u\left(\frac{d^2}{N}\right)$$

$$= \begin{cases} \pi^{\frac{3}{2}s_0} L_\infty(\mathrm{sym}^2, s_0)\zeta_N(2s_0) + O_{\sigma_0,k}\left(\mathbf{d}(N) N^{\frac{1}{2}-\sigma_0}\right) & \text{if } u = s_0 \\ O_{\sigma_0,k}\left(\mathbf{d}(N) N^{\sigma_0-\frac{1}{2}}\right) & \text{if } u = 1-s_0 \\ \frac{\pi^{\frac{3}{4}}}{2} L_\infty(\mathrm{sym}^2, \frac{1}{2}) \prod_{p|N}\left(1-\frac{1}{p}\right) \log N + O_k\left(N^{-\frac{1}{2}}\right) & \text{if } u = \frac{1}{2} \end{cases}$$

as $N \to \infty$. *Here* $\mathbf{d}(N)$ *denotes the number of divisors of* N.

Proof. From the definition of $W_u(y)$ we have

$$\sum_{\substack{d \\ (d,N)=1}} \frac{1}{d^{2u}} W_u(\frac{d^2}{N}) = \frac{1}{2\pi i} \int_{(2)} \pi^{\frac{3}{2}u} L_\infty(\mathrm{sym}^2, s+u)\zeta_N(2s+2u) N^s \frac{ds}{s}.$$

For $u = s_0$, we move the line of integration from (2) to the left of $(\frac{1}{2} - \sigma_0)$ and calculate the residues of the integrand at $s = 0$ and $s = \frac{1}{2} - \sigma_0$ to get the first identity. For $u = 1 - s_0$, we move the line of integration to the left of $(\sigma_0 - \frac{1}{2})$ and calculate the residue of the integrand at $s_0 = \sigma_0 - \frac{1}{2}$. Finally, for $u = \frac{1}{2}$, we move the line of integration to $(-\frac{1}{2})$ and calculate the residue at $s = 0$. $\square$

Lemma 2.3. *Let* α, β *and* γ *be real numbers such that* $-1 < \alpha < min\{1, \frac{1+\beta}{2}, \gamma + 2\}$, *then*

$$\sum_{\substack{d,e \\ (d,N)=1}} \frac{\mathbf{d}(e^2)}{e^\alpha d^\beta} W_\gamma\left(\frac{d^2 e}{N}\right) \sim \frac{3}{\pi^2} \frac{\pi^{\frac{3}{2}\gamma}}{1-\alpha} \zeta_N(\beta - 2\alpha + 2) L_\infty(\mathrm{sym}^2, 1-\alpha+\gamma) N^{1-\alpha} \log^2 N$$

and

$$\sum_{\substack{d,e \\ (d,N)=1}} \frac{1}{e^\alpha d^\beta} W_\gamma\left(\frac{d^2 e}{N}\right) \sim \frac{6}{\pi^2} \frac{\pi^{\frac{3}{2}\gamma}}{1-\alpha} \zeta_N(\beta - 2\alpha + 2) L_\infty(\mathrm{sym}^2, 1-\alpha+\gamma) N^{1-\alpha}$$

as $N \to \infty$.

Proof. First note that $\sum_{e=1}^{\infty}\frac{\mathbf{d}(e^2)}{e^s}=\frac{\zeta^3(s)}{\zeta(2s)}$ for $Re(s)>1$. Now by this identity and the definition of $W_\gamma(.)$, the above sum is equal to

$$\sum_{\substack{d\\(d,N)=1}}\frac{1}{d^\beta}\frac{1}{2\pi i}\int_{(2)}\pi^{\frac{3}{2}\gamma}L_\infty(\mathrm{sym}^2,s+\gamma)\frac{\zeta^3(s+\alpha)}{\zeta(2s+2\alpha)}\left(\frac{N}{d^2}\right)^s\frac{ds}{s}.$$

Moving the line of integration to the left of $(1-\alpha)$ and calculating the residue at $s=1-\alpha$ yields the result. The proof of the second asymptotic is similar. □

We need one more lemma before stating the main result of this section. Recall that we have an inner product on $S_k(N)$ called Petersson inner product defined by

$$\langle f,g\rangle_N=\int_{D_0(N)}f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}$$

where $D_0(N)$ is a fundamental domain for $\Gamma_0(N)$. Using this inner product we define

$$\omega_f=\frac{(4\pi)^{k-1}}{\Gamma(k-1)}\langle f,f\rangle_N.$$

The following estimation for the weighted trace of the Hecke operators acting on the space of newforms is fundamental in the proof of the main result of this section.

Lemma 2.4. *For prime N, we have*

$$\sum_{f\in\mathcal{F}_N}\omega_f^{-1}a_f(n)=\delta_{n1}+O_k\left(N^{-\frac{3}{2}}n^{\frac{1}{2}}+N^{-1}\mathbf{d}(n)\right)$$

where δ_{n1} denotes the Kronecker delta.

Proof. We outline the proof given in [Luo99]. Recall that $\mathcal{F}_N$ is the set of normalized newforms of weight k and level N. We consider the following orthogonal basis for $S_k(N)$.

$$\mathcal{B}_N=\mathcal{F}_N\cup\mathcal{F}_1\cup\{g=\alpha_f f(z)+f(Nz)|\ f\in\mathcal{F}_1\}=\mathcal{F}_N\cup\mathcal{F}_1\cup\mathcal{G}_N.$$

Here $\alpha_f=-\frac{\langle f(Nz),f(z)\rangle_N}{\langle f(z),f(z)\rangle_N}$. We have

$$\sum_{f\in\mathcal{F}_N}\omega_f^{-1}a_f(n)a_f(m)=\sum_{f\in\mathcal{B}_N}-\sum_{f\in\mathcal{F}_1}-\sum_{g\in\mathcal{G}_N}.$$

By taking $m=1$ in the above identity we have

$$\begin{aligned}\sum_{f\in\mathcal{F}_N}\omega_f^{-1}a_f(n) &= \sum_{f\in\mathcal{B}_N}\omega_f^{-1}a_f(n)a_f(1)-\sum_{f\in\mathcal{F}_1}\omega_f^{-1}a_f(n)\\ &-\sum_{f\in\mathcal{F}_1}\omega^{-1}_{\alpha_f f(z)+f(Nz)}\alpha_f\left(\alpha_f a_f(n)+N^{-\frac{k-1}{2}}a_f(\frac{n}{N})\right).\end{aligned}\tag{2}$$

Note that $a_f(\frac{n}{N})=0$ if N does not divide n. By Petersson formula (see [Mur95], Proposition 2), for any orthogonal basis $\mathcal{B}_N$ of $S_k(N)$ we have

$$\sum_{f\in\mathcal{B}_N}\omega_f^{-1}a_f(n)a_f(1)=\delta_{n1}+O_k\left(N^{-\frac{3}{2}}n^{\frac{1}{2}}\right).$$

Also for $f\in\mathcal{F}_1$ we have the following

$$-\alpha_f=\frac{N^{-\frac{k-1}{2}}}{N+1}a_f(N),\ \omega_f^{-1}=\frac{\Gamma(k-1)}{(N+1)(4\pi)^{k-1}\langle f,f\rangle_1},\ \omega^{-1}_{\alpha_f f(z)+f(Nz)}\ll_k\frac{N^{k-1}}{\langle f,f\rangle_1}$$

(see [Luo99], p. 596 for details). Now applying these relations together with Petersson formula and Deligne's bound ($|a_f(n)| \le \mathbf{d}(n)$) in (2) yields the result. □

Here we prove the main result of this section.

Theorem 2.5. *Let N be prime, for any $s_0 = \sigma_0 + it_0$ with $\frac{1}{2} < \sigma_0 < 1$ or $s_0 = \frac{1}{2}$, we have*

$$\sum_{f \in \mathcal{F}_N} \omega_f^{-1} L(\mathrm{sym}^2 f, s_0) = \begin{cases} \zeta_N(2s_0) + O_{\sigma_0,k}\left(\frac{N^{\frac{1}{2}-\sigma_0}}{|L_\infty(\mathrm{sym}^2, s_0)|}\right) & \text{if } \frac{1}{2} < \sigma_0 < 1 \\ \log N + O_k(1) & \text{if } s_0 = \frac{1}{2} \end{cases}.$$

Proof. From Lemmas 2.1 and 2.4 we have

$$\pi^{\frac{3}{2}s_0} L_\infty(\mathrm{sym}^2, s_0) \sum_{f \in \mathcal{F}_N} L(\mathrm{sym}^2 f, s_0) = \sum_{\substack{d,e \\ (d,N)=1}} \frac{1}{d^{2s_0}} W_{s_0}\left(\frac{d^2 e}{N}\right)$$

$$(3) \qquad + \left(\pi^{-\frac{3}{2}} N\right)^{1-2s_0} \sum_{\substack{d,e \\ (d,N)=1}} \frac{1}{d^{2(1-s_0)}} W_{1-s_0}\left(\frac{d^2 e}{N}\right) + N^{-\frac{3}{2}} S_1 + N^{-\frac{1}{2}-2\sigma_0} S_2$$

where

$$(4) \qquad S_1 \ll \sum_{\substack{d,e \\ (d,N)=1}} \frac{1}{e^{\sigma_0 - 1} d^{2\sigma_0}} W_{\sigma_0}\left(\frac{d^2 e}{N}\right) + N^{\frac{1}{2}} \sum_{\substack{d,e \\ (d,N)=1}} \frac{\mathbf{d}(e^2)}{e^{\sigma_0} d^{2\sigma_0}} W_{\sigma_0}\left(\frac{d^2 e}{N}\right)$$

and

(5)
$$S_2 \ll \sum_{\substack{d,e \\ (d,N)=1}} \frac{1}{e^{(1-\sigma_0)-1} d^{2(1-\sigma_0)}} W_{1-\sigma_0}\left(\frac{d^2 e}{N}\right) + N^{\frac{1}{2}} \sum_{\substack{d,e \\ (d,N)=1}} \frac{\mathbf{d}(e^2)}{e^{1-\sigma_0} d^{2(1-\sigma_0)}} W_{1-\sigma_0}\left(\frac{d^2 e}{N}\right).$$

Applying Lemma 2.2 in (3) and Lemma 2.3 in (4) and (5) yield the result. □

3. Mean Square Estimate

In [IM01] Iwaniec and Michel found an upper bound for the mean square values of $L(\mathrm{sym}^2 f, s)$ on the critical line $Re(s) = \frac{1}{2}$. In this section we re-write the arguments of [IM01] for a general point in the critical strip and as a result we derive a similar estimate for the mean square values of $L(\mathrm{sym}^2 f, s)$ on the critical strip $\frac{1}{2} \le Re(s) \le 1$. The following Theorem of [IM01] plays a fundamental role in the arguments.

Theorem 3.1. *Let s_0 be a point inside the critical strip. Let g be a smooth function with support $[1,\ 2]$ satisfying*

$$g^{(j)}(x) \ll |s_0|^j$$

for any $j \ge 0$ (the implied constant depending on j only). For $X \ge 1$ we define the partial sums

$$S_f(X) = \sum_n a_f(n^2) g\left(\frac{n}{X}\right)$$

and their mean square

$$S(X) = \sum_{f \in \mathcal{F}_N} \omega_f^{-1} |S_f(X)|^2.$$

Then we have

$$S(X) \ll |s_0|^{3+\epsilon}(NX)^{\epsilon}(N^{-1}X^2 + X)$$

for any $\epsilon > 0$. *The implied constant depends only on* ϵ.

Proof. See [IM01], Theorem 5.1. □

We also need to use an adjusted version of Lemma 2.1 in our proof.

Lemma 3.2. *Let* N *be square-free. Let* $A > 2$ *be an integer and let* $G(s) = \cos\left(\frac{\pi s}{4A}\right)^{-3A}$. *For any* s_0 *with* $0 \le Re(s_0) \le 1$, *we have*

$$L(\mathrm{sym}^2 f, s_0) = \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right) + \varepsilon(s_0) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^{1-s_0}} V_{1-s_0}\left(\frac{n}{N}\right)$$

where

$$V_{s_0}(y) = \int_{(2)} G(s) \frac{L_\infty(\mathrm{sym}^2, s_0 + s)}{L_\infty(\mathrm{sym}^2, s_0)} \zeta_N(2s_0 + 2s) y^{-s} \frac{ds}{s}$$

and $\varepsilon(s_0) = N^{1-2s_0} L_\infty(\mathrm{sym}^2, 1 - s_0)/L_\infty(\mathrm{sym}^2, s_0)$.

Proof. see Lemma 3.1 in [IM01]. □

We have the following estimations for $V_{s_0}(y)$ and its j-th derivative $V_{s_0}^{(j)}(y)$ (see [IM01] for proofs).

$$V_{s_0}(y) \ll \mathbf{d}(N) \left(1 + \frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2 + \frac{1}{y}\right) \tag{6}$$

and

$$V_{s_0}^{(j)}(y) \ll \mathbf{d}(N) y^{-j} \left(1 + \frac{y}{|s_0|^{\frac{3}{2}}}\right)^{-A} \log\left(2 + \frac{1}{y}\right) \tag{7}$$

where the implied constants are depending only on j, A and k.

We are now ready to prove the main result of this section.

Theorem 3.3. *Let* N *be square-free. Let* s_0 *be a point in the strip* $\frac{1}{2} \le Re(s_0) \le 1$. *Then,*

$$\sum_{f \in \mathcal{F}_N} \omega_f^{-1} |L(\mathrm{sym}^2 f, s_0)|^2 \ll |s_0|^8 N^{\epsilon}$$

for any $\epsilon > 0$. *The implied constant depends only on* ϵ *and* k.

Proof. Let ϵ be the reciprocal of a natural number bigger than 2 and let $A = 3 + \frac{2}{\epsilon}$. It is plain that

$$A \in \mathbb{N},\ 0 < \epsilon < \frac{1}{2},\ \frac{A + \epsilon}{A - 2} = 1 + \epsilon.$$

Let $\alpha = |s_0|^{\frac{3}{2}}$ and $\beta = |1 - s_0|^{\frac{3}{2}}$. Now by using Lemma 3.2, we write $L(\mathrm{sym}^2 f, s_0) = L_1(f, s_0) + L_2(f, s_0) + L_3(f, s_0) + L_4(f, s_0)$, where

$$L_1(f, s_0) = \sum_{n \le \alpha N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right),$$

$$L_2(f, s_0) = \sum_{n \ge \alpha N^{1+\epsilon}} \frac{a_f(n^2)}{n^{s_0}} V_{s_0}\left(\frac{n}{N}\right),$$

$$L_3(f,s_0)=N^{1-2s_0}\frac{L_\infty(\mathrm{sym}^2,1-s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\sum_{n\le\beta N^{1+\epsilon}}\frac{a_f(n^2)}{n^{1-s_0}}V_{1-s_0}\left(\frac{n}{N}\right),$$

$$L_4(f,s_0)=N^{1-2s_0}\frac{L_\infty(\mathrm{sym}^2,1-s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\sum_{n\ge\beta N^{1+\epsilon}}\frac{a_f(n^2)}{n^{1-s_0}}V_{1-s_0}\left(\frac{n}{N}\right).$$

Our first goal is to estimate $L_i(f,s_0)$'s. We start with $L_2(f,s_0)$. By Deligne's bound ($|a_f(n^2)|\le\mathbf{d}(n^2)$) and (6),

$$L_2(f,s_0)\ll\sum_{n\ge\alpha N^{1+\epsilon}}\frac{\mathbf{d}(n^2)}{n^{\sigma_0}}\mathbf{d}(N)\frac{|s_0|^{\frac{3}{2}A}}{(|s_0|^{\frac{3}{2}}+\frac{n}{N})^A}\log\left(2+\frac{N}{n}\right).$$

Since $\dfrac{A+\epsilon}{A-2}=1+\epsilon$, we have

$$n\ge\alpha N^{1+\epsilon}\iff\frac{n}{N}\ge\alpha^{\frac{A-2}{A}}\left(n^2N^\epsilon\right)^{\frac{1}{A}}.$$

By using the inequality $\mathbf{d}(n)\ll_\delta n^\delta$ for any $\delta>0$, we get

$$\begin{aligned}L_2(f,s_0)&\ll\mathbf{d}(N)|s_0|^{\frac{3}{2}A}\sum_{n\ge\alpha N^{1+\epsilon}}\frac{n^\epsilon}{n^{\sigma_0}}\frac{1}{(\frac{n}{N})^A}\\&\ll|s_0|^3\frac{\mathbf{d}(N)}{N^\epsilon}\sum_{n\ge\alpha N^{1+\epsilon}}\frac{1}{n^{2+\sigma_0-\epsilon}}\ll|s_0|^3.\end{aligned}\tag{8}$$

With a similar argument we attain

$$L_4(f,s_0)\ll|1-s_0|^3\frac{\mathbf{d}(N)N^{1-2\sigma_0}}{N^\epsilon}\left|\frac{L_\infty(\mathrm{sym}^2,1-s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\right|\ll|1-s_0|^3.$$

This is true, since by Stirling's formula, the ratio of the L_∞-factors is bounded.

Now we consider a smooth partition of unity, which is a C^∞ function h with support $[1,2]$ such that for any $x>0$,

$$\sum_{k=-\infty}^{\infty}h\left(\frac{x}{2^{\frac{k}{2}}}\right)=1.$$

To estimate $L_1(f,s_0)$, we first re-write it as a new sum involving the function h. For simplicity, we use X for $2^{\frac{k}{2}}$. We have

$$L_1(f,s_0)=\sum_{n\le\alpha N^{1+\epsilon}}\sum_k\frac{a_f(n^2)}{n^{s_0}}V_{s_0}\left(\frac{n}{N}\right)h\left(\frac{n}{X}\right).$$

Note that the support of h is [1, 2], so, we can assume that $X<n<2X$. Also, since $n\ge1$, k is in fact ≥-1. Therefore, by interchanging the order of the addition we have

$$\begin{aligned}L_1(f,s_0)&=\sum_{k\ge-1}\sum_{X<n<2X}\frac{a_f(n^2)}{n^{s_0}}V_{s_0}\left(\frac{n}{N}\right)h\left(\frac{n}{X}\right)\\&=\sum_{k\ge-1}\left(\frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{X^{s_0}\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^A}\sum_{X<n<2X}a_f(n^2)g\left(\frac{n}{X}\right)\right),\end{aligned}$$

where

$$g(x)=\frac{1}{\mathbf{d}(N)\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{-A}\log\left(2+\frac{N}{X}\right)}x^{-s_0}V_{s_0}\left(\frac{X}{N}x\right)h(x).$$

To be consistent with the notations of Theorem 3.1, we put

$$S_f(X)=\sum a_f(n^2)g\left(\frac{n}{X}\right).$$

So,

$$L_1(f,s_0)=\sum_{k\geq -1}\frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{A}}\frac{S_f(X)}{X^{s_0}}.$$

In a similar fashion

$$L_3(f,s_0)=N^{1-2s_0}\frac{L_\infty(\mathrm{sym}^2,1-s_0)}{L_\infty(\mathrm{sym}^2,s_0)}\sum_{k\geq -1}\frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|1-s_0|^{\frac{3}{2}}}\right)^{A}}\frac{S_f(X)}{X^{1-s_0}}.$$

By the Cauchy-Schwarz inequality

$$\left|L(\mathrm{sym}^2 f,s_0)\right|^2 \quad = \quad \left|\sum_{i=1}^{4}L_i(f,s_0)\right|^2\leq 4\sum_{i=1}^{4}\left|L_i(f,s_0)\right|^2.$$

So, we have

$$\sum_{f\in\mathcal{F}_N}\omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2$$

$$\ll\sum_f\omega_f^{-1}|L_1(f,s_0)|^2+\sum_f\omega_f^{-1}|L_2(f,s_0)|^2+\sum_f\omega_f^{-1}|L_3(f,s_0)|^2+\sum_f\omega_f^{-1}|L_4(f,s_0)|^2.$$

Now we estimate the above four sums. For the first sum, by applying the Cauchy-Schwarz inequality, we deduce

(9)

$$\begin{aligned}\sum_f\omega_f^{-1}\left|L_1(f,s_0)\right|^2 &= \sum_f\omega_f^{-1}\left|\sum_{k\geq -1}\frac{\mathbf{d}(N)\log\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{A}}\frac{S_f(X)}{X^{s_0}}\right|^2\\ &\ll \sum_f\omega_f^{-1}\left(\sum_{k\geq -1}\frac{\mathbf{d}^2(N)\log^2\left(2+\frac{N}{X}\right)}{\left(1+\frac{X}{N|s_0|^{\frac{3}{2}}}\right)^{2A}}\sum_{k\geq -1}\frac{|S_f(X)|^2}{X^{2\sigma_0}}\right)\\ &\ll \mathbf{d}^2(N)\sum_{k\geq -1}\frac{\log^2(2+\frac{N}{X})}{(1+\frac{X}{N|s_0|^{\frac{3}{2}}})^{2A}}\sum_{k\geq -1}\frac{1}{X^{2\sigma_0}}\left(\sum_f\omega_f^{-1}|S_f(X)|^2\right).\end{aligned}$$

By (6) and (7) it can be shown that the function $g(x)$ satisfies the conditions of Theorem 3.1. Moreover, note that the conditions $n\leq\alpha N^{1+\epsilon}$ and $X<n<2X$ imply that

$$-1\leq k<2(\log_2\alpha+(1+\epsilon)\log_2 N).$$

So, by applying the result of Theorem 3.1 in (9), we deduce

(10)

$$\begin{aligned}\sum_f \omega_f^{-1}\left|L_1(f,s_0)\right|^2 &\ll \mathbf{d}^2(N)N^{\epsilon}\sum_{k\geq -1}\frac{1}{X^{2\sigma_0}}|s_0|^{3+\epsilon}(NX)^{\epsilon}\left(N^{-1}X^2+X\right)\\ &\ll |s_0|^{3+\epsilon}\mathbf{d}^2(N)N^{\epsilon}(\alpha N^{2+\epsilon})^{\epsilon}\left(N^{-1}\sum_{k\geq -1}X^{2-2\sigma_0}+\sum_{k\geq -1}X^{1-2\sigma_0}\right)\\ &\ll |s_0|^{\frac{9}{2}+\frac{5}{2}\epsilon}N^{7\epsilon}\end{aligned}$$

Note that by Lemma 2.4 $\sum_f \omega_f^{-1} = 1 + O(N^{-1})$, so from (8) we have

$$\sum_f \omega_f^{-1}|L_2(f,s_0)|^2 \ll |s_0|^3\sum_f \omega_f^{-1} \ll |s_0|^3. \tag{11}$$

In a similar fashion we derive the following inequalities

$$\sum_f \omega_f^{-1}|L_3(f,s_0)|^2 \ll |1-s_0|^{6+\frac{5}{2}\epsilon}N^{7\epsilon}, \tag{12}$$

and

$$\sum_f \omega_f^{-1}|L_4(f,s_0)|^2 \ll |1-s_0|^3. \tag{13}$$

Considering (10), (11), (12) and (13), we arrive at

$$\sum_{f\in\mathcal{F}_N}\omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2 \ll |s_0|^{6+\frac{5}{2}\epsilon}N^{7\epsilon}.$$

The proof is now complete. □

4. Proof of Theorem 1.1

Proof. Let $\frac{1}{2} < \sigma_0 < 1$. By the asymptotic formulae of Theorems 2.5 and 3.3 together with the upper bound $\frac{\log N}{N}$ for ω_f^{-1} (see [GHL94]), and by Cauchy-Schwarz inequality we can write (for large N)

$$\begin{aligned}|\zeta_N(2s_0)|^2 &\ll \left|\sum_{f\in\mathcal{F}_N}\omega_f^{-1}L(\mathrm{sym}^2 f,s_0)\right|^2\\ &\leq \#\left\{f\in\mathcal{F}_N:\ L(\mathrm{sym}^2 f,s_0)\neq 0\right\}\frac{\log N}{N}\sum_{f\in\mathcal{F}_N}\omega_f^{-1}|L(\mathrm{sym}^2 f,s_0)|^2\\ &\ll \#\left\{f\in\mathcal{F}_N:\ L(\mathrm{sym}^2 f,s_0)\neq 0\right\}\frac{\log N}{N}|s_0|^8N^{\epsilon}.\end{aligned}$$

Thus,

$$\#\left\{f\in\mathcal{F}_N:\ L(\mathrm{sym}^2 f,s_0)\neq 0\right\} \gg \frac{|\zeta(2s_0)|^2}{|s_0|^8}\frac{N^{1-\epsilon}}{\log N}.$$

The proof in case $s_0 = \frac{1}{2}$ is similar. □

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