Computer Science 1820

Solutions for Recommended Exercises

Section 1.6

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2. Let m and n be any two even integers (possibly the same). Then, there exist integers k and l such that $m = 2k$ and $n = 2l$. Consequently,

$$m + n \;=\; 2k + 2l \;=\; 2(k + l), \;\text{ where }\; k + l \in \mathbb{Z},$$

so the sum of m and n is even, as required.

4. Let m be any even integer. Then, there exists an integer k such that $m = 2k$. Accordingly,

$$-m \;=\; -2k \;=\; 2(-k), \;\text{ where }\; -k \in \mathbb{Z},$$

so the negative of m is even, as required.

6. Let m and n be any two odd integers (possibly the same), so that there exist integers k and l such that $m = 2k + 1$ and $n = 2l + 1$. Then,

$$mn \;=\; (2k+1)(2l+1) \;=\; 4kl + 2k + 2l + 1 \;=\; 2(2kl + k + l) + 1, \;\text{ where }\; 2kl + k + l \in \mathbb{Z},$$

so the product of m and n is odd, as required.

8. First, let n be an integer. To prove the given statement, we use a proof by contradiction: assume that n is a perfect square *and* $n + 2$ is a perfect square. Then, $n = k^2$ and $n + 2 = l^2$ for some integers k and l; clearly, $n + 2 > n \geq 0$. We now let $p = |k|$ and $q = |l|$, so that $n = p^2$, $n + 2 = q^2$, and $q > p \geq 0$. Consequently,

$$p^2 + 2 = q^2 \;\rightarrow\; 2 = q^2 - p^2 \;\rightarrow\; 2 = (q - p)(q + p) \;\rightarrow\; (q + p = 1 \lor q + p = 2) \rightarrow\; p = 0$$

(consider the positive integer factors of 2, then consider that $q > p \geq 0$). However,

$$n + 2 \;=\; p^2 + 2 \;=\; 0^2 + 2 \;=\; 2,$$

which is not a perfect square. This a contradiction, thereby proving the original statement.

10. In "if-then" form, the given statement is "If x is rational and y is rational, then xy is rational." To do a direct proof, we assume that x and y are both rational. Then, there exist integers p, q, m, and n with q and n being nonzero such that $x = p/q$ and $y = m/n$. Accordingly,

$$xy = \left(\frac{p}{q}\right)\left(\frac{m}{n}\right) = \frac{pm}{qn},$$

where pm and qn are integers and qn is nonzero; by definition, xy is rational, as required.

12. The "if-then" form of the given statement is "If x is a nonzero rational and y is irrational, then xy is irrational," where the domain is $\mathbb{R}$. This statement is *true.* To prove it, we use a proof by contradiction: assume that x is a nonzero rational and y is irrational *and* xy is *rational.* Then, there exist integers p, q, m, and n with q, n, *and* p being nonzero such that $x = p/q$ and $xy = m/n$. Accordingly,

$$\left(\frac{p}{q}\right)y = \left(\frac{m}{n}\right) \to y = \left(\frac{q}{p}\right)\left(\frac{m}{n}\right) = \frac{qm}{pn},$$

where qm and pn are integers and qn is nonzero i.e. y is rational, which is a contradiction, as required.

14. We use a direct proof: let x be a nonzero rational, so that there exist integers p and q *both* nonzero such that $x = p/q$. Then

$$\frac{1}{x} = \frac{1}{p/q} = \frac{q}{p},$$

where q and p are integers and p is nonzero, so $1/x$ is rational, as required.

16. Let the domain be the integers, and let $E(n)$ be "n is even." Then, the given statement is

$$(\forall m)(\forall n)(E(mn) \to (E(m) \vee E(n))).$$

Using the Contrapositive Law and De Morgan's Law, it is logically equivalent to

$$(\forall m)(\forall n)(\neg(E(m) \vee E(n)) \to \neg E(mn)) \equiv (\forall m)(\forall n)((\neg E(m) \wedge \neg E(n)) \to \neg E(mn)).$$

Consider: an integer is either even or odd, so if it is *not* even, it *must* be odd. Ergo, the last statement may be expressed in English as "If m is odd and n is odd, then mn is odd," *which we proved in Exercise 6!* ($\ddot\smile$)

30. As shown in Example 13, it suffices to prove that $(i) \to (ii)$, $(ii) \to (iii)$, and $(iii) \to (i)$:

***(i)* → *(ii)*:** "If a is less than b, then the average of a and b is greater than a:"

We use a direct proof: let a be less than b. Then,

$$a < b \;\to\; b > a \;\to\; a+b > a+a \;\to\; a+b > 2a \;\to\; \frac{a+b}{2} > a,$$

so the average of a and b is greater than a, as required.

***(ii)* → *(iii)*:** "If the average of a and b is greater than a, then the average of a and b is less than b:"

Again, we use a direct proof: let the average of a and b be greater than a. Then,

$$\frac{a+b}{2} > a \to a < \frac{a+b}{2} \to 2a < a+b \to 2a+b-a < a+b+b-a \to a+b < 2b \to \frac{a+b}{2} < b,$$

so the average of a and b is less than b, as required.

***(iii)* → *(i)*:** "If the average of a and b is less than b, then a is less than b:"

Let the average of a and b be less than b. Then,

$$\frac{a+b}{2} < b \;\to\; a+b < 2b \;\to\; a+b-b < 2b-b \;\to\; a < b,$$

as required.

Therefore, (i), (ii), and (iii) are equivalent.

32. As in Exercise 30, we prove three implications:

***(i)* → *(ii)*:** "If x is rational, then $x/2$ is rational:"

We use a direct proof: let x be rational. Then, there exists an integer p and a nonzero integer q such that $x = p/q$. Consequently,

$$\frac{x}{2} \;=\; \frac{p/q}{2/1} \;=\; \frac{p}{q}\cdot\frac{1}{2} \;=\; \frac{p}{2q},$$

where p is an integer and $2q$ is a nonzero integer i.e. $x/2$ is rational, as required.

***(ii)* → *(iii)*:** "If $x/2$ is rational, then $3x-1$ is rational:"

Again, we use a direct proof: let $x/2$ be rational. Then, there exists an integer p and a nonzero integer q such that $x/2 = p/q$; we rewrite this as $x = 2p/q$. Accordingly,

$$3x-1 \;=\; 3\left(\frac{2p}{q}\right)-1 \;=\; \frac{6p}{q}-\frac{q}{q} \;=\; \frac{6p-q}{q},$$

where $6p-q$ is an integer and q is a nonzero integer i.e. $3x-1$ is rational, as required.

***(ii)* → *(iii)*:** "If $3x-1$ is rational, then x is rational:"

Let $3x-1$ be rational, so that there exists an integer p and a nonzero integer q such that $3x-1 = p/q$. Then,

$$3x-1=\frac{p}{q} \;\rightarrow\; 3x=\frac{p}{q}+1 \;\rightarrow\; 3x=\frac{p}{q}+\frac{q}{q} \;\rightarrow\; 3x=\frac{p+q}{q} \;\rightarrow\; x=\frac{p+q}{3q},$$

where $p+q$ is an integer and $3q$ is a nonzero integer, so x is rational, as required.

Thus, (i), (ii), and (iii) are equivalent.

42. This time, we need to prove at least four implications. To avoid a difficult subproof, let us prove five: $(i) \to (ii)$, $(ii) \to (iv)$, $(iv) \to (i)$, $(ii) \to (iii)$, and $(iii) \to (ii)$:

***(i)* → *(ii)*:** "If n^2 is odd, then $1-n$ is even:"

This time, we use a contrapositive proof: let $1-n$ be odd, so that $1-n = 2k+1$ for some integer k. Then, $n = -2k$, and

$$n^2 \;=\; (-2k)^2 \;=\; 4k^2 \;=\; 2\left(2k^2\right),$$

where $2k^2$ is an integer, so n^2 is even, as required.

***(ii)* → *(iv)*:** "If $1-n$ is even, then n^2+1 is even:"

We use a direct proof: let $1-n$ be even, so that $1-n = 2k$ for some integer k. Then, $n = 1-2k$, and

$$n^2+1 = (1-2k)^2 = (1-2k)(1-2k)+1 = 1-4k+4k^2+1 = 2-4k+4k^2 = 2\left(1-2k+2k^2\right),$$

where $1-2k+2k^2$ is an integer, so n^2+1 is even, as required.

***(iv)* → *(i)*:** "If n^2+1 is even, then n^2 is odd:"

Let us use a contrapositive proof: let n^2 be even, so that $n^2 = 2k$ for some integer k. Then, $n^2+1 = 2k+1$, where k is an integer, so n^2+1 is odd, as required.

(continued)

(continued) So far, we've proved that (i), (ii), and (iv) are equivalent. We will now prove that (iii) is equivalent to them by proving that it is equivalent to (ii):

***(ii)* → *(iii)*:** "If $1-n$ is even, then n^3 is odd:"

We use a direct proof: let $1-n$ be even, so that $1-n=2k$ for some integer k. Then, $n=1-2k$, and

$$n^3=(1-2k)^3=(1-2k)(1-2k)(1-2k)=1-6k+12k^2-8k^3=2\left(-3k+6k^2-4k^3\right)+1,$$

where $-3k+6k^2-4k^3$ is an integer, so n^3 is odd, as required.

***(iii)* → *(ii)*:** "If n^3 is odd, then $1-n$ is even:"

We use a contrapositive proof: let $1-n$ be odd, so that $1-n=2k+1$ for some integer k. Then, $n=-2k$, and

$$n^3 \;=\; (-2k)^3 \;=\; -8k^3 \;=\; 2\left(-4k^3\right),$$

where $-4k^3$ is an integer, so n^3 is even, as required.

Ergo, all four statements are equivalent.