# Computer Science 1820

## Solutions for Recommended Exercises

# Section 3.2

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2. (a) If $x > 4$, then $x^2 > 16 \;\rightarrow\; x^2 \geq 11 \;\rightarrow\; 11 \leq x^2$.

   Also, if $x > 17$, then $x^2 > 17x \;\rightarrow\; x^2 \geq 17x \;\rightarrow\; 17x \leq x^2$.

   Seventeen is greater than four, so if $x > 17$, we have

   $$0 < 17x + 11 \leq 17x + x^2 \leq x^2 + x^2 = 2x^2 \;\rightarrow\; |17x + 11| \leq 2\left|x^2\right|.$$

   So, we let $C = 2$ and $k = 17$. Accordingly, $17x + 11$ $\boxed{\textit{is } O\left(x^2\right).}$

   (b) If $x > 32$, then $x^2 > 1024 \;\rightarrow\; x^2 \geq 1000 \;\rightarrow\; 1000 \leq x^2$

   $\rightarrow\; x^2 + 1000 \leq x^2 + x^2 \;\rightarrow\; 0 < x^2 + 1000 \leq 2x^2 \;\rightarrow\; \left|x^2 + 1000\right| \leq 2\left|x^2\right|$.

   So, $C = 2$ and $k = 32$ are our "witnesses" to the fact that $x^2 + 1000$ $\boxed{\textit{is } O\left(x^2\right).}$

   (c) Using the Principle of Mathematical Induction (discussed in this course), we can prove that $n < 2^n$ for all positive integers $n$. Using calculus, we can prove that $x < 2^x$ for all real $x$. We can also use calculus to prove that $\log x$ (we assume that the base of this logarithm is 2) is an *increasing* function, which essentially means that it preserves order. For now, we will assume that these results are true. Accordingly, for all $x \geq 1$,

   $$1 \leq x < 2^x \;\rightarrow\; \log 1 \leq \log x < \log 2^x \;\rightarrow\; 0 \leq \log x < x$$

   $$\rightarrow\; 0 \leq x \log x < x^2 \rightarrow\; |x \log x| < \left|x^2\right| \;\rightarrow\; |x \log x| \leq 1\left|x^2\right|.$$

   So, $x \log x$ $\boxed{\textit{is } O\left(x^2\right),}$ with our witnesses being $C = k = 1$.

(d) Assume that $x^4/2$ *is* $O(x^2)$. Then, there exist real constants $C$ and $k$ such that for all $x > k$,

$$\left|x^4/2\right| \le C\left|x^2\right| \;\rightarrow\; \left|x^2\right| \le 2C.$$

Clearly, $C$ must be positive. Now, let $x_0$ be the *maximum* of $k+1$ and $\sqrt{2C+1}$. Then, $x_0 > k$, so by the statement above,

$$\left|x_0^2\right| \;\le\; 2C.$$

On the other hand,

$$x_0 \ge \sqrt{2C+1} \ge 0 \;\rightarrow\; x_0^2 \ge 2C+1 \ge 0 \;\rightarrow\; x_0^2 > 2C \ge 0 \;\rightarrow\; \left|x_0^2\right| > 2C,$$

which is a *contradiction*. Thus, $x^4/2$ is $\boxed{\textit{not}\;\; O(x^2)\,.}$

(e) Recall that $x < 2^x$ for all real $x$. Like in (e), we assume that $2^x$ *is* $O(x^2)$, so that we can find real constants $C$ and $k$ such that for all $x > k$,

$$|2^x| \le C\left|x^2\right| \;\rightarrow\; 2^x \le Cx^2.$$

Of course, $C$ must be positive. Now, let $x_0$ be the maximum of $k+1$ and $27(C+1)$. Then, $x_0 > k$, so by the statement above,

$$2^{x_0} \;\le\; Cx_0^2.$$

However, $\quad 2^{x_0} \;=\; \left(2^{x_0/3}\right)^3 \;>\; (x_0/3)^3 \;=\; \dfrac{x_0}{27}\,x_0^2 \;\ge\; \dfrac{27(C+1)}{27}\,x_0^2 \;=\; (C+1)x_0^2 \;>\; Cx_0^2$

i.e. $2^{x_0} > Cx_0^2$, which is a contradiction. Thus, $2^x$ is $\boxed{\textit{not}\;\; O(x^2)\,.}$

(f) $\lfloor x \rfloor$, known as the *floor* function, returns the largest integer $n$ that is less than or equal to $x$ i.e. $n$ is the unique integer satisfying

$$x-1 < n \le x \;\rightarrow\; n \le x \;\rightarrow\; \lfloor x \rfloor \le x.$$

$\lceil x \rceil$, known as the *ceiling* function, returns the smallest integer $n$ that is greater than or equal to $x$ i.e. $n$ is the unique integer satisfying

$$x \le n < x+1 \;\rightarrow\; n < x+1 \;\rightarrow\; \lceil x \rceil < x+1.$$

Then, for all $x \ge 1$,

$$0 \;\le\; \lfloor x \rfloor \cdot \lceil x \rceil \;\le\; x\cdot(x+1) \;=\; x^2+x \;\le\; x^2+x\cdot x \;=\; 2x^2,$$

so $|\,\lfloor x \rfloor \cdot \lceil x \rceil\,| \le 2\left|x^2\right|$. Ergo, $\lfloor x \rfloor \cdot \lceil x \rceil$ $\boxed{\textit{is}\;\; O(x^2)\,,}$ with witnesses $C = 2$ and $k = 1$.

6. For all $x \geq 1$, $\;0 \;<\; \dfrac{x^3+2x}{2x+1} \;<\; \dfrac{x^3+2x}{2x} \;=\; \dfrac{x^3}{2x}+\dfrac{2x}{2x} \;=\; \dfrac{1}{2}x^2+1 \;\leq\; \dfrac{1}{2}x^2+x^2 \;<\; 2x^2$,

so $\left|\dfrac{x^3+2x}{2x+1}\right| \leq 2\left|x^2\right|$. Choosing $C=2$ and $k=1$, we obtain the desired result.

8. (a) For all $x \geq 1$, $\;0 \;<\; 2x^2+x^3\log x \;<\; 2x^2+x^3(x) \;\leq\; 2x^2\cdot x^2+x^4 \;=\; 3x^4$

$\rightarrow \left|2x^2+x^3\log x\right| \leq 3\left|x^4\right|$, so $\boxed{n=4.}$

(b) For all $x \geq 1$, $\;0 \;<\; 3x^5+(\log x)^4 \;\leq\; 3x^5+(x)^4 \;\leq\; 3x^5+x^4\cdot x \;=\; 4x^5$

$\rightarrow \left|3x^5+(\log x)^4\right| \leq 4\left|x^5\right|$, so $\boxed{n=5.}$

(c) For all $x \geq 1$, $\;0 \;<\; \dfrac{x^4+x^2+1}{x^3+1} \;\leq\; \dfrac{x^4+x^2\cdot x^2+1\cdot x^4}{x^3+1} \;=\; \dfrac{3x^4}{x^3+1} \;<\; \dfrac{3x^4}{x^3} \;=\; 3x$

$\rightarrow \left|\dfrac{x^4+x^2+1}{x^3+1}\right| \leq 3|x|$, so $\boxed{n=1.}$

(d) For all $x \geq 1$, $\;0 \;<\; \dfrac{x^3+5\log x}{x^4+1} \;<\; \dfrac{x^3+5(x)}{x^4+1} \;<\; \dfrac{x^3+5x}{x^4} \;=\; \dfrac{x^3}{x^4}+\dfrac{5x}{x^4} \;=\; \dfrac{1}{x}+\dfrac{5}{x^3} \;\leq\; 1+5 \;=\; 6$

$\rightarrow \left|\dfrac{x^3+5\log x}{x^4+1}\right| \leq 6|1|$, so $\boxed{n=0.}$

10. $x \geq 1 \;\rightarrow\; 0<1\leq x \;\rightarrow\; 0<1\cdot x^3 \leq x\cdot x^3 \;\rightarrow\; 0<x^3\leq x^4$,

so for all $x>1$, $\left|x^3\right| \leq 1\left|x^4\right|$. Thus, $x^3$ *is* $O\left(x^4\right)$ with witnesses $C=k=1$.

Now assume that $x^4$ is $O\left(x^3\right)$, so that we can find real constants $C$ and $k$ such that for all $x>k$,

$$\left|x^4\right| \leq C\left|x^3\right| \;\rightarrow\; |x| \leq C.$$

Picking any $x$ that is larger than both $k$ and $C$ yields a contradiction, so $x^4$ is *not* $O\left(x^3\right)$.

12. For all $x \geq 1$, $0 \leq \log x < x \rightarrow 0 \leq x \cdot \log x \leq x \cdot x \rightarrow 0 \leq x \log x < x^2 \rightarrow |x \log x| \leq 1\left|x^2\right|$

so $x \log x$ *is* $O\left(x^2\right)$ with witnesses $C = k = 1$.

Now assume that $x^2$ is $O(x \log x)$, so that we can find real constants $C$ and $k$ such that for all $x > k$,

$$\left|x^2\right| \leq C|x \log x| \rightarrow |x| \leq C|\log x|.$$

Again, $C$ must be positive. Next, let $x_0$ be the maximum of $k+1$ and $2^{4C}$. Then, $x_0 > k$, so by our assumption,

$$|x_0| \leq C|\log x_0|.$$

However, $|x_0| = \left|2^{\log x_0}\right| = \left|2^{(\log x_0)/2}\right|^2$

$$> \left|\frac{\log x_0}{2}\right|^2 = \frac{|\log x_0|}{4}|\log x_0| > \frac{\left|\log 2^{4C}\right|}{4}|\log x_0| = \frac{|4C|}{4}|\log x_0| = C|\log x_0|$$

i.e. $|x_0| > C\,|\log x_0|$, which is a contradiction. Therefore, $x^2$ is *not* $O(x \log x)$.

14. (a) Assume that $x^3$ is $O\left(x^2\right)$, so that we can find real constants $C$ and $k$ such that for all $x > k$, $\left|x^3\right| \leq C\left|x^2\right| \rightarrow |x| \leq C$. Choosing any $x$ that is greater than both $k$ and $C$ produces a contradiction, so the answer is $\boxed{\text{no.}}$

(b) For *all* real $x$, $\left|x^3\right| \leq 1\left|x^3\right|$, so the answer is $\boxed{\text{yes.}}$

(c) For *all* real $x$, $x^3 \leq x^2 + x^3$, and for all nonnegative $x$, $\left|x^3\right| \leq \left|x^2 + x^3\right| = 1\left|x^2 + x^3\right|$, so the answer is $\boxed{\text{yes.}}$

(d) For all $x \geq 1$, $0 < x^3 \leq x^4 \leq x^2 + x^4 \rightarrow \left|x^3\right| \leq 1\left|x^2 + x^4\right|$, so the answer is $\boxed{\text{yes.}}$

(e) $$|3^x| = \left|2^{\log 3}\right|^x = \left|2^{x \log 3}\right| = \left|2^{(x \log 3)/3}\right|^3 > \left|\frac{x \log 3}{3}\right|^3 = \left(\frac{\log 3}{3}\right)^3 \left|x^3\right|$$

$$\rightarrow \left|x^3\right| \leq \left(\frac{3}{\log 3}\right)^3 |3^x|,$$ so the answer is $\boxed{\text{yes.}}$

(f) For *all* real $x$, $\left|\frac{x^3}{2}\right| \leq \frac{1}{2}\left|x^3\right|$, so the answer is $\boxed{\text{yes.}}$

20. (a) We use Theorems 2 and 3: $\left(n^3+n^2\log n\right)(\log n+1)+(17\log n+19)\left(n^3+2\right)$

$$= O\big(\left(n^3+n^2\log n\right)(\log n)+(17\log n+19)\left(n^3+2\right)\big)$$

$$= O\big(\left(n^3+n^2\log n\right)(\log n)+(17\log n)\left(n^3+2\right)\big) = O\big(\left(n^3+n^2\log n\right)(\log n)+(17\log n)\left(n^3\right)\big)$$

$$= O\left(n^2(n+\log n)(\log n)+(17\log n)\left(n^3\right)\right) = O\left(n^2(n)(\log n)+(17\log n)\left(n^3\right)\right)$$

$$= O\left(n^3\log n+17n^3\log n\right) = \boxed{O\left(n^3\log n\right).}$$

(b) $\left(2^n+n^2\right)\left(n^3+3^n\right) = O\left((2^n)\left(n^3+3^n\right)\right) = O((2^n)(3^n)) = O((2\cdot 3)^n) = \boxed{O(6^n).}$

(c) $(n^n+n2^n+5^n)(n!+5^n) = O((n^n+n2^n+5^n)(n!)) = O((n^n)(n!)) = \boxed{O(n!\,n^n).}$

24. (a) We first show that $3x+7$ is $O(x)$: for all $x \geq 7$,

$$0 < 3x+7 \leq 3x+x = 4x \rightarrow |3x+7| \leq 4|x|,$$

so our witnesses are $C=4$ and $k=7$.

Next, we show that $3x+7$ is $\Omega(x)$: for all $x \geq 1$,

$$3x+7 > 3x > 0 \rightarrow |3x+7| \geq 3|x|,$$

so our witnesses are $C=3$ and $k=1$.

Since $3x+7$ is both $O(x)$ and $\Omega(x)$, it is $\Theta(x)$, as required.

(b) $\left|2x^2+x-7\right| = \left|\left(2x^2\right)+(x-7)\right| \leq \left|2x^2\right| + |x-7|$ (Triangle Inequality)

$= \left|2x^2\right| + |(x)+(-7)| \leq \left|2x^2\right| + |x| + |-7|$ (Triangle Inequality again)

$= \left|2x^2\right| + |x| + 7 = 2x^2 + x + 7$ (provided that $x$ is nonnegative)

(continued)

(continued) $\leq\ 2x^2 + x^2 + 7$ (provided that $x \geq 1$)

$\leq\ 2x^2 + x^2 + x^2$ (provided that $x \geq 3$)

$=\ 4x^2\ =\ 4|x^2|$. Accordingly, $2x^2+x-7$ is $O(x^2)$ with witnesses $C=4$ and $k=3$.

Next, $2x^2+x-7\ =\ 2x^2+(x-7)\ \geq\ 2x^2$ (provided that $x \geq 7$)

$\geq\ 0\ \ \rightarrow\ \ |2x^2+x-7|\ \geq\ 2|x^2|$, so $2x^2+x-7$ is $\Omega(x^2)$ with witnesses $C=2$ and $k=7$.

Consequently, $2x^2+x-7$ is $\Theta(x^2)$, as required.

(c) $\left\lfloor x+\frac{1}{2}\right\rfloor$ is the unique integer $n$ satisfying

$$\left(x+\frac{1}{2}\right)-1\ <\ n\ \leq\ x+\frac{1}{2}\ \ \rightarrow\ \ x-\frac{1}{2}\ <\ \left\lfloor x+\frac{1}{2}\right\rfloor\ \leq\ x+\frac{1}{2}.$$

Then, for all $x \geq 1$, $\ 0\ <\ \frac{1}{2}x\ <\ \frac{1}{2}x+\frac{1}{2}(x-1)\ =\ x-\frac{1}{2}\ <\ \left\lfloor x+\frac{1}{2}\right\rfloor$

$\rightarrow\ \left|\left\lfloor x+\frac{1}{2}\right\rfloor\right| \geq \frac{1}{2}|x|$, so $\left\lfloor x+\frac{1}{2}\right\rfloor$ is $\Omega(x)$ with witnesses $C=1/2$ and $k=1$.

Also, for all $x \geq 1$, $\ 0\ <\ \left\lfloor x+\frac{1}{2}\right\rfloor\ \leq\ x+\frac{1}{2}\ <\ x+x\ =\ 2x$

$\rightarrow\ \left|\left\lfloor x+\frac{1}{2}\right\rfloor\right| \leq 2|x|$, so $\left\lfloor x+\frac{1}{2}\right\rfloor$ is $O(x)$ with witnesses $C=2$ and $k=1$.

Ergo, $\left\lfloor x+\frac{1}{2}\right\rfloor$ is $\Theta(x)$, as required.

(d) For all $x \geq 2$, $\ x\ =\ \sqrt{x^2}\ <\ \sqrt{x^2+1}\ <\ \sqrt{x^2+x^2}\ =\ \sqrt{2x^2}\ =\ \left(\sqrt{2}\right)x\ <\ (x)x\ =\ x^2.$

In other words, for all $x \geq 2$, $\ 1\ <\ x\ <\ \left(x^2+1\right)^{1/2}\ <\ x^2$

$\rightarrow\ \ \log 1\ <\ \log x\ <\ \log\left(x^2+1\right)^{1/2}\ <\ \log x^2$

$\rightarrow\ \ 0\ <\ \log x\ <\ \frac{1}{2}\log\left(x^2+1\right)\ <\ 2\log x$ (continued)

(continued) $\rightarrow\ 0 < 2\log x < \log\left(x^2+1\right) < 4\log x$

$\rightarrow\ 2|\log x| \leq \left|\log\left(x^2+1\right)\right| \leq 4|\log x|$.

To show that $\log\left(x^2+1\right)$ is $O(\log x)$, we choose $C=4$ and $k=2$, and to show that it is $\Omega(\log x)$, we make $C=2$ and $k=2$; both of these show that it is $\Theta(\log x)$, as required.

(e) For all positive $x$, $x = 2^{\log_2 x} \rightarrow \log_{10} x = \log_{10} 2^{\log_2 x} \rightarrow \log_{10} x = (\log_2 x)\log_{10} 2$

$\rightarrow \log_{10} x = (\log_{10} 2)\log_2 x \rightarrow |\log_{10} x| = (\log_{10} 2)\,|\log_2 x|$.

Note that if $a=b$, then $a \leq b$ and $a \geq b$ are both true! Thus, $\log_{10} x$ is $\Theta\left(\log_2 x\right)$ with witnesses $C = \log_{10} 2$ and $k=1$.

26. To prove this biconditional statement, we must prove each implication. First, assume that $f(x)$ is $O(g(x))$. Then, there exist real constants $C$ and $k$ such that for all $x>k$, $|f(x)| \leq C|g(x)|$. Clearly, $C$ must be posi…waitaminute! It *could* be zero! Precisely, $C$ must be *nonnegative.*

Say that $C=0$. Then, for all real $x$, $f(x)=0$, so

$$|g(x)| \geq 0 \ \rightarrow\ |g(x)| \geq |f(x)| \ \rightarrow\ |g(x)| \geq 1|f(x)|.$$

Hence, $g(x)$ is $\Omega(f(x))$ with witnesses $C' = k' = 1$.

Now say that $C>0$. Then, for all $x>k$,

$$|f(x)| \leq C|g(x)| \ \rightarrow\ C|g(x)| \geq |f(x)| \ \rightarrow\ |g(x)| \geq \frac{1}{C}|f(x)|.$$

Thus, $g(x)$ is $\Omega(f(x))$ with witnesses $C' = \dfrac{1}{C}$ and $k' = |k|+1$ (this ensures that $k'>0$ and $k'>k$).

Either way, $g(x)$ is $\Omega(f(x))$.

Next, assume that $g(x)$ is $\Omega(f(x))$. Then, there are *positive* numbers $C$ and $k$ such that for all $x>k$,

$$|g(x)| \geq C|f(x)| \ \rightarrow\ C|f(x)| \leq |g(x)| \ \rightarrow\ |f(x)| \leq \frac{1}{C}|g(x)|.$$

Ergo, $f(x)$ is $O(f(x))$ with witnesses $C' = \dfrac{1}{C}$ and $k'=k$.