Computer Science 1820

Solutions for Recommended Exercises

Section 3.8

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2. (a) To add two matrices *of the same size,* we add *corresponding entries:*

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-1) & 0+3 & 4+5 \\ -1+2 & 2+2 & 2+(-3) \\ 0+2 & -2+(-3) & -3+0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}.}$$

(b) $$\mathbf{A}+\mathbf{B} = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+9 & 5+(-3) & 6+4 \\ -4+0 & -3+(-2) & 5+(-1) & -2+2 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}.}$$

4. To find the $(i,\, j)$th entry in the product $\mathbf{AB}$, we pair up the entries in the ith row of $\mathbf{A}$ with those in the jth row of $\mathbf{B}$, multiply each pair, then add the results.

(a) $$\mathbf{AB} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(0)+0(1)+1(-1) & 1(1)+0(-1)+1(0) & 1(-1)+0(0)+1(1) \\ 0(0)-1(1)-1(-1) & 0(1)-1(-1)-1(0) & 0(-1)-1(0)-1(1) \\ -1(0)+1(1)+0(-1) & -1(1)+1(-1)+0(0) & -1(-1)+1(0)+0(1) \end{bmatrix}$$

(continued)

$$\text{(continued)} = \begin{bmatrix} 0+0-1 & 1+0+0 & -1+0+1 \\ 0-1+1 & 0+1-0 & 0-0-1 \\ 0+1+0 & -1-1+0 & 1+0+0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}.}$$

(b) $$\mathbf{AB} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)-3(-1)+0(-3) & 1(-1)-3(0)+0(-2) & 1(2)-3(3)+0(0) & 1(3)-3(-1)+0(2) \\ 1(1)+2(-1)+2(-3) & 1(-1)+2(0)+2(-2) & 1(2)+2(3)+2(0) & 1(3)+2(-1)+2(2) \\ 2(1)+1(-1)-1(-3) & 2(-1)+1(0)-1(-2) & 2(2)+1(3)-1(0) & 2(3)+1(-1)-1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+0 & -1-0+0 & 2-9+0 & 3+3+0 \\ 1-2-6 & -1+0-4 & 2+6+0 & 3-2+4 \\ 2-1+3 & -2+0+2 & 4+3-0 & 6-1-2 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -1 & -7 & 6 \\ -7 & -5 & 8 & 5 \\ 4 & 0 & 7 & 3 \end{bmatrix}.}$$

(c) $$\mathbf{AB} = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 0-0 & 0-3 & 0-4 & 0-1 \\ 28-4 & -7+0 & 14+6 & 21+8 & 0+2 \\ -16+6 & 4-0 & -8-9 & -12-12 & 0-3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix}.}$$

10. The product of an $m \times n$ matrix and a $p \times q$ matrix (in that order) is defined only if $n = p$; if it is defined, its size is $m \times q$.

(a) $4 = 4$, so **AB** $\boxed{\textit{is}\text{ defined,}}$ and its size is $\boxed{3 \times 5.}$

(b) $5 \neq 3$, so **BA** is $\boxed{\textit{not}\text{ defined.}}$

(continued)

(continued)

(c) $4 = 4$, so $\mathbf{AC}$ $\boxed{\textit{is}\text{ defined,}}$ and its size is $\boxed{3 \times 4.}$

(d) $4 \neq 3$, so $\mathbf{CA}$ is $\boxed{\textit{not}\text{ defined.}}$

(e) $5 \neq 4$, so $\mathbf{BC}$ is $\boxed{\textit{not}\text{ defined.}}$

(f) $4 = 4$, so $\mathbf{CB}$ $\boxed{\textit{is}\text{ defined,}}$ and its size is $\boxed{4 \times 5.}$

18. Two matrices are *(multiplicative) inverses* of each other if their product (in any order) is the (same) *(multiplicative) identity*, $\mathbf{I}_n$. So, we calculate the product of the two matrices in both orders:

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14-12-1 & -16+15+1 & 10-9-1 \\ 7-8+1 & -8+10-1 & 5-6+1 \\ -7+4+3 & 8-5-3 & -5+3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3.$$

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14-8-5 & 21-16-5 & -7-8+15 \\ -8+5-3 & -12+10+3 & 4+5-9 \\ 2-1-1 & 3-2-1 & -1-1+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3.$$

Accordingly, the given matrices *are* inverses of each other.

20. (a) Using the formula in Exercise 19,

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{(3)}{(-1)(3)-(2)(1)} & \dfrac{-(2)}{(-1)(3)-(2)(1)} \\[2ex] \dfrac{-(1)}{(-1)(3)-(2)(1)} & \dfrac{(-1)}{(-1)(3)-(2)(1)} \end{bmatrix} = \boxed{\begin{bmatrix} -.6 & .4 \\ .2 & .2 \end{bmatrix}.}$$

(continued)

(continued)

(b) $\mathbf{A}^3 = \mathbf{A}(\mathbf{A}\mathbf{A}) = \mathbf{A}\left(\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\right)$

$= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix}\right) = \boxed{\begin{bmatrix} 1 & 18 \\ 9 & 37 \end{bmatrix}.}$

(c) $\left(\mathbf{A}^{-1}\right)^3 = \mathbf{A}^{-1}(\mathbf{A}^{-1}\mathbf{A}^{-1}) = \mathbf{A}^{-1}\left(\begin{bmatrix} -.6 & .4 \\ .2 & .2 \end{bmatrix}\begin{bmatrix} -.6 & .4 \\ .2 & .2 \end{bmatrix}\right)$

$= \begin{bmatrix} -.6 & .4 \\ .2 & .2 \end{bmatrix}\left(\begin{bmatrix} .44 & -.16 \\ -.08 & .12 \end{bmatrix}\right) = \boxed{\begin{bmatrix} -.296 & .144 \\ .072 & -.008 \end{bmatrix}.}$

(d) $\begin{bmatrix} 1 & 18 \\ 9 & 37 \end{bmatrix}\begin{bmatrix} -.296 & .144 \\ .072 & -.008 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -.296 & .144 \\ .072 & -.008 \end{bmatrix}\begin{bmatrix} 1 & 18 \\ 9 & 37 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

so $\left(\mathbf{A}^{-1}\right)^3$ is the inverse of $\mathbf{A}^3$, as expected.

28. (a) $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\vee 0 & 1\vee 1 \\ 0\vee 1 & 1\vee 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.}$

(b) $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\wedge 0 & 1\wedge 1 \\ 0\wedge 1 & 1\wedge 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.}$

(c) The *Boolean product* of two zero-one matrices is computed like their ordinary matrix product, but with $\wedge$ replacing multiplication and $\vee$ replacing addition:

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0) \\ (0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(1\wedge 0) \end{bmatrix}$$

(continued)

$$\text{(continued)} = \begin{bmatrix} 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 0 \vee 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.}$$

30. $$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \vee 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.}$$

36. Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ zero-one matrix, so that the (i, j)th entry of $\mathbf{A}$ is a_{ij}. Similarly, let $\mathbf{I}_n = [\partial_{ij}]$, so that ∂_{ij} is one if $i = j$ and zero otherwise. Then, the (i, j)th entry of $\mathbf{A} \odot \mathbf{I}_n$ is

$$(a_{i1} \wedge \partial_{1j}) \vee (a_{i2} \wedge \partial_{2j}) \vee \cdots \vee (a_{ij} \wedge \partial_{jj}) \vee \cdots \vee (a_{in} \wedge \partial_{nj})$$

$$= (a_{i1} \wedge 0) \vee (a_{i2} \wedge 0) \vee \cdots \vee (a_{ij} \wedge 1) \vee \cdots \vee (a_{in} \wedge 0) = 0 \vee 0 \vee \cdots \vee a_{ij} \vee \cdots \vee 0 = a_{ij}.$$

Since i and j were arbitrary, *every* entry in $\mathbf{A} \odot \mathbf{I}_n$ is equal to its corresponding entry in $\mathbf{A}$; in other words, $\mathbf{A} \odot \mathbf{I}_n = \mathbf{A}$.

Next, the (i, j)th entry of $\mathbf{I}_m \odot \mathbf{A}$ is

$$(\partial_{i1} a_{1j}) \vee (\partial_{i2} a_{2j}) \vee \cdots \vee (\partial_{ii} a_{ij}) \vee \cdots \vee (\partial_{im} a_{mj})$$

$$= (0 \wedge a_{1j}) \vee (0 \wedge a_{2j}) \vee \cdots \vee (1 \wedge a_{ij}) \vee \cdots \vee (0 \wedge a_{mj}) = 0 \vee 0 \vee \cdots \vee a_{ij} \vee \cdots \vee 0 = a_{ij}.$$

So, every entry in $\mathbf{I}_m \odot \mathbf{A}$ is equal to its corresponding entry in $\mathbf{A}$, meaning that $\mathbf{I}_m \odot \mathbf{A} = \mathbf{A}$.

Combining these results, we have $\mathbf{A} \odot \mathbf{I} = \mathbf{I} \odot \mathbf{A} = \mathbf{A}$, as required.