Computer Science 1820

Solutions for Recommended Exercises

Section 4.1

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4. (a) $P(1)$ is $1^3 = (1(1+1)/2)^2$.

   (b) The right-hand side of the equation in $P(1)$ is

   $$(1(1+1)/2)^2 \;=\; (1(2)/2)^2 \;=\; (2/2)^2 \;=\; (1)^2 \;=\; 1 \;=\; 1^3,$$

   which is the left-hand side of the equation in $P(1)$; ergo, $P(1)$ is true.

   (c) The inductive hypothesis is that for some integer $k \geq 1$, $P(k)$ is true

   $$\text{i.e. } 1^3+2^3+3^3+\cdots+k^3 \;=\; (k(k+1)/2)^2.$$

   (d) In the inductive step, we need to prove $P(k+1)$, or

   $$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 \;=\; ((k+1)((k+1)+1)/2)^2.$$

   (e) By our inductive hypothesis, the left-hand side of the equation in $P(k+1)$ may be rewritten as

   $$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 \;=\; \left(1^3+2^3+3^3+\cdots+k^3\right)+(k+1)^3$$

   $$=\; (k(k+1)/2)^2+(k+1)^3 \;=\; \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \;=\; \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}$$

   $$=\; \frac{k^2(k+1)^2+4(k+1)^3}{4} \;=\; \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4} \;=\; \frac{(k+1)^2\left[k^2+4k+4\right]}{4}$$

   $$=\; \frac{(k+1)^2\left[(k+2)(k+2)\right]}{4} \;=\; \frac{(k+1)^2(k+2)^2}{4} \;=\; \left(\frac{(k+1)(k+2)}{2}\right)^2$$

   $$=\; ((k+1)((k+1)+1)/2)^2,$$

   which is the right-hand side of the equation in $P(k+1)$, making that statement true. By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer n.

(f) We proved that if $P(k)$ is true for any positive integer k, then $P(k+1)$ is also true. We also proved that $P(1)$ is true; since 1 is a positive integer, k could have been 1, which means that we proved that $P(2)$ is also true. 2 is also a positive integer, so k could have been 2, which means that $P(3)$ is also true. k could have been 3, which means that $P(4)$ is also true. k could have been 5, so $P(6)$ is true, ad infinitum.

6. **Base Case:** $1 \cdot 1! = 1 \cdot 1 = 1 = 2-1 = 2!-1 = (1+1)!-1$, so $P(1)$ is true.

Inductive Step: Assume that for some integer $k \geq 1$, $P(k)$ is true

$$\text{i.e. } 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots k \cdot k! = (k+1)! - 1.$$

Then, $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots k \cdot k! + (k+1) \cdot (k+1)!$

$= (1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots k \cdot k!) + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)!$

$= (k+1)! + (k+1) \cdot (k+1)! - 1 = [1 + (k+1)] \cdot (k+1)! - 1$

$= [k+2] \cdot (k+1)! - 1 = [k+2]! - 1 = [(k+1)+1]! - 1$, so $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

8. **Base Case:** $2(-7)^0 = 2(1) = 2 = 8/4 = (1-(-7))/4 = \left(1-(-7)^{0+1}\right)/4$, so $P(0)$ is true.

Inductive Step: Assume that for some integer $k \geq 0$, $P(k)$ is true

$$\text{i.e. } 2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k = \left(1-(-7)^{k+1}\right)/4.$$

Then, $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k + 2(-7)^{k+1}$

$= \left(2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k\right) + 2(-7)^{k+1} = \left(1-(-7)^{k+1}\right)/4 + 2(-7)^{k+1}$

$$= \frac{1-(-7)^{k+1}}{4} + \frac{8(-7)^{k+1}}{4} = \frac{1-(-7)^{k+1}+8(-7)^{k+1}}{4} = \frac{1+7(-7)^{k+1}}{4} = \frac{1-(-7)(-7)^{k+1}}{4}$$

$= \left(1-(-7)^{(k+1)+1}\right)/4$, so $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all nonnegative integers n.

10. (a) $\underline{n=1}$: $\dfrac{1}{1\cdot 2} = \dfrac{1}{2} = \dfrac{1}{1+1}$.

$\underline{n=2}$: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3} = \dfrac{1}{2}+\dfrac{1}{6} = \dfrac{3}{6}+\dfrac{1}{6} = \dfrac{4}{6} = \dfrac{2}{3} = \dfrac{2}{2+1}$.

$\underline{n=3}$: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4} = \dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{12} = \dfrac{6}{12}+\dfrac{2}{12}+\dfrac{1}{12} = \dfrac{9}{12} = \dfrac{3}{4} = \dfrac{3}{3+1}$.

We guess that for all positive integers n, $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

(b) **Base Case:** We proved above that $P(1)$ is true.

Inductive Step: Assume that for some integer $k \geq 1$, $P(k)$ is true

$$\text{i.e.}\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Then, $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)((k+1)+1)}$

$$= \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}\right)+\frac{1}{(k+1)((k+1)+1)} = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1},\ \text{so } P(k+1) \text{ is true.}$$

By the Principle of Mathematical Induction,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all positive integers n.

20. **Base Case:** $3^7 = 2187 < 5040 = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 = 7!$, so $P(7)$ is true.

Inductive Step: Assume that for some integer $k > 6$, $P(k)$ is true i.e. $3^k < k!$.

Then, $3^{k+1} = 3\cdot 3^k < 3\cdot k! < (k+1)\cdot k! = (k+1)!$, so $P(k+1)$ is true.

By the Principle of Mathematical Induction, $3^n < n!$ for all integers $n > 6$.

22. We do a small table of values:

n	n^2	$n!$
0	$0^2 = 0$	$0! = 0$
1	$1^2 = 1$	1
2	$2^2 = 4$	$1 \cdot 2 = 2$
3	$3^2 = 9$	$1 \cdot 2 \cdot 3 = 6$
4	$4^2 = 16$	$1 \cdot 2 \cdot 3 \cdot 4 = 24$
5	$5^2 = 25$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

We guess than if n is a nonnegative integer and $n^2 \leq n!$, then $n = 0$, $n = 1$, or $n \geq 4$.

We proved $P(0)$ and $P(1)$ above, so we need to prove that $P(n)$ is true for all integers $n \geq 4$:

Base Case: We proved above that $P(4)$ is true.

Inductive Step: Assume that for some integer $k \geq 4$, $P(k)$ is true i.e. $k^2 \leq k!$.

Then, $(k+1)^2 \;=\; (k+1)(k+1) \;=\; k^2 + 2k + 1 \;<\; k^2 + k^2 + k^2$

$=\; 3 \cdot k^2 \;\leq\; 3 \cdot k! \;<\; (k+1) \cdot k! \;=\; (k+1)!$, so $P(k+1)$ is true.

By the Principle of Mathematical Induction, $n^2 \leq n!$ for all integers $n \geq 4$.

32. **Base Case:** $(1)^3 + 2(1) \;=\; 1 + 2 \;=\; 3$, and 3 divides 3, so $P(1)$ is true.

Inductive Step: Assume that for some integer $k \geq 1$, $P(k)$ is true i.e. 3 divides $k^3 + 2k$, which means that we can find an integer m such that $k^3 + 2k = 3m$.

Then, $(k+1)^3 + 2(k+1) \;=\; (k+1)\left[(k+1)^2 + 2\right] \;=\; (k+1)[(k+1)(k+1) + 2]$

$=\; (k+1)\left[k^2 + 2k + 1 + 2\right] \;=\; (k+1)\left[k^2 + 2k + 3\right] \;=\; k\left[k^2 + 2k + 3\right] + \left[k^2 + 2k + 3\right]$

$=\; k^3 + 2k^2 + 3k + k^2 + 2k + 3 \;=\; k^3 + 2k + 3k^2 + 3k + 3 \;=\; \left(k^3 + 2k\right) + 3k^2 + 3k + 3$

$=\; 3m + 3k^2 + 3k + 3 \;=\; 3\left(m + k^2 + k + 3\right)$, where $m + k^2 + k + 3$ is an integer.

Consequently, 3 divides $(k+1)^3 + 2(k+1)$, so $P(k+1)$ is true. By the Principle of Mathematical Induction, 3 divides $n^3 + 2n$ for all positive integers n.

34. **Base Case:** $(0)^3 - (0) \;=\; 0$, and 6 divides 0, so $P(0)$ is true.

Inductive Step: Assume that for some integer $k \geq 0$, $P(k)$ is true i.e. 6 divides $k^3 - k$, which means that we can find an integer m such that $k^3 - k = 6m$.

Then, $(k+1)^3 - (k+1) \;=\; (k+1)\left[(k+1)^2 - 1\right] \;=\; (k+1)[(k+1)(k+1) - 1]$

$= \; (k+1)\left[k^2 + 2k + 1 - 1\right] \;=\; (k+1)\left[k^2 + 2k\right] \;=\; k\left[k^2 + 2k\right] + \left[k^2 + 2k\right] \;=\; k^3 + 2k^2 + k^2 + 2k$

$= \; k^3 + 3k^2 + 2k \;=\; \left(k^3 - k\right) + 3k^2 + 3k \;=\; 6m + 3\left(k^2 + k\right) \;=\; 6m + 3 \cdot k(k+1).$

Consider: if k is even, then $k(k+1)$ is even, but if k is odd, then $k+1$ is even, so $k(k+1)$ is even; either way, we can find an integer ℓ such that $k(k+1) = 2\ell$. Then, the expression above becomes

$\left(k^3 - k\right) + 3 \cdot k(k+1) \;=\; 6m + 3 \cdot 2\ell \;=\; 6m + 6\ell \;=\; 6(m+\ell)$, where $m + \ell$ is an integer.

Accordingly, 6 divides $(k+1)^3 - (k+1)$, so $P(k+1)$ is true. By the Principle of Mathematical Induction, 6 divides $n^3 - n$ for all nonnegative integers n.