NAME

STUDENT NUMBER

TOTAL MARKS: 50
TOTAL TIME: 50 minutes

Problem 1: **[5 pts]** Determine whether each of the following statements are true of false:

a) $2 + 2 > 5$ if 6 is a prime integer.

SOLUTION:
The statement is equivalent to (6 is prime $\rightarrow 2 + 2 > 5$), which is true since 6 is not a prime.

b) 0.5 is rational if and only if 4 is integer.

SOLUTION:
The statement is equivalent to (0.5 is rational $\leftrightarrow$ 4 is integer). Both component statements are true, and the biconditional is also true.

c) If $1 + 1 = 3$ then 3 is integer.

SOLUTION:
Because $1 + 1 \neq 3$, the implication is true.

d) 14 is a prime integer and $0 < 1$.

SOLUTION:
Since 14 is not a prime and we have the logical and connective, the result is false.

e) $1 + 1 = 2$ or $2 + 2 = 4$.

SOLUTION:
Because $1 + 1 = 2$ is true and we use the logical or connective, the result is true.

Problem 2: **[8 pts]** Construct a truth table for each of the following compound statements:

a) $(p \wedge q) \rightarrow (p \vee q)$.

SOLUTION:

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

b) $p \to \neg p$.

SOLUTION:

p	$\neg p$	$p \to \neg p$
0	1	1
1	0	0

Problem 3: **[12 pts]** Let $P(x, y)$ be the statement $x - y \leq 2$. If the domain of variables x and y is the set of all real numbers, then determine the truth value of the following statements and briefly argue your answer.

a) $P(1, 2)$.

SOLUTION:
$P(1, 2)$ is true because $1 - 2 = -1 \leq 2$.

b) $\forall x\ P(x, 2)$.

SOLUTION:
False, because for $x = 5$, $x - 2 = 3 \not\leq 2$.

c) $\exists y\ ((y > 4) \wedge P(1, y))$.

SOLUTION:
True. Consider $y = 5$. Then $y > 4$ and $1 - y = -4 \leq 2$.

d) $\forall x \forall y\ ((x < y) \to P(x, y))$.

SOLUTION:
True. If $x < y$ then $x - y < 0 \leq 2$.

e) $\forall x \exists y\ P(x, y)$.

SOLUTION:
True. Given some arbitrary x, take $y = x$ and $x - y = 0 \leq 2$.

f) $\exists y \forall x\ P(x, y)$.

SOLUTION:
False. Suppose such y exists. Take $x = y + 3$. Then $x - y = 3 \not\leq 2$.

Problem 4: **[5 pts]** Prove that if x is rational and $x \neq 0$ then $\frac{1}{x}$ is rational.

SOLUTION:
See solution to Exercise 14, Section 1.6.

Problem 5: **[10 pts]** Let a and b be real numbers. Prove that the following statements are equivalent:
A: a is less than b.
B: The average of a and b is greater than a.
C: The average of a and b is less than b.

SOLUTION:
See solution to Exercise 30, Section 1.6.

Problem 6: **[5 pts]** Prove the following statement by contraposition. If x and y are integers and xy is even, then x is even or y is even.

SOLUTION:
To simplify the hypothesis, we can consider the domain of variables x and y to be the integers. In this case, the hypothesis is: "xy is even", and the conclusion: "x is even or y is even". (Another solution is to consider the hypothesis: "x is integer AND y is integer AND xy is even".)
For the former approach, the contrapositive is: "if x is not even and y is not even, then xy is not even". Since x and y are integer, it must be that x and y are odd. Then there exist $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$ satisfying

$$x = 2k + 1 \text{ and } y = 2l + 1.$$
$$\to xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

This means xy is odd, thus xy is not even.

Problem 7: **[5 pts]** Prove the following statement. If n is an integer and n^2 is even, then n^3 is even.

SOLUTION:
The domain of variable n is the integers.
Hypothesis: n^2 is even.
Conclusion: n^3 is odd.
From the previous problem, taking $x = y = n$, we conclude that n is even. Thus there exists $k \in \mathbb{Z}$, satisfying $n = 2k$. It follows that $n^3 = 8k^3 = 2(4k^3)$, and n^3 is even.

FINAL SCORE (out of 50)