NAME

STUDENT NUMBER

TOTAL MARKS: 50
TOTAL TIME: 50 minutes

Problem 1: **[total 20 pts]** Suppose that a die is loaded such that a six occurs three times as often as the other 5 faces which are equally likely.

a) **[2 pts]** Describe the sample space if the experiment consists of a single roll of the die.

SOLUTION:
$S = \{1, 2, 3, 4, 5, 6\}$.

b) **[4 pts]** Give the probability distribution for the experiment in subproblem 1a.

SOLUTION:
Let $p(1) = \ldots = p(5) = x$, $p(6) = 3x$.

$$\to 5x + 3x = 1$$
$$\to x = \frac{1}{8}.$$

c) **[2 pts]** Describe the sample space if the experiment consists of three successive rolls of the die.

SOLUTION:
$S' = \{(a, b, c) : a, b, c \in \{1, \ldots 6\}\}$. (the set of all three tuples of integers from 1 to 6).

d) **[4 pts]** What is the probability of not rolling a "6" in four rolls of the die? Write an expression involving powers.

SOLUTION:
The probability of not rolling "6" in one roll is $\frac{5}{8}$. Since the successive rolls are independent, the probability of not rolling "6" in four rolls is the product of the probabilities of not rolling "6" in each roll, which is $\left(\frac{5}{8}\right)^4$.

e) **[4 pts]** What is the probability of rolling a "6" twice in four rolls of the die? Write an expression involving powers and binomials.

SOLUTION:
Think of Bernoulli experiments ("6":success, "not 6": failure). We are rolling "6" twice and "not 6" twice, so the probability for a particular sequence is $\left(\frac{3}{8}\frac{5}{8}\right)^2$. There are $C(4, 2)$ ways for choosing the sequence number for rolling the "6", so the answer is $C(4, 2)\left(\frac{3}{8}\frac{5}{8}\right)^2$.

f) **[4 pts]** What is the probability of rolling a "1" twice in four rolls of the die? Write an expression involving powers and binomials.

Solution:
The answer is similar, but the probability of success is "1": $\frac{1}{8}$, and of failure is "not 1": $\frac{7}{8}$. The answer is $C(4,2)\left(\frac{1}{8}\frac{7}{8}\right)^2$.

Problem 2: **[6 pts]** What is the conditional probability that exactly three heads appear when a fair coin is flipped five times given that the first flip came up tails? Justify you work.

Solution:
We can justify directly as follows. Since first flip is tail, we look for the event of tossing heads three times in four tosses. Using the binomial distribution (see answer above as well), the result is $C(4,3)\left(\frac{1}{2}\right)^3\frac{1}{2} = \frac{1}{4}$.

Problem 3: **[6 pts]** Suppose that Brunhilda selects a ball by first choosing a box and then choosing a ball from this box. Box 1 contains 7 orange balls and 2 white balls. Box 2 contains 5 white balls and 4 orange balls. What is the probability that Brunhilda picked Box 2 given that she has selected an orange ball? Recall Bayes' formula:

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}.$$

Solution:
Let F be the event of picking Box 2.
Let E be the event of picking an orange ball.
We know: $p(F) = p(\overline{F}) = 0.5$. $p(E|F)$ is the probability of picking an orange ball from Box 2 which is $\frac{4}{9}$. $p(E|\overline{F}$ is the probability of picking an orange ball from Box 1 which is $\frac{7}{9}$. Substitute in the formula:

$$p(F|E) = \frac{2/9}{2/9 + 7/18} = \frac{4}{11}.$$

Problem 4: **[6 pts]** What is the expected sum of the numbers that appear when a fair die is rolled two times? Justify your work.

Solution:
One solution involves listing all 36 possibilities with their sum, and computing the expected value using the definition.
We can also define two random variables, X_1: the value of the die in the first toss, and X_2, the value of the die in the second toss. $E(X_1) = E(X_2) = \frac{1+2+3+4+5+6}{6} = 3.5$. Let $X = X_1 + X_2$. Then $E(X) = E(X_1) + E(X_2) = 7$.

Problem 5: **[12 pts]** A string of 6 characters representing a number in base 4 is generated randomly by choosing the digits 0, 1, 2, and 3 with equal probability. (a) What is the expected number of times digit 3 is generated? (b) What is the variance of the number of times digit 3 is generated?
Hint: use a Bernoulli experiment or the definition for the expected value and variance.

Solution:
a) This is again a Bernoulli experiment, with generating "3" the successful outcome. Like in previous questiuon, let X_i be the random variable returning the number of times "3" occurs in the i-th position of the string. $E(X_i) = 0.25$ (probability of generating "3"). Let $X = X_1 + \ldots + X_6$. Then, $E(X) = E(X_1) + \ldots + E(X_6) = 6\frac{1}{4} = 1.5$.
b) Since the number of times "3" appears in some position i is independent from the number of times "3" appears in another position j, we have $V(X) = V(X_1) + \ldots + V(X_6)$. But $V(X_i) = E(X_i^2) - E(X_i)^2$. The

random variable X_i^2 has values 1 with probability 0.25 and 0 with probability 0.75, so $E(X_i^2) = E(X_i) = 0.25$. Then $V(X_i) = \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$. Therefore $V(X) = 6 \cdot \frac{3}{16} = \frac{9}{8}$.

FINAL SCORE (out of 50)