GENERALIZING THE TITCHMARSH DIVISOR PROBLEM

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Abstract. Let a be a natural number different from 0. In 1963, Linnik proved the following unconditional result about the Titchmarsh divisor problem

$$\sum_{p\leq x} d(p-a) = cx + O\left(\frac{x\log\log x}{\log x}\right)$$

where c is a constant dependent on a. Titchmarsh proved the above result assuming GRH for Dirichlet L-functions in 1931.

We establish the following asymptotic relation:

$$\sum_{\substack{p\leq x\\ p\equiv a \mod k}} d\left(\frac{p-a}{k}\right) = C_k x + O\left(\frac{x}{\log x}\right)$$

where C_k is a constant dependent on k and a and the implied constant is dependent on k. We also apply it a question related to Artin's conjecture for primitive roots.

1. Introduction

Let $d(n)$ denote the number of positive divisors of $n \in \mathbb{N}$. Let p denote a prime number. Then, the Titchmarsh divisor problem [2, §9.3] is concerned with the following summation:

$$\sum_{p\leq x} d(p-a)$$

where a is a fixed non-zero integer. The above summation was first investigated by Titchmarsh [15], who proved the following theorem:

Theorem 1.1 (Titchmarsh)**.** *Suppose the Generalized Riemann Hypothesis holds for Dirichlet L-functions. Then*

$$\sum_{a<p\leq x} d(p-a) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p|a}\left(1-\frac{p}{p^2-p+1}\right)x + O\left(\frac{x\log\log x}{\log x}\right) \tag{1.1}$$

as $x \to \infty$.

In 1961, Linnik [10] proved the above asymptotic formula with his dispersion method, thereby eliminating the use of the generalized Riemann hypothesis from the above theorem.

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Rodriquez [14] and Halberstam [7] independently showed that the above can be proven unconditionally using the Bombieri-Vinogradov Theorem. In fact, Fouvry [6, Corollaire 2], and Bombieri, Friedlander and Iwaniec [1, Corollary 1] have shown for any $A > 1$,

$$\sum_{p \le x} d(p-a) = cx + c_1 \mathrm{li}(x) + O\left(\frac{x}{(\log x)^A}\right) \tag{1.2}$$

where c and c_1 are effectively computable constants dependent on a and the implied constant depends only on a and A, and $\mathrm{li}(x)$ is the usual logarithmic integral.

1.1. **Notation.** The letter p will denote a prime number. The letter k will denote a positive integer. The function $d(n)$ will denote the number of positive divisors of $n \in \mathbb{N}$. The Euler totient function, which will be denoted by $\varphi(n)$ with $n \in \mathbb{N}$, is the number of coprime residue classes in $\mathbb{Z}/n\mathbb{Z}$. Let $a, k \in \mathbb{N}$ with $\gcd(a,k) = 1$ and $x \in \mathbb{R}$ with $x \ge 1$. Then, by $\pi(x; k, a)$ denote by the number $\#\{p \le x : p \equiv a \bmod k\}$, and by $\psi(x; k, a)$ we denote by the summation

$$\psi(x; k, a) := \sum_{\substack{n \le x \\ n \equiv a \bmod k}} \Lambda(n) \tag{1.3}$$

where $\Lambda(n) = \log p$ if $n = p^\alpha$ for some prime p and $\alpha \in \mathbb{N}$ and $\Lambda(n) = 0$ otherwise. That is, Λ is the von Mangoldt function. For $p \nmid a$, define $f_a(p) = \min\{d \in \mathbb{N} : a^d \equiv 1 \bmod p\}$ and for $p | a$, define $f_a(p) = \infty$. For $p \nmid a$, define $i_a(p) = \frac{p-1}{f_a(p)}$ and for $p|a$, define $i_a(p) = 0$. We call $f_a(p)$ the order of a modulo p and $i_a(p)$ the index of a modulo p. The logarithmic integral, denoted by $\mathrm{li}(x)$, is the integral

$$\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}. \tag{1.4}$$

Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}_{\ge 0}$. We will write $f(x) = O(g(x))$ if there is a constant C such that $|f(x)| \le Cg(x)$ for all $x \in \mathbb{R}$. Equivalently, we will write $f(x) \ll g(x)$ for the same relation. We will write $f(x) \asymp g(x)$ if $f(x) \ll g(x) \ll f(x)$. Here f has codomain $\mathbb{R}_{\ge 0}$. We will write $f(x) \sim g(x)$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1. \tag{1.5}$$

We will write $f(x) = o(g(x))$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0. \tag{1.6}$$

1.2. **Statement of Results.** We wish to consider

$$\sum_{\substack{p \le x \\ p \equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right) \tag{1.7}$$

where $k \in \mathbb{N}$ is fixed. We will also consider the above summation with $d((p-1)/k)$ replaced by $d((p-a)/k)$ as p ranges over $p \le x$ with $p \equiv a \bmod k$. However, the case $a = 1$ has applications to Artin's conjecture for primitive roots.

In §2, we will prove the following theorems:

Theorem 1.2. *For any $k \in \mathbb{N}$, $k > 1$ be an integer, and let $a \in \mathbb{Z}$ such that* $\gcd(a,k) = 1$, *and $A > 0$. We have the following results uniformly in $k \leq (\log x)^{A+1}$:*

$$\sum_{\substack{d \leq x \\ \gcd(a,d)=1}} \frac{1}{\varphi(kd)} = \frac{c_k}{k} \log x + O\left(\frac{\log k}{k}\right) \tag{1.8}$$

where the O-constant is dependent only on a, and

$$\sum_{\substack{p \leq x \\ p \equiv a \bmod k}} d\left(\frac{p-a}{k}\right) = \frac{c_k}{k} x + O\left(\frac{x(\log k)(1+c_k)}{k \log x}\right) + O\left(\frac{x}{(\log x)^A}\right) \tag{1.9}$$

where

$$\begin{aligned} c_k := c_k(a) &= \sum_{\substack{w \geq 1 \\ \gcd(w,a)=1}} \frac{\mu^2(w) \gcd(w,k)}{w\varphi(w)} \\ &= \prod_{\substack{p \nmid a \\ p \nmid k}} \left(1 + \frac{1}{p(p-1)}\right) \prod_{p \mid k} \left(1 + \frac{1}{p-1}\right) \prod_{p \mid a} \left(1 - \frac{1}{p}\right) \\ &= \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p \mid a} \left(1 - \frac{p}{p^2-p+1}\right) \prod_{p \mid k} \left(1 + \frac{p-1}{p^2-p+1}\right) \end{aligned} \tag{1.10}$$

and the first O-constant is absolute and the second O-constant is dependent only on a and A.

In §3, we will prove

Lemma 1.1. *Let $a = 1$. Let $c_k := c_k(1)$ be defined as above. Then, we have*

$$\sum_{k \leq x} \frac{\mu(k) c_k}{k^2} = 1 + O\left(\frac{1}{x}\right). \tag{1.11}$$

and

Theorem 1.3. *We have*

$$\sum_{k \leq x} \frac{\varphi(k)}{k} \pi(x; k, 1) = x + O\left(\frac{x}{\log x}\right) \tag{1.12}$$

as $x \to \infty$.

These above theorems will then give an application to a problem related to Artin's conjecture for primitive roots:

Theorem 1.4. *Let y be a function of x such that $\frac{x}{\log x} = o(y)$. Then,*

$$\frac{1}{y}\sum_{a\le y}\sum_{p\le x}\frac{1}{f_a(p)} = \log x + O(\log\log x) + O\left(\frac{x}{y}\right). \tag{1.13}$$

and

Theorem 1.5. *Let y be a function of x such that $\frac{x}{\log x} = o(y)$. Then,*

$$\frac{1}{y}\sum_{a\le y}\sum_{p\le x}\frac{1}{\varphi(f_a(p))} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x + c_1\log\log x + O\left(\frac{x}{y}\right) \tag{1.14}$$

where

$$c_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(-2\sum_{p \text{ prime}}\frac{\log p}{p^2-p+1} + 2\gamma - 1\right) \tag{1.15}$$

where γ is Euler's constant:

$$\gamma := \lim_{x\to\infty}\left(\sum_{1\le n\le x}\frac{1}{n} - \log x\right). \tag{1.16}$$

2. Proof of Theorem 1.2

Let

$$\delta(n) := \begin{cases} 1 & \text{if } \sqrt{n}\in\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}. \tag{2.1}$$

So, we have

$$d(n) = 2\sum_{\substack{d\mid n \\ d<\sqrt{n}}} 1 + \delta(n). \tag{2.2}$$

Then,

$$\sum_{\substack{p\le x \\ p\equiv a \bmod k}} d\left(\frac{p-a}{k}\right) = 2\sum_{\substack{p\le x \\ p\equiv a \bmod k}}\sum_{\substack{d\le\sqrt{\frac{p-a}{k}} \\ d\mid\frac{p-a}{k}}} 1 + \sum_{\substack{p\le x \\ p\equiv a \bmod k}}\delta\left(\frac{p-a}{k}\right)$$

$$= 2\sum_{d\le 2\sqrt{\frac{x}{k}}}\pi(x;kd,a) + O\left(\sum_{n\le x}\delta(n)\right). \tag{2.3}$$

Now,

$$\sum_{n\le x}\delta(n) = \#\{n\le x : n = m^2 \text{ for some } m\in\mathbb{N}\}$$

$$= \#\{m^2\le x\} \le \sqrt{x}. \tag{2.4}$$

So, we have

$$\sum_{\substack{p\le x\\ p\equiv a \bmod k}} d\left(\frac{p-a}{k}\right) = 2\sum_{d\le 2\sqrt{\frac{x}{k}}} \pi(x;kd,a) + O\left(\sqrt{x}\right). \tag{2.5}$$

However, since $\pi(x;kd,a)\le 1$ if $\gcd(a,d)>1$, we have

$$\begin{aligned}\sum_{d\le 2\sqrt{\frac{x}{k}}} \pi(x;kd,a) &= \sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \left(\pi(x;kd,a) - \frac{\mathrm{li}(x)}{\varphi(kd)} + \frac{\mathrm{li}(x)}{\varphi(kd)}\right) + O\left(\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)\ne 1}} 1\right)\\ &= \mathrm{li}(x)\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} + O\left(\left|\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \left(\pi(x;kd,a) - \frac{\mathrm{li}(x)}{\varphi(kd)}\right)\right|\right)\\ &\quad + O(\sqrt{x}).\end{aligned} \tag{2.6}$$

Hence, we have

$$\begin{aligned}\sum_{d\le 2\sqrt{\frac{x}{k}}} \pi(x;kd,a) &= \mathrm{li}(x)\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} + O\left(\left|\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \left(\pi(x;kd,a) - \frac{\mathrm{li}(x)}{\varphi(kd)}\right)\right|\right)\\ &\quad + O(\sqrt{x})\\ &= \mathrm{li}(x)\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} + O\left(\frac{x}{(\log x)^A}\right),\end{aligned} \tag{2.7}$$

where we have used [1, Theorem 9] which states the following: choose $\varepsilon>0$. If we assume $k<x^{\frac{1}{10}-\varepsilon}$ (which is true since, in our case, $k<(\log x)^{A+1}$), then, there exists $C:=C(A)$ such that for any $Q\le x/k(\log x)^C$ we have

$$\sum_{\substack{m\le k\\ \gcd(a,m)=1}} \left|\sum_{\substack{n\le Q\\ \gcd(n,a)=1}} \left(\psi(x;nm,a) - \frac{x}{\varphi(qr)}\right)\right| \ll \frac{x}{(\log x)^A} \tag{2.8}$$

where the implied constant depends on at most ε, a, and A, and C is dependent only on A. We note that $\frac{x}{k(\log x)^B} > \sqrt{x/k}$ as $\sqrt{k} < (\log x)^{(A+1)/2} < \sqrt{x}/(\log x)^C$. Therefore, by partial summation and concerning ourselves only with $m=k$, which satisfies $\gcd(a,k)=1$ by hypothesis, in the above summation, we have

$$\sum_{d\le 2\sqrt{\frac{x}{k}}} \pi(x;kd,a) = \mathrm{li}(x) \sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} + O\left(\frac{x}{(\log x)^A}\right). \tag{2.9}$$

So, we just need to deal with

$$\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)}.$$

We will evaluate this summation by using [2, Exercises 6.3 and 6.4] as a guide. We have

$$\begin{aligned}
\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)} &= \sum_{\substack{d\le x\\ \gcd(d,a)=1}} \sum_{w|kd} \frac{\mu^2(w)}{\varphi(w)} = \sum_{w\le kx} \frac{\mu^2(w)}{\varphi(w)} \sum_{\substack{d\le x\\ w|kd\\ \gcd(d,a)=1}} 1\\
&= \sum_{w\le kx} \frac{\mu^2(w)}{\varphi(w)} \sum_{\substack{d\le x\\ \frac{w}{\gcd(w,k)}|d\\ \gcd(d,a)=1}} 1 = \sum_{\substack{w\le kx\\ \gcd\left(\frac{w}{\gcd(w,k)},a\right)=1}} \frac{\mu^2(w)}{\varphi(w)} \sum_{\substack{d\le x\\ \frac{w}{\gcd(w,k)}|d\\ \gcd(d,a)=1}} 1\\
&= \sum_{\substack{w\le kx\\ \gcd(w,a)=1}} \frac{\mu^2(w)}{\varphi(w)} \sum_{\substack{d\le \frac{x\gcd(w,k)}{w}\\ \gcd(d,a)=1}} 1 \qquad (2.10)
\end{aligned}$$

since $\gcd(a,k) = 1$ implies $\gcd\left(\frac{w}{\gcd(w,k)},a\right) = \gcd(w,a)$, and $\gcd\left(\frac{w}{\gcd(w,k)},a\right) = 1$ implies $\gcd\left(\frac{w}{\gcd(w,k)}d,a\right) = \gcd(d,a)$. Also, since (*mutatis mutandis* the proof of [11, Exercise 1.5.8])

$$\sum_{\substack{d\le y\\ \gcd(d,a)=1}} 1 = \frac{\varphi(a)}{a}y + O(d(a)), \tag{2.11}$$

we have

$$\begin{aligned}\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)} &= \sum_{\substack{w\le kx\\ \gcd(w,a)=1}} \frac{\mu^2(w)}{\varphi(w)}\left(\frac{\varphi(a)}{a}\frac{x\gcd(w,k)}{w} + O(d(a))\right)\\ &= \frac{\varphi(a)}{a}x\sum_{\substack{w\le kx\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} + O\left(d(a)\sum_{w\le kx}\frac{1}{\varphi(w)}\right)\\ &= x\frac{\varphi(a)}{a}\sum_{\substack{w\ge 1\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} + O\left(x\sum_{w>kx}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)}\right)\\ &\quad + O\left(d(a)\log(kx)\right). \end{aligned} \tag{2.12}$$

However,

$$\sum_{w>kx}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} \le k\sum_{w>kx}\frac{\mu^2(w)}{w\varphi(w)} \ll \frac{1}{x} + O\left(\frac{k\log(kx)}{(kx)^2}\right) \ll \frac{1}{x}. \tag{2.13}$$

So

$$\begin{aligned}\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)} &= x\frac{\varphi(a)}{a}\sum_{\substack{w\ge 1\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} + O\left(x\sum_{w>kx}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)}\right)\\ &\quad + O\left(\log(kx)\right)\\ &= x\frac{\varphi(a)}{a}\sum_{\substack{w\ge 1\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} + O(\log(kx)). \end{aligned} \tag{2.14}$$

We notice that

$$\sum_{\substack{w\ge 1\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} \le k\sum_{w\ge 1}\frac{\mu^2(w)}{w\varphi(w)}, \tag{2.15}$$

which converges absolutely for k fixed. Define

$$c_k := c_k(a) := \frac{\varphi(a)}{a}\sum_{\substack{w\ge 1\\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)}. \tag{2.16}$$

Thus, we have

$$\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)} = c_k x + O(\log(kx)). \tag{2.17}$$

Note that

$$\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} = \frac{1}{k}\sum_{d\le x}\frac{kd}{d\varphi(kd)}. \tag{2.18}$$

However, by partial summation, we have

$$\begin{aligned}\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{d\varphi(kd)} &= \frac{1}{x}\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)} + \int_1^x \frac{1}{u^2}\sum_{\substack{d\le u\\ \gcd(d,a)=1}} \frac{kd}{\varphi(kd)}\, du\\ &= c_k + O\left(\frac{\log(kx)}{x}\right) + \int_1^x \frac{c_k}{u}\, du + O\left(\int_1^x \frac{\log(ku)}{u^2}\, du\right)\\ &= c_k\log x + O(\log k) \qquad (2.19)\end{aligned}$$

since $f(w) := \gcd(w,k)$ is a multiplicative function implies that we have

$$\begin{aligned}c_k(a) &\le \sum_{w\ge 1}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} = \prod_p\left(1+\frac{1}{p(p-1)}\right)\prod_{p|k}\frac{p^2(p-1)}{(p-1)(p^2-p+1)}\\ &\ll \prod_{p|k}\left(1+\frac{p-1}{p(p-1)+1}\right) \le \prod_{p|k}\left(1+\frac{1}{p-1}\right)\\ &= \sum_{d|k}\frac{\mu^2(d)}{\varphi(d)} \ll \sum_{d\le k}\frac{1}{\varphi(d)} \ll \log k. \qquad (2.20)\end{aligned}$$

So, we have

$$\sum_{\substack{d\le x\\ \gcd(d,a)=1}} \frac{1}{\varphi(kd)} = \frac{1}{k}(c_k\log x + O(\log k)) = \frac{c_k}{k}\log x + O\left(\frac{\log k}{k}\right). \tag{2.21}$$

Thus,

$$\begin{aligned}\sum_{d\le 2\sqrt{\frac{x}{k}}}\pi(x;kd,a) &= \mathrm{li}(x)\sum_{\substack{d\le 2\sqrt{\frac{x}{k}}\\ \gcd(d,a)=1}}\frac{1}{\varphi(kd)} + O\left(\frac{x}{(\log x)^A}\right)\\ &= \frac{c_k}{2k}x - \frac{c_k\log k}{2k}\mathrm{li}(x) + \frac{c_k}{2k}\log 2 + O\left(\frac{x\log k}{k\log x}\right) + O\left(\frac{x}{(\log x)^A}\right)\\ &= \frac{c_k}{2k}x + O\left(\frac{x(\log k)(1+c_k)}{k\log x}\right) + O\left(\frac{x}{(\log x)^A}\right). \qquad (2.22)\end{aligned}$$

Thus,

$$\sum_{\substack{p\le x\\ p\equiv a \bmod k}} d\left(\frac{p-a}{k}\right) = \frac{c_k}{k}x + O\left(\frac{x(\log k)(1+c_k)}{k\log x}\right) + O\left(\frac{x}{(\log x)^A}\right). \tag{2.23}$$

So we just need to show that the Euler products are true. To see this note that $f(w) := \gcd(w,k)$ is a multiplicative function. Therefore,

$$\begin{aligned} c_k(a) &= \frac{\varphi(a)}{a} \sum_{\substack{w\geq 1 \\ \gcd(w,a)=1}} \frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} = \prod_{p|a}\left(1-\frac{1}{p}\right)\prod_{p\nmid a}\left(\sum_{n\geq 0}\frac{\mu(p^n)\gcd(p^n,k)}{p^n\varphi(p^n)}\right) \\ &= \prod_{p|a}\left(1-\frac{1}{p}\right)\prod_{p\nmid a}\left(1+\frac{\gcd(p,k)}{p(p-1)}\right) \\ &= \prod_{p|a}\left(1-\frac{1}{p}\right)\prod_{p|k}\left(1+\frac{1}{p-1}\right)\prod_{\substack{p\nmid a\\ p\nmid k}}\left(1+\frac{1}{p(p-1)}\right) \\ &= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p|a}\left(1-\frac{p}{p^2-p+1}\right)\prod_{p|k}\left(1+\frac{p-1}{p^2-p+1}\right). \end{aligned} \tag{2.24}$$

Therefore, Theorem 1.2 holds.

We note that we obtain an improvement in this theorem by using Fiorilli's work [4, Theorem 3.4] on extending Bombieri, Friedlander, and Iwaniec's work [1]. This will give us improved results in the next section.

3. Application to a Generalization of Artin's Conjecture

Recall for $p \nmid a$, the order of a modulo p is

$$f_a(p) = \min\{d \in \mathbb{N} : a^d \equiv 1 \bmod p\}, \tag{3.1}$$

and for convenience, $f_a(p) = \infty$ for $p|a$.

We want to consider

$$\sum_{p\leq x}\frac{1}{f_a(p)} = \sum_{p\leq x}\frac{i_a(p)}{p-1}. \tag{3.2}$$

We note that if the above summation is $\ll x^{1/4}$, then there are infinitely many primes p for which a is a primitive root (see Murty and Srinivasan [12]). We will show that, on average, the above summation is $\log x$.

Let us consider the following summation:

$$\frac{1}{y}\sum_{a\leq y}\sum_{p\leq x}\frac{1}{f_a(p)}. \tag{3.3}$$

We will also see that this summation is related to the Titchmarsh divisor problem (see [2, §9.3]):

$$\sum_{p\leq x} d(p-a) = cx + c_1\mathrm{li}(x) + O\left(\frac{x}{(\log x)^A}\right) \tag{3.4}$$

for any $A > 0$, where c and c_1 are constants, and the implied constant is dependent only on a and A. More precisely, this is related to the summation

$$\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right)$$

discussed in the previous section.

Since $f_{kp}(p)=\infty$ for any $k\in\mathbb{Z}$ and $y=\frac{y+1}{p}p-1\le\left\lceil\frac{y+1}{p}\right\rceil p-1$, where $\lceil x\rceil$ is the smallest integer greater than $x\in\mathbb{R}$, we have

$$\begin{aligned}\sum_{p\le x}\sum_{a\le y}\frac{1}{f_a(p)} &\le \sum_{p\le x}\left(\sum_{1\le a\le p-1}\frac{1}{f_a(p)}+\cdots+\sum_{(\lceil\frac{y+1}{p}\rceil-1)p+1\le a\le\lceil\frac{y+1}{p}\rceil p-1}\frac{1}{f_a(p)}\right)\\ &= \sum_{p\le x}\left\lceil\frac{y+1}{p}\right\rceil\sum_{a\le p-1}\frac{1}{f_a(p)} \qquad (3.5)\end{aligned}$$

since $f_a(p)=f_{a+kp}(p)$ by definition. Therefore, since $\left\lceil\frac{y+1}{p}\right\rceil=\frac{y+1}{p}+O(1)$, we have

$$\begin{aligned}\sum_{p\le x}\sum_{a\le y}\frac{1}{f_a(p)} &\le \sum_{p\le x}\left(\frac{y+1}{p}+O(1)\right)\sum_{a\le p-1}\frac{1}{f_a(p)}\\ &= (y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}+O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{f_a(p)}\right). \qquad (3.6)\end{aligned}$$

Similarly, using $[x]$ the greatest integer smaller than x and truncating the above summations to $a\le[(y+1)/p]$ instead of extending them to $a\le\lceil(y+1)/p\rceil$, we have

$$\sum_{p\le x}\sum_{a\le y}\frac{1}{f_a(p)}\ge(y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}+O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{f_a(p)}\right). \qquad (3.7)$$

In particular,

$$\begin{aligned}\sum_{p\le x}\sum_{a\le y}\frac{1}{f_a(p)} &= (y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}+O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{f_a(p)}\right)\\ &= y\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}+O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{f_a(p)}\right) \qquad (3.8)\end{aligned}$$

Let us consider the error term first. We note that the number of elements of $(\mathbb{Z}/p\mathbb{Z})^*$ that have order k is $\varphi(k)$ provided $k|p-1$ and 0 otherwise. So, we have

$$\sum_{a\le p-1}\frac{1}{f_a(p)}=\sum_{k|p-1}\frac{\#\{a\in(\mathbb{Z}/p\mathbb{Z})^*: f_a(p)=k\}}{k}=\sum_{k|p-1}\frac{\varphi(k)}{k}. \qquad (3.9)$$

Therefore,

$$\sum_{p\le x}\sum_{a\le p-1}\frac{1}{f_a(p)}=\sum_{p\le x}\sum_{k|p-1}\frac{\varphi(k)}{k}\le\sum_{p\le x}d(p-1)\ll x \tag{3.10}$$

by the Titchmarsh divisor problem [2, §9.3].

Let us now consider the main term. We first note that we have

$$\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}=\sum_{k\le x}\frac{\varphi(k)}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}}\frac{1}{p} \tag{3.11}$$

since

$$\sum_{a\le p-1}\frac{1}{f_a(p)}=\sum_{k|p-1}\frac{\varphi(k)}{k}. \tag{3.12}$$

By partial summation, we have

$$\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}}\frac{1}{p}=\frac{\pi(x;k,1)}{x}+\int_k^x\frac{\pi(u;k,1)}{u^2}\ du. \tag{3.13}$$

Therefore,

$$\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{f_a(p)}=\sum_{k\le x}\frac{\varphi(k)}{k}\left(\frac{\pi(x;k,1)}{x}+\int_k^x\frac{\pi(u;k,1)}{u^2}\ du\right). \tag{3.14}$$

We will evaluate each of these sums separately.

The first summation is dealt with as follows:

$$\sum_{k\le x}\frac{\varphi(k)}{k}\frac{\pi(x;k,1)}{x}\le\frac{1}{x}\sum_{k\le x}\pi(x;k,1)=\frac{1}{x}\sum_{p\le x}d(p-1)\ll 1 \tag{3.15}$$

by the Titchmarsh divisor problem [2, §9.3].

In order to evaluate the second summation notice that

$$\sum_{k\le x}\frac{\varphi(k)}{k}\int_k^x\frac{\pi(u;k,1)}{u^2}\ du=\int_2^x\frac{1}{u^2}\sum_{k\le u}\frac{\varphi(k)}{k}\pi(u;k,1)\ du. \tag{3.16}$$

So, in order to evaluate our desired sum, we need to evaluate

$$\sum_{k\le x}\frac{\varphi(k)}{k}\pi(x;k,1). \tag{3.17}$$

We have

$$\begin{aligned}\sum_{k\le x}\frac{\varphi(k)}{k}\pi(x;k,1) &= \sum_{p\le x}\sum_{k|p-1}\sum_{m|k}\frac{\mu(m)}{m} = \sum_{m\le x}\frac{\mu(m)}{m}\sum_{\substack{p\le x\\ p\equiv 1 \bmod m}}\sum_{\substack{k|p-1:\\ m|k}} 1\\
&= \sum_{k\le x}\frac{\mu(k)}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right)\\
&= \sum_{k\le(\log x)^{A+2}}\frac{\mu(k)}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right)\\
&\quad + \sum_{(\log x)^{A+2}<k\le x}\frac{\mu(k)}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right).\end{aligned}\tag{3.18}$$

Let us consider the second summation above. We have

$$\begin{aligned}\sum_{(\log x)^{A+2}<k\le x}\frac{\mu(k)}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right) &\ll \sum_{(\log x)^{A+2}<k\le x}\frac{1}{k}\sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right)\\
&\le \sum_{(\log x)^{A+2}<k\le x}\frac{1}{k}\sum_{n\le x/(\log x)^{A+2}} d(n)\\
&\ll \frac{x}{(\log x)^{A+1}}\sum_{(\log x)^{A+2}<k\le x}\frac{1}{k}\\
&\ll \frac{x}{(\log x)^{A}}.\end{aligned}\tag{3.19}$$

For the first summation, we have

$$\begin{aligned}
\sum_{k\le(\log x)^{A+2}} \frac{\mu(k)}{k} \sum_{\substack{p\le x\\ p\equiv 1 \bmod k}} d\left(\frac{p-1}{k}\right)
&= \sum_{k\le(\log x)^{A+2}} \frac{\mu(k)}{k}\left(\frac{c_k}{k}x + O\left(\frac{x(\log k)(1+c_k)}{k\log x}\right) + O\left(\frac{x}{(\log x)^{A+1}}\right)\right)\\
&= x\sum_{k\le(\log x)^{A+2}} \frac{\mu(k)c_k}{k^2} + O\left(\frac{x}{\log x}\sum_{k\le(\log x)^{A+2}} \frac{(1+c_k)\log k}{k^2}\right)\\
&\qquad + O\left(\frac{x}{(\log x)^{A+1}}\sum_{k\le(\log x)^{A+2}} \frac{1}{k}\right)\\
&= x\sum_{k\le(\log x)^{A+2}} \frac{\mu(k)c_k}{k^2} + O\left(\frac{x}{\log x}\sum_{k\le(\log x)^{A+2}} \frac{(1+c_k)\log k}{k^2}\right)\\
&\qquad + O\left(\frac{x}{(\log x)^{A}}\right) \qquad (3.20)
\end{aligned}$$

by Theorem 1.2. Also,

$$\begin{aligned}
\sum_{k\le x}\frac{\mu(k)c_k}{k^2} &= \sum_{k\ge 1}\frac{\mu(k)c_k}{k^2} - \sum_{k>x}\frac{\mu(k)c_k}{k^2}\\
&= \sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}\sum_{w=1}^{\infty}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)} - \sum_{k>x}\frac{\mu(k)}{k^2}\sum_{w=1}^{\infty}\frac{\mu^2(w)\gcd(w,k)}{w\varphi(w)}\\
&= \prod_p\left(1+\frac{1}{p(p-1)}\right)\left(\sum_{k=1}^{\infty}\mu(k)\prod_{p\mid k}\frac{1}{p^2-p+1}\right)\\
&\qquad + O\left(\sum_{k>x}|\mu(k)|\prod_{p\mid k}\frac{1}{p(p-1)}\right)\\
&= \prod_p\left(1+\frac{1}{p(p-1)}\right)\prod_p\left(1-\frac{1}{p^2-p+1}\right) + O\left(\sum_{k>x}\frac{|\mu(k)|}{k\varphi(k)}\right)\\
&= \prod_p\left(\frac{(p(p-1)+1)(p^2-p+1-1)}{p(p-1)(p^2-p+1)}\right) + O\left(\frac{1}{x}\right)\\
&= 1 + O\left(\frac{1}{x}\right). \qquad (3.21)
\end{aligned}$$

This proves Lemma 1.1.

As $c_k \ll \log k$, we have

$$\sum_{k\le(\log x)^{A+2}} \frac{(1+c_k)\log k}{k^2} \ll 1. \tag{3.22}$$

So,

$$\sum_{k\le x} \frac{\varphi(k)}{k}\pi(x;k,1) = x\left(1+O\left(\frac{1}{(\log x)^{A+2}}\right)\right)+O\left(\frac{x}{\log x}\right) \tag{3.23}$$

$$= x + O\left(\frac{x}{\log x}\right) \tag{3.24}$$

as $A>0$ can be chosen to be arbitrarily large. Therefore, Theorem 1.3 holds.

It can now be shown that

$$\frac{1}{y}\sum_{a\le y}\sum_{p\le x}\frac{1}{f_a(p)} = \log x + O(\log\log x) + O\left(\frac{x}{y}\right) \tag{3.25}$$

assuming $\frac{x}{\log x} = o(y)$. To see this, we need to evaluate

$$\int_2^x \frac{1}{u^2}\sum_{k\le u}\frac{\varphi(k)}{k}\pi(u;k,1)\ du. \tag{3.26}$$

We have

$$\int_2^x \frac{1}{u^2}\sum_{k\le u}\frac{\varphi(k)}{k}\pi(u;k,1)\ du = \int_2^x \frac{1}{u^2}\left(u+O\left(\frac{u}{\log u}\right)\right)\ du$$

$$= \log x + O(\log\log x) \tag{3.27}$$

by Theorem 1.3. This proves Theorem 1.4 since $\frac{x}{\log x} = o(y)$ forces $\frac{x}{y} = o(\log x)$ and so our summation becomes

$$\frac{1}{y}\sum_{a\le y}\sum_{p\le x}\frac{1}{f_a(p)} = \log x + O(\log\log x) + O\left(\frac{x}{y}\right). \tag{3.28}$$

To see that Theorem 1.5 holds, note that all of the previous error terms and justifications work for this case as well. To see this, consider the following: since $f_{kp}(p)=\infty$ for any $k\in\mathbb{Z}$ and $y = \frac{y+1}{p}p - 1 \le \left\lceil\frac{y+1}{p}\right\rceil p - 1$, we have

$$\sum_{p\le x}\sum_{a\le y}\frac{1}{\varphi(f_a(p))} \le \sum_{p\le x}\left(\sum_{1\le a\le p-1}\frac{1}{\varphi(f_a(p))}+\cdots+\sum_{(\lceil\frac{y+1}{p}\rceil-1)p+1\le a\le\lceil\frac{y+1}{p}\rceil p-1}\frac{1}{\varphi(f_a(p))}\right)$$

$$= \sum_{p\le x}\left\lceil\frac{y+1}{p}\right\rceil\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} \tag{3.29}$$

since $f_a(p) = f_{a+kp}(p)$ by definition. Therefore, since $\left\lceil \frac{y+1}{p} \right\rceil = \frac{y+1}{p} + O(1)$, we have

$$\sum_{p\le x}\sum_{a\le y}\frac{1}{\varphi(f_a(p))} \le \sum_{p\le x}\left(\frac{y+1}{p}+O(1)\right)\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}$$
$$= (y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}$$
$$+ O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}\right). \tag{3.30}$$

Similarly, we have

$$\sum_{p\le x}\sum_{a\le y}\frac{1}{\varphi(f_a(p))} \ge (y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}$$
$$+ O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}\right). \tag{3.31}$$

In particular,

$$\sum_{p\le x}\sum_{a\le y}\frac{1}{\varphi(f_a(p))} = (y+1)\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} + O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}\right)$$
$$= y\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} + O\left(\sum_{p\le x}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))}\right) \tag{3.32}$$

Since

$$\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} = \sum_{k|p-1}\frac{\#\{1\le a\le p-1 : f_a(p)=k\}}{\varphi(k)} = \sum_{k|p-1} 1 = d(p-1), \tag{3.33}$$

we have

$$\sum_{p\le x}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} = \sum_{p\le x} d(p-1) \ll x \tag{3.34}$$

by the Titchmarsh divisor problem.

Similarly,

$$\sum_{p\le x}\frac{1}{p}\sum_{a\le p-1}\frac{1}{\varphi(f_a(p))} = \sum_{k\le x}\left(\frac{\pi(x;k,1)}{x} + \int_k^x \frac{\pi(u;k,1)}{u^2}\,du\right). \tag{3.35}$$

The first summation is bounded by $\ll 1$ using the Titchmarsh Divisor problem as before. The integral becomes

$$\sum_{k\le x}\int_k^x \frac{\pi(u;k,1)}{u^2}\,du = \int_2^x \frac{1}{u^2}\sum_{k\le u}\pi(u;k,1)\,du. \tag{3.36}$$

However, this inner summation becomes

$$\sum_{k\le u}\pi(u;k,1)=\sum_{p\le u}d(p-1)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}u+c_1\frac{u}{\log u}+O\left(\frac{u}{(\log u)^2}\right). \tag{3.37}$$

Hence,

$$\sum_{k\le x}\int_k^x\frac{\pi(u;k,1)}{u^2}\,du=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x+c_1\log\log x+O(1). \tag{3.38}$$

Therefore, Theorem 1.5 holds.

Remarks

We note that in both of Theorems 1.4 and 1.5, there is a $\log\log x$ term. We also have

$$\sum_{p\le x}\frac{1}{f_a(p)}=\sum_{p\le x}\frac{i_a(p)}{p-1}. \tag{3.39}$$

We could apply partial summation the right-hand side but an estimate on the summation

$$\sum_{p\le x}i_a(p)$$

would be needed. Currently, if we assume GRH for the Dedekind zeta functions of $\mathbb{Q}(\zeta_n,a^{1/n})$ as n ranges over $\mathbb{N}$, we get lower bounds of the form

$$\sum_{p\le x}i_a(p)\gg x \tag{3.40}$$

(see [3, Chapter 7, §1]). Unconditionally, we have

$$\sum_{p\le x}i_a(p)\gg\frac{x\log\log x}{\log x}. \tag{3.41}$$

To see this let $\pi_d(x):=\#\{p\le x : d|i_a(p)\}$. Then, by [8, Page 213], $d|i_a(p)$ if and only if p splits completely in $\mathbb{Q}(\zeta_d,a^{1/d})$ where ζ_d is a primitive d^{th} root of unity. So, by the unconditional effective Chebotarev density theorem ([9, Theorem 1.4] or [13, Page 376]), we have for any

$A > 1$

$$\begin{aligned}\sum_{p\le x} i_a(p) &= \sum_{d\le x}\varphi(d)\pi_d(x) \ge \sum_{d\le(\log x)^{1/7}}\varphi(d)\pi_d(x)\\ &= \sum_{d\le(\log x)^{1/7}}\varphi(d)\left(\frac{\mathrm{li}(x)}{[K_d:\mathbb{Q}]}+O\left(\frac{x}{(\log x)^A}\right)\right)\\ &\gg \mathrm{li}(x)\sum_{d\le(\log x)^{1/7}}\frac{1}{d}+O\left(\frac{x}{(\log x)^A}\right) \qquad \text{(by [16, Proposition 4.1])}\\ &\gg \frac{x\log\log x}{\log x}. \qquad (3.42)\end{aligned}$$

The belief [5, Conjecture 1(a)] is that we have

$$\sum_{p\le x} i_a(p) \sim c_a x \qquad (3.43)$$

where c_a is a positive constant dependent on a. Theorem 1.4 indicates that we may have

$$\sum_{p\le x} i_a(p) = c_a x + O\left(\frac{x}{\log x}\right). \qquad (3.44)$$

We note that in Theorem 1.4 and 1.5, there is a $\log\log x$ term. In fact, in Theorem 1.5, this term contributes to the summation. This suggests that we may have

$$\sum_{p\le x} i_a(p) = c_a x + \Omega\left(\frac{x}{\log x}\right). \qquad (3.45)$$

As mentioned in the previous section, Fiorilli [4, Theorem 3.4] allow us to improve Theorem 1.4 by replacing $O(\log\log y)$ by $c\log\log y + O(1)$ where c is a constant. We relegate further analysis and computation of this summation to future research.

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