1. Chow Rings

Let R be a ring with fraction field k. Let $S = \operatorname{Spec}(R)$ be the base scheme. Let X be a scheme over S. Let $U_i = \operatorname{Spec}(A_i)$ be an open affine cover of X. For a point $x \in U_i$ let $\mathfrak{p}_x$ be the associated ideal. Define $k(x)$ to be the fraction field of $A_i/\mathfrak{p}_x$.

For this section, assume X is regular noetherian separated finite dimensional.

Let $Z^p(X)$ be the free abelian group on points $x \in X$ of co-dimension p. Note that $Z^p(X)$ is the same as free abelian group on closed sub-schemes of co-dimension p, via the association between generic points and their closures.

For $f \in k(x)$ define $\operatorname{div}(f) = \sum_{y \in \overline{x}} \operatorname{ord}_y(f) y$. This makes sense either in $Z^1(\overline{x})$ or $Z^{p+1}(X)$.

Let $B^p(X)$ be the free abelian group generated by $\operatorname{div}(f)$ for $f \in k(x)$, x of co-dimension $p-1$.

Finally, define $CH^p(X) = Z^p(X)/B^p(X)$.

Notice that $CH^0(X)$ is generated by the irreducible components of X. Notice that $CH^1(X)$ is the usual divisor class group or Picard group. Notice that $CH^{\dim(X)}(X)$ come from points.

Notice that we really want to only every be considering pieces of constant dimension at any given time.

Finally, notice we have made little fuss about fields or rings of definition for things. It is implicit that all of our objects are defined over the base ring, the whole theory should be functorial with respect to base extension.

1.1. Chow Rings with Support.

Let $Y \subset X$ be a closed subscheme, we define $Z^p_Y(X)$ to be the subgroup of $Z^p(X)$ generated by those elements contained in Y. We let $B^p_Y(X)$ be the subgroup of $Z^p_Y(X)$ generated by principal divisors on $Z^{p-1}_Y(X)$. We then naturally have:

$$CH^p_Y(X) = Z^p_Y(X)/B^p_Y(X)$$

This is functorial in Y, but is not by definition injective as Y is enlarged.

2. Intersection Theory

Given $x, y \in X$ it makes sense to ask about $\overline{x} \cap \overline{y} \subset X$. One naturally may think of the intersection scheme theoretically as being $\overline{x} \times_X \overline{y}$. These two notions agree scheme theoretically pointwise.

However, when considering this intersection, one loses track of where it came from. A slightly more refined notion of intersection can try to keep track of information about the varieties being intersected in a neighborhood of the intersection. For example, we might be interested in knowing about the order of vanishing of functions on the defining varieties at the point of intersection. The intersection multiplicity doesn't give us quite this, but it is a step in this direction.

2.1. Desired Properties.

One expects that if $x, y \in X$ have codimensions p, q respectively then $x \cap y$ will have codimension $p+q$. Thus we would like to define an intersection pairing:

$$Z^p(X) \times Z^q(X) \to Z^{p+q}(X)$$

With the property that $(x, y) \mapsto \sum_{z \in x \cap y} i_z(x, y) z$, and the numbers $i_z(x, y)$ have something to do with orders of vanishing of x, y at z.

We would like for this pairing to descend to a map:

$$CH^p(X) \times CH^q(X) \to CH^{p+q}(X)$$

We would like such a pairing to be natural and functorial.

There are of course several non-obvious steps. Firstly, how to handle x, y which have intersections in co-dimension less than $p+q$. Equally importantly, what $i_z(x, y)$ should actually be.

2.2. **Definitions and Existence Theorems.**

Definition 2.1. Let M be a finitely generated R module, then there exists a filtration:

$$M = M_0 \supset M_1 \supset \cdots \supset M_l = 0$$

such that $M_{i-1}/M_i \simeq R/\mathcal{P}_i$ where $\mathcal{P}_i$ are prime ideal of R.

M has finite length if $\mathcal{P}_i$ are all maximal, in which case $l = l_R(M)$ is independent of the choices. (This occurs if and only if the M is supported on maximal ideals).

Definition 2.2. For a/b in the fraction field of R define $\mathrm{ord}_R(a/b) = l_R(R/a) - l_R(R/b)$.

Notice, that this generalizes the valuation of an element for a regular local ring.

Definition 2.3. If x, y intersect properly, we define:

$$i_z(x, y) = \sum (-1)^i l_{\mathcal{O}_{X,z}}(\mathrm{Tor}_i^{\mathcal{O}_{X,z}}(\mathcal{O}_{\overline{x},z}, \mathcal{O}_{\overline{y},z}))$$

(Recall that Tor is the homology of the total complex for the double complex coming from the tensor product of projective resolutions of the modules involved).

Remark. This makes sense because the length generalizes the valuation of defining elements, and for 'nice' rings the Tor functor will vanish for $i \geq 1$.

It is actually a hard theorem that these values are typically positive.

Example. Compute $\mathrm{Tor}_.(\mathbb{Z}/n, \mathbb{Z}/m)$.

$$0 \to \mathbb{Z} \stackrel{[n]}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Is a projective resolution.

$$0 \to \mathbb{Z}/m \stackrel{[n]}{\to} \mathbb{Z}/m \to 0$$

Thus:

$$\begin{aligned} \mathrm{Tor}_0 &= \mathbb{Z}/(m,n) \\ \mathrm{Tor}_1 &= \tfrac{m}{(n,m)}\mathbb{Z}/m \cong \mathbb{Z}/(m/(m,n)) \ \mathrm{Tor}_i &= 0 \end{aligned}$$

Compute $\mathrm{Tor}_.^{k[x,y]_(0,0)}(k[x,y]/(y-x^2), k[x,y]/y)$.

$$0 \to k[x,y]_(0,0) \stackrel{[y-x^2]}{\to} k[x,y]_(0,0) \to (k[x,y]/(y-x^2))_(0,0) \to 0$$

Thus look at:

$$0 \to k[x]_0 \stackrel{[x^2]}{\to}\to k[x]_(0) \to 0$$

Thus:

$$\begin{aligned} \mathrm{Tor}_0 &= k[x]_0/x^2 \\ \mathrm{Tor}_1 &= 0 \end{aligned}$$

Theorem 2.4. *Let X be a regular scheme, Y, Z closed subschemes, then there exists a pairing:*

$$CH^p_Y(X) \times CH^q_Z(X) \to CH^{p+q}_{Y\cap Z}(X)$$

That makes the Chow ring into a graded ring with unity, which is functorial with respect to changes of support and which is natural with respect to cycles with proper intersection.

2.3. **K-Theory.** Let $K_0(X)$ (resp $K_0'(X)$) be the group of coherent, locally free (resp coherent) $\mathcal{O}_X$-modules. For $Y \subset X$ closed let $K_0^Y(X)$ denote the group of finite subgroup of complexes:

$$\mathcal{F} : 0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

which are a-cyclic outside of Y. Consider these up to equivalence of additivity on short exact sequences.

Lemma 2.5. *For X regular $K_0(X) = K_0'(X)$.*

We can define a filtration on the $K_0(X)$ in terms of the size of the support.

We can then define the Grothendick group as the 'homology' of the filtration.

These $Gr^p K_0^Y(X)$ are naturally isomorphic to the chow groups, and these naturally acquire an intersection pairing.

3. (Logarithmic) Green's Currents

We now switch to the context of complex geometry, we shall be considering $X = X(\mathbb{C})$ a smooth projective complex equidimensional variety.

(Note that projective implies Kahler, a condition we shall need)

3.1. **Definitions.** Let $X(\mathbb{C})$ be a complex projective variety of dimension d. Let $A^{p,q}(X(\mathbb{C}))$ be the differential forms of type p, q.

Let $\partial, \overline{\partial}$ be differentiation with respect to z and $\overline{z}$ respectively (even in multi-dimensional sense). $d = \partial + \overline{\partial}$.

We let $D_{p,q}(X) = A^{p,q}(X)^*$ be the space of continuous linear functionals. Then set $D^{p,q}(X) = D_{d-p,d-q}(X)$. We then have a mapping:

$$A^{p,q}(X) \to D^{p,q}(X)$$

Let $\partial', \overline{\partial}', d'$ be the naturally induced maps on $D^{p,q}$. That is $\partial', \overline{\partial}', d'$ act as $(\partial' g)(\omega) = g(\partial(\omega))$

And by abuse let $\partial = (-1)^{p+q}\partial'$ (et. al.). So that we get commutativity with respect to the inclusion above. (the reason for this is that under the natural map $A^{p,q} \to D^{p,q}$ one finds that $[d\omega] = (-1)^{p+q} d'[\omega]$.)

For $Y \subset X$ of codimension p we have $\delta_Y \in D^{p,p}(X)$:

$$\delta_Y(\alpha) = \int_Y i^*(\alpha)$$

given by integration on Y.

Let $d^c = (4\pi i)^{-1}(\partial - \overline{\partial})$ so $dd^c = -(2\pi i)^{-1}\partial\overline{\partial}$.

Definition 3.1. A **Green current** for Y is a current $g \in D^{p-1,p-1}$ such that $dd^c g + \delta_Y = [\omega]$ for some $\omega \in A^{p,p}$.

Example. On $\mathbb{P}_1$ we can consider $\mathcal{O}_{\mathbb{P}_1}(1)$ the twisted sheaf of Serre, and we may consider the standard section x.

The claim is that $[-\log |x|^2]$ is a green current for $[0:1] - [1:0]$.

Firstly, being a meromorphic function it gives an element of $D^{0,0}$. The coordinate on $\mathbb{P}^1$ near $[0:1]$ is x.

$$\begin{aligned}\int_{\mathbb{P}^1} \log|x|^2\, dd^c f(x) &= \int_{\mathbb{P}^1} dd^c(\log|x|^2) f(x)\\ &= \int_\gamma (d^c \log(x\overline{x})) f(x)\\ &= \tfrac{i}{4\pi}\int_\gamma f(x)(\tfrac{1}{x}dx + \tfrac{1}{\overline{x}}d\overline{x})\\ &= f([0:1]) - f([1:0])\\ &= \delta_{\mathrm{div}(x)}(f)\end{aligned}$$

Definition 3.2. A smooth form α on $X-Y$ is said to be logarithmic on Y if in a nhd of Y, there exist local functions z_i defining Y and:

$$\alpha = \sum \alpha_i \log|z_i|^2 + \gamma$$

Where α_i, γ are all smooth forms.

3.2. **Properties.**

Theorem 3.3. *Let L be a holomorphic line bundle with hermitian metric $|\cdot|$ and s a meromorphic section. Then $[-\log|s|^2] \in D^{0,0}$ is a green current for $\operatorname{div} s$ and:*

$$dd^c[-log|s|^2] + \delta_{\operatorname{div} s} = [c_1(L, |\cdot|)]$$

Theorem 3.4. *If X is Kahler, then every $Y \subset X$ has a greens current and for any two g_1, g_2 green currents for Y we have:*

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

for $\eta \in A^{p-1,p-1}, S_1 \in D^{p-2,p-1}, S_2 \in D^{p-1,p-2}$.

Definition 3.5. Let $Y, Z \subset X$ intersect properly and g_Y be a green current for Y of logarithmic type, let $q : \tilde{Z} \to X$ be a resolution of the singularities of Z.

Define $[g_Y] \wedge \delta_Z = q_*[q^* g_Y]$. This then extends to a product $[g_Y] * g_Z = [g_Y] \wedge \delta_Z + [w_Y] \wedge \delta g_Z$. Where $[w_Y] = dd^c g_Y - \delta_Y$.

3.3. **Existence.**

Theorem 3.6. *If $X(\mathbb{C})$ is Kahler, then every $Y \subset X$ has a Green current and for any two such:*

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

Theorem 3.7. *If $X(\mathbb{C})$ is Kahler, then every $Y \subset X$ has a Green current of the form $[g_Y]$ where g_Y is a logarithmic form. Moreover, every Green current is of logarithmic type up to $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$.*

Theorem 3.8. *The $*$-product works like an intersection pairing. In particular it takes a green current for Y and Z to one for $Y \cap Z$, it is symmetric, associative, commutative (provided we use forms of log type), the image mod $\partial, \overline{\partial}$ only depends on source mod $\partial, \overline{\partial}$.)*

Remark. The key property of forms of logarithmic type is that we via functionality and knowing how to handle integrals of logs show that the $*$-product gives a greens current.

In particular we know the pullback $q^* g_Y$ will be of logarithmic type, thus we can use our understanding of log derivatives there and push the result forward.

The details are quite technical.

Example. $X = \mathbb{P}^d$, Y given by $x_0, \dots, x_{p-1} = 0$.
$\tau = \log(\sum_{i=0}^d |x_i|^2)$, $\alpha = dd^c\tau$ on X.
$\sigma = \log(\sum_{i=0}^p |x_i|^2)$, $\beta = dd^c\sigma$ on $X - Y$.
$\Lambda = (\tau - \sigma)(\sum_{i=0}^{p-1} \alpha^i \wedge \beta^{p-1-i})$ on $X - Y$.
There is a theorem that says:

$$dd^c[\Lambda] + \delta_Y = [\alpha^p]$$

moreover, Λ has logarithmic growth.

4. Arithmetic Chow Rings

The basic idea, is to to combine the information about an arithmetic scheme arising from the finite places with information arising from considering the scheme at the infinite places.

I mentally think of this as doing things 'adelically'.

Let the base scheme be $S = \operatorname{Spec}(\mathcal{O}_K)$. We shall require that X be flat, quasi-projective, over S with regular generic fibre.

(Note that $X(\mathbb{C}) = \sqcup_\sigma X_\sigma(\mathbb{C})$, so by talking about $X(\mathbb{C})$ we are treating all the infinite places at the same time).

Definition 4.1. We define the following objects: Let F be complex conjugation on $X(\mathbb{C})$.

- $A^{p,p}(X) = \{\omega \in A^{p,p}(X(\mathbb{C})) | \omega \text{ real}, F\omega = (-1)^p\omega\}$
- $Z^{p,p}(X) = \ker(d : A^{p,p}(X) \to A^{2p+1}(X(\mathbb{C})))$
- $H^{p,p}(X)\{\omega \in H^{p,p}(X(\mathbb{C})) | \omega \text{ real}, F\omega = (-1)^p\omega\}$
- $\tilde{A}^{p,p}(X) = A^{p,p}(X)/(\operatorname{im}\partial + \operatorname{im}\overline{\partial})$
- $D^{p,p}(X) = \{T \in D^{p,p}(X(\mathbb{C})) | T \text{ real}, FT = (-1)^pT\}$

Note that $H^{p,p}(X) = \ker dd^c/(\operatorname{im}\partial + \operatorname{im}\overline{\partial}) \subset \tilde{A}^{p,p}(X)$.

We define arithmetic cycles $\hat{Z}^p(X) = \{(Z, g_Z)\}$ where $Z \in Z^p(X)$ and g_Z is a green current for Z.

The boundaries are: $\hat{R}^p(X)$ generated by $(\operatorname{div} f, -[\log|f|^2]$ and $(0, \operatorname{im}\partial + \operatorname{im}\overline{\partial})$.

Arithmetic Chow groups are then:

$$\hat{CH}^p(X) = \hat{Z}^p(X)/\hat{R}^p(X)$$

We have the following exact sequences:

$$CH^{p-1,p}(X) \xrightarrow{\rho} H^{p-1,p-1} \xrightarrow{a} \hat{CH}^p(X) \xrightarrow{\zeta,\omega} CH^p(X) \oplus Z^{p,p}(X) \xrightarrow{cl} H^{p,p}(X) \to 0$$

$$CH^{p-1,p}(X) \xrightarrow{\rho} \tilde{A}^{p-1,p-1} \xrightarrow{a} \hat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

Theorem 4.2 (Functoriality)**.** *If X, Y are regular, projective, flat over $\mathbb{Z}$ and $f : Y \to X$ is a morphism, then there exists a pullback $f^* : \hat{CH}^p(X) \to \hat{CH}^p(Y)$.*

If X, Y are equidimensional, f is proper, $f_{\mathbb{Q}} : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$. then there is a pushforward map $f_* : \hat{CH}^p(Y) \to \hat{CH}^{p-\delta}$. ($\delta = \dim(Y) - \dim(X)$).

5. Arakelov Intersection Theory

5.1. **Desired Properties.** We would like to have a bilinear pairing

$$\hat{CH}^p(X) \otimes \hat{CH}^q(X) \to \hat{CH}^{p+q}(X)$$

which is compatible with the that on the regular Chow groups.

5.2. **Existence Theorem.**

Theorem 5.1. *The intersection pairing exists.*

$$\hat{CH}^p(X) \otimes \hat{CH}^q(X) \to \hat{CH}^{p+q}(X)_{\mathbb{Q}}$$

(except there is a $\mathbb{Q}$).

If X is smooth over a Dedekind domain, or $q = 1$ then we don't need the $\mathbb{Q}$.

6. Faltings Heights

Definition 6.1. Let X be a scheme over $\mathcal{O}_K$. We define a bilinear pairing $\hat{CH}^{\cdot}(X) \times Z_{\cdot}(X) \to \hat{CH}^{\cdot}(\mathrm{Spec}(\mathcal{O}_K))$.

Let $y = (Y, g_Y) \in \hat{CH}^p(X)$ and $U \in Z_q(X)$, let $\tilde{U}$ be a resolution of the singularities of U, and $\tilde{pi} : \tilde{U} \to \mathrm{Spec}(\mathcal{O}_K)$ be the structure morphism.

$$(y|U) = \tilde{\pi}_*(y \cap [\tilde{U}])$$

One can also define it by cases ($p = q, p = q + 1$ (trivial in other cases)). For $p = q$ get $\pi_*(\pi_*(Y.U)\pi_*(g\delta_Y))$. For $p = q + 1$ get $\pi_*(Y.U) \in CH^0(\mathrm{Spec}(\mathcal{O}_K)) = CH^0(\hat{\mathrm{Spec}}(\mathcal{O}_K)) = \mathbb{Z}$.

Definition 6.2. We wish to define the degree maps on $\hat{CH}^{\cdot}(\mathrm{Spec}(\mathcal{O}_K))$. The **algebraic degree map** is $\deg_K \hat{CH}^{\cdot}(\mathrm{Spec}(\mathcal{O}_K)) \to CH^0(\mathrm{Spec}(\mathcal{O}_K)) = \mathbb{Z}$. The arithmetic degree map is induced by $\pi : \mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z})$ and is given by:

$$\hat{\deg} : \hat{CH}^{\cdot}(\mathrm{Spec}(\mathcal{O}_K)) \to \hat{CH}^1(\mathrm{Spec}(\mathcal{O}_K)) \to \hat{CH}^1(\mathrm{Spec}(\mathbb{Z})) = \mathbb{R}$$

Theorem 6.3. *Let $z = (Z, g) \in \hat{Z}^q(X)$ and $Y \in Z_q$ be such that $Z_K \cap Y_K = \varnothing$. Write $[Z] \cdot [Y] = \sum_\alpha m_\alpha W_\alpha$ with $M_\alpha \in Z \cap Y$.*

$$\hat{\deg}(z|Y) = \sum_\alpha m_\alpha \log(|k(W_a)|) + \frac{1}{2} \int_{X(\mathbb{C})} g\delta_Y$$

7. Example, Metrized Line Bundles

Definition 7.1. Let $\overline{L} = (L, h)$ be a hermitian line bundle on X, we associate to it a canonical element $\hat{c}_1(\overline{L}) \in \hat{CH}^1(X)$ by considering any non-vanishing rational section s of L, and $(\mathrm{div}(s), -\log|s|^2)$ where $|\cdot| = h(\cdot)$.

Definition 7.2. The height associated to a hermitian line bundle $\overline{L} = (L, h)$ on X is a map $Z_p(X) \to \mathbb{R}$ given by:

$$h_{\overline{L}} Z \mapsto \hat{\deg}(\hat{c}_1(\overline{L})^p|Z)$$

The is the arithmetic analog of:

$$\deg_{L_K}(Z) = \pi_*(c_1(L_K)^{p-1}[Z_K]) = \deg_K(\hat{c}_1(\overline{L}^{p-1}|Z)$$

TODO-this agrees with previous notions

8. Example, Borcherds Lift

Theorem 8.1 (Borcherds)**.** *Let L be an even lattice of signature $(2,l)$ where $l \geq 3$ and $z \in L$ is a primitive isotropic vector. Let $z' \in L'$, $K = L \cap z^{\perp} \cap z'^{\perp}$. As well, assume K has an isotropic vector. Furthermore let f be a nearly holomorphic modular form of weight $k = 1 - \frac{l}{2}$ with integral Fourier coefficients $c(\gamma, n)$ when $n < 0$. Define $\Psi : \mathcal{H}_l \to \mathbb{C}$ by*

$$\Psi(Z) = \prod_{\beta \in L'/L} \prod_{\substack{m \in \mathbb{Z}+Q(\beta) \\ m<0}} \Psi_{\beta,m}(Z)^{\frac{c(\beta,m)}{2}}.$$

Then we have the following properties.

(1) *The function Ψ is a modular form of weight $\frac{c(0,0)}{2}$ for the orthogonal group $\Gamma(L)$ with a multiplier system χ of finite order. In particular, if $c(0,0) \in 2\mathbb{Z}$ then χ is a character.*

(2) *The divisor of $\Psi(Z)$ on $\mathcal{H}_l$ is*

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+Q(\beta) \\ m<0}} c(\beta, m) H(\beta, m).$$

Moreover, $\Psi(Z)$ defines a 'green's current' $\Phi(Z)$ via $\Phi(Z) = -2\log|\Psi(Z)|^2$ with log-log *growth for $Z(f)$.*

$$dd^c[\Psi(Z)] + \delta_{\mathrm{div}(\Psi(Z))} = [dd^c\Psi(Z)]$$

(3) *Let $m_0 = \min\{n \in \mathbb{Q} \,|\, c(\gamma, n) \neq 0\}$. Given a Weyl chamber $W \subset \mathcal{H}_l$ with respect to f let $\varrho_f(W) \in K \otimes \mathbb{R}$ denote the Weyl vector attached to W and f. On the set $Z \in \mathcal{H}_l$ which satisfy $q(Y) > |m_0|$ and which belong to the complement of the set of poles of $\Psi(Z)$, the function $\Psi(Z)$ has the normally convergent Borcherds product expansion*

$$\Psi(Z) = Ce((\varrho_f(W), Z)) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in L_0'/L \\ p(\delta) = \lambda + K}} (1 - e((\delta, z') + (\lambda, Z)))^{c(\delta, q(\lambda))}$$

where C is a constant of absolute value 1.

9. Application, Brunier, Yang

Theorem 9.1. *For f a harmonic weak mass form with vanishing constant coefficient we have, U a negative definite rational subspace (in the space where* O *shall come from).*

$$\Phi(Z(U), f) = \deg(Z(U)) \cdot (CT(\langle f^+, \theta_p \otimes \varepsilon_N^+ \rangle) + L'(\xi(f), U, 0))$$

The $\cdot^+$ denotes the holomorphic part, $CT(\cdot)$ is the constant term. ξ is a certain anti-linear differential operator whose kernel is the weakly holomorphic forms.

$Z(U)$ is the CM-cycle associated to U. θ_P is the theta series associated to the orthogonal complement of U in L. $\varepsilon_N = \frac{d}{ds}E_N(\tau, s; 1)|_{s=0}$, ($E_N$ is an eisenstein series associated to $N = U \cap L$). The tensor product is a rankin-type convolution. $L(\xi(f), U, s)$ is a rankin type convolution of the θ_P, and $\xi(f)$.

$\Phi(Z(U), f)$ literally means the evaluation of $\Phi(\cdot, f)$ at the cycle $Z(U)$, that is the archimedian contribution to the faltings height of $Z(U)$ at the class we would associate to f via the Borcherds lift.

The $\langle , \rangle$ is the inner product on the space the vector valued forms land.

Conjecture 9.2. *The Faltings height pairing is:*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{Fal} = \frac{\deg(Z(U))}{2} L'(\xi(f), U, 0)$$

Conjecture 9.3. $\langle \mathcal{Z}(m,\mu), \mathcal{Z}(U)\rangle_{fin}$ *is* $-\frac{\deg(Z(U))}{2}$ *times the* (m,μ) *coefficient of* $\theta_P \otimes \varepsilon_N^+$.