Notation

Let $d, D \in \mathbb{Z}^+$ be a square free positive integers.

Let $K = \mathbb{Q}(\sqrt{D})$. Let $H = \mathbb{Q}(\sqrt{d})$.

Let $\delta = \delta_1 + \delta_2\sqrt{D} \in K$ be a totally positive element of K ($\delta' = \delta_1 - \delta_2\sqrt{D}$ and $\delta, \delta' > 0$)

Let $\lambda \in K$.

Let $F = K(\sqrt{-\delta})$

The Torus T_F

We next consider the algebraic torus $T_F := R_{K/\mathbb{Q}}(R^{(1)}_{F/K}(\mathbb{C}_m))$.

$$R^{(1)}_{F/K}(\mathbb{C}_m) = \{a + b\sqrt{-\delta} | a, b \in K, (a + b\sqrt{-\delta})(a - \sqrt{-\delta}) = a^2 + \delta b^2 d = 1\}$$

The computation:

$$\begin{pmatrix} a & -\delta b \\ b & a \end{pmatrix} \begin{pmatrix} a' & -\delta b' \\ b' & a' \end{pmatrix} = \begin{pmatrix} aa' - \delta bb' & -\delta(ab' + a'b) \\ ab' + a'b & aa' - \delta bb' \end{pmatrix}$$

gives the isomorphism $(a + b\sqrt{-\delta}) \mapsto \begin{pmatrix} a & -\delta b \\ b & a \end{pmatrix}$ which gives:

$$R^{(1)}_{F/K}(\mathbb{C}_m) \simeq \{M = \begin{pmatrix} a & -\delta b \\ b & a \end{pmatrix} \in SL_2(K)\}$$

The computation:

$$\det \frac{1}{2} \begin{pmatrix} t + \frac{1}{t} & \sqrt{-\delta}(t - \frac{1}{t}) \\ \frac{1}{\sqrt{-\delta}}(t - \frac{1}{t}) & t + \frac{1}{t} \end{pmatrix} = (\frac{1}{4})(t^2 + 2 + (\frac{1}{t})^2 - [t^2 - 2 + (\frac{1}{t})^2]) = 1$$

then yields:

$$R^{(1)}_{F/K}(\mathbb{C}_m) \simeq \{\frac{1}{2} \begin{pmatrix} t + \frac{1}{t} & \sqrt{-\delta}(t - \frac{1}{t}) \\ \frac{1}{\sqrt{-\delta}}(t - \frac{1}{t}) & t + \frac{1}{t} \end{pmatrix}\}$$

With $R^{(1)}_{F/K}(\mathbb{C}_m)$ acting naturally on F as a K v.s. with basis $\{1, \sqrt{-\delta}\}$. Moreover $M_t \in R^{(1)}_{F/K}(\mathbb{C}_m)$ acts as t on the vector $(\sqrt{-\delta}, 1) \in F \otimes_K \mathbb{C}_m$ (where F was a K v.s). and acts as $\frac{1}{t}$ on $(-\sqrt{-\delta}, 1)$. So then, if we take $\{(\sqrt{-\delta}, 1), (-\sqrt{-\delta}, 1)\}$ for a basis of $F \otimes_K \mathbb{C}_m$ we get $R^{(1)}_{F/K}(\mathbb{C}_m) = \{\frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}\}$.

If we then take a basis of $K/\mathbb{Q}$ $\{1, \sqrt{D}\}$, then perform the restriction of scalars we have:

$$t = \frac{r}{2} \begin{pmatrix} s + \frac{1}{s} & \sqrt{D}(s - \frac{1}{s}) \\ \frac{1}{\sqrt{D}}(s - \frac{1}{s}) & s + \frac{1}{s} \end{pmatrix}$$

$$\frac{1}{t} = \frac{1}{2r} \begin{pmatrix} s + \frac{1}{s} & \sqrt{D}(\frac{1}{s} - s) \\ \frac{-1}{\sqrt{D}}(s - \frac{1}{s}) & s + \frac{1}{s} \end{pmatrix}$$

$$\sqrt{-\delta} = \begin{pmatrix} \sqrt{-\delta} + \sqrt{-\delta'} & \sqrt{D}(\sqrt{-\delta} - \sqrt{-\delta'}) \\ \frac{1}{\sqrt{D}}(\sqrt{-\delta} - \sqrt{-\delta'}) & \sqrt{-\delta} + \sqrt{-\delta'} \end{pmatrix}$$

We then get that if we had taken as a basis for $F/\mathbb{Q}$ the elements $\{1, \sqrt{D}, \sqrt{-\delta}, \sqrt{-\delta D}\}$ we would have T_F is the matricies:

$$\frac{1}{4}\begin{pmatrix} rs+\frac{r}{s}+\frac{s}{r}+\frac{1}{sr} & \sqrt{D}(rs-\frac{r}{s}-\frac{s}{r}+\frac{1}{sr}) & \sqrt{-\delta}(rs-\frac{1}{sr})+\sqrt{-\delta'}(\frac{r}{s}-\frac{s}{r}) & \sqrt{-\delta D}(r \\ \frac{1}{\sqrt{D}}(rs-\frac{r}{s}-\frac{s}{r}+\frac{1}{sr}) & rs+\frac{r}{s}+\frac{s}{r}+\frac{1}{sr} & \frac{\sqrt{-\delta}}{\sqrt{D}}(rs-\frac{1}{sr})+\frac{\sqrt{-\delta'}}{\sqrt{D}}(-\frac{r}{s}+\frac{s}{r}) & \sqrt{-\delta}(\\ \frac{1}{\sqrt{-\delta}}(rs-\frac{1}{sr})+\frac{1}{\sqrt{-\delta'}}(\frac{r}{s}-\frac{s}{r}) & \frac{\sqrt{D}}{\sqrt{-\delta}}(rs-\frac{1}{sr})+\frac{\sqrt{D}}{\sqrt{-\delta'}}(-\frac{r}{s}+\frac{s}{r}) & rs+\frac{r}{s}+\frac{s}{r}+\frac{1}{sr} & \sqrt{} \\ \frac{1}{\sqrt{-\delta D}}(rs-\frac{1}{sr})+\frac{1}{\sqrt{-\delta' D}}(-\frac{r}{s}+\frac{s}{r}) & \frac{1}{\sqrt{-\delta}}(rs-\frac{1}{sr})+\frac{1}{\sqrt{-\delta'}}(+\frac{r}{s}-\frac{s}{r}) & \frac{1}{\sqrt{D}}(rs-\frac{r}{s}-\frac{s}{r}+\frac{1}{sr}) & \end{pmatrix}$$

Consider the element $w = w_1 + w_2\sqrt{D} + w_3\sqrt{-\delta} + w_4\sqrt{-\delta D} \in F$ with $(w_1^2 + 2Dw_1w_2 + w_2^2) + \delta(w_3^2 + 2Dw_3w_4 + w_4^2) = 1$. then w gives us an element $t_w \in T_F$:

First in $R^{(1)}_{F/K}(\mathbb{C}_m)$ we get:

$$t_w = w = \begin{pmatrix} w_1 + w_2\sqrt{D} & -\delta(w_3 + w_4\sqrt{D}) \\ w_3 + w_4\sqrt{D} & w_1 + w_2\sqrt{D} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} w + \frac{1}{w} & \sqrt{-\delta}(w - \frac{1}{w}) \\ \frac{1}{\sqrt{-\delta}}(w - \frac{1}{w}) & w + \frac{1}{w} \end{pmatrix}$$

To perform the restriction down to $\mathbb{Q}$ is to to do it componentwise, this then gives :

$$t_w = \begin{pmatrix} w_1 & Dw_2 & -\delta_1 w_3 - D\delta_2 w_4 & -D(\delta_1 w_4 + \delta_2 w_3) \\ w_2 & w_1 & -\delta_1 w_4 - \delta_2 w_3 & -\delta_1 w_3 - D\delta_2 w_4 \\ w_3 & Dw_4 & w_1 & Dw_2 \\ w_4 & w_3 & w_2 & w_1 \end{pmatrix}$$

To compare this to the previous version, set:

$$\begin{aligned} r_w^2 = \text{"}N_{K/Q}(w)\text{"} &= (w_1 + w_3\sqrt{-\delta} + w_2\sqrt{D} + w_4\sqrt{-\delta D})(w_1 + w_3\sqrt{-\delta'} - w_2\sqrt{D} - w_4\sqrt{-\delta' D}) \\ &= (w_1 + w_3\sqrt{-\delta})^2 - D(+w_2 + w_4\sqrt{-\delta})^2 \end{aligned}$$

$s_w = w/r_w$
set $w' = (w_1 + w_3\sqrt{-\delta'} - w_2\sqrt{D} - w_4\sqrt{-\delta' D})$
then $1/w' = \overline{w'}$
then we see:

$$\begin{aligned} r_w s_w + \frac{r_w}{s_w} + \frac{s_w}{r_w} + \frac{1}{s_w r_w} &= \frac{r_w w}{r_w} + \frac{r_w^2}{w} + \frac{w}{r_w^2} + \frac{r_w}{r_w w} \\ &= w + ww'/w + w/ww' + 1/w \\ &= w + w' + 1/w' + 1/w \\ &= 2w_1 + 2w_2\sqrt{D} + w' + \overline{w'} \\ &= 4w_1 \end{aligned}$$

One can check that the other components also agree.

We now wish to split the torus, to do this, we need to work with $F \otimes_{\mathbb{Q}} F$ since it is over this field that the torus splits.

If we choose instead the basis of $K \otimes_{\mathbb{Q}} K$ of $\{(\sqrt{D}, 1), (-\sqrt{D}, 1)\}$ The t's would become:

$$t = \frac{r}{2}\begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}, \frac{1}{t} = \frac{1}{2r}\begin{pmatrix} \frac{1}{s} & 0 \\ 0 & s \end{pmatrix}$$

So, if we choose as a basis of $F \otimes_{\mathbb{Q}} \mathbb{C}_m$ the elements:

$$\{v_1 = (\sqrt{-\delta D}, \sqrt{-\delta}, \sqrt{D}, 1), \qquad v_2 = (\sqrt{-\delta' D}, -\sqrt{-\delta'}, \sqrt{D}, -1)$$
$$v_3 = (\sqrt{-\delta D}, \sqrt{-\delta}, -\sqrt{D}, -1), \qquad v_4 = (\sqrt{-\delta' D}, -\sqrt{-\delta'}, -\sqrt{D}, 1)\}$$

Then we have

$$T_F = \{\frac{1}{4}\begin{pmatrix} rs & 0 & 0 & 0 \\ 0 & \frac{r}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{rs} & 0 \\ 0 & 0 & 0 & \frac{s}{r} \end{pmatrix}\} = \frac{1}{4}\begin{pmatrix} w & 0 & 0 & 0 \\ 0 & w' & 0 & 0 \\ 0 & 0 & \overline{w} & 0 \\ 0 & 0 & 0 & \overline{w'} \end{pmatrix}\}$$

In particular then, if $T_F \subset O_q$, then:

$$q(x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4) = ax_1x_3 + bx_2x_4$$

For example the form $Tr_{K/\mathbb{Q}}(N_{F/K})$ is a gives a quadratic form on $F \otimes_{\mathbb{Q}} F$ which is preserved by T_F. (this is easily seen when we interpret things with the usual basis of $F \otimes_{\mathbb{Q}} F$). We now compute what this quardratic form looks like in terms of terms of this other basis.

$$\begin{aligned}
&\text{"}Tr_{K/\mathbb{Q}}(N_{F/K})\text{"}(x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4) \\
&\quad= \text{"}Tr_{K/\mathbb{Q}}\text{"}((\sqrt{-\delta D}(x_1+x_3) + \sqrt{-\delta' D}(x_2+x_4) + (\sqrt{-\delta}(x_1+x_3) - \sqrt{-\delta'}(x_2+x_4))\text{"}\sqrt{D}\text{"})^2 \\
&\quad+ \delta(\sqrt{D}(x_1+x_2-x_3-x_4) + (x_1-x_2-x_3+x_4)\text{"}\sqrt{D}\text{"}))^2 \\
&\quad= (\sqrt{-\delta D}(x_1+x_3) + \sqrt{-\delta' D}(x_2+x_4))^2 + D(\sqrt{-\delta}(x_1+x_3) - \sqrt{-\delta'}(x_2+x_4))^2 \\
&\quad+ \delta_1((\sqrt{D}(x_1+x_2-x_3-x_4))^2 + D(x_1-x_2-x_3+x_4)^2) \\
&\quad+ \delta_2(2D\sqrt{D}(x_1+x_2-x_3-x_4)(x_1-x_2-x_3+x_4)) \\
&\quad= D(-\delta(x_1+x_3)^2 + 2\sqrt{\delta\delta'}(x_1+x_3)(x_2+x_4) - \delta'(x_2+x_4)^2 \\
&\quad- \delta(x_1+x_3)^2 - 2\sqrt{\delta\delta'}(x_1+x_3)(x_2+x_4) - \delta'(x_2+x_4)^2) \\
&\quad+ \delta_1(D(x_1+x_2-x_3-x_4)^2 + D(x_1-x_2-x_3+x_4)^2) \\
&\quad+ 2\delta_2D\sqrt{D}(x_1+x_2-x_3-x_4)(x_1-x_2-x_3+x_4) \\
&\quad= D(-2\delta(x_1+x_3)^2 - 2\delta'(x_2+x_4)^2) \\
&\quad+ \delta_1D((x_1+x_2-x_3-x_4)^2 + (x_1-x_2-x_3+x_4)^2) \\
&\quad+ 2\delta_2D\sqrt{D}(x_1+x_2-x_3-x_4)(x_1-x_2-x_3+x_4) \\
&\quad= -2\delta_1D((x_1+x_3)^2 + (x_2+x_4)^2) - 2\delta_2D\sqrt{D}((x_1+x_3)^2 - (x_2+x_4)^2) \\
&\quad+ \delta_1D((x_1+x_2-x_3-x_4)^2 + (x_1-x_2-x_3+x_4)^2) \\
&\quad+ 2\delta_2D\sqrt{D}(x_1+x_2-x_3-x_4)(x_1-x_2-x_3+x_4) \\
&\quad= \delta_1D((x_1+x_2-x_3-x_4)^2 + (x_1-x_2-x_3+x_4)^2 - 2(x_1+x_3)^2 - 2(x_2+x_4)^2) \\
&\quad+ 2\delta_2D\sqrt{D}((x_1+x_2-x_3-x_4)(x_1-x_2-x_3+x_4) - (x_1+x_3)^2 + (x_2+x_4)^2) \\
&\quad= \delta_1D(-8x_1x_3 - 8x_2x_4) + 2\delta_2D\sqrt{D}(-4x_1x_3 - 4x_2x_4) \\
&\quad= -16D\delta(x_1x_3 + x_2x_4)
\end{aligned}$$

Which we note, is of the allowable form.

We will now try to "undo" the above computation and express the candidate q in terms of the usual rational basis $e_1 = (1,0,0,0), ..., e_4 = (0,0,0,1)$

$e_1 = \frac{1}{4}(\frac{1}{\sqrt{-\delta D}}(v_1+v_3) + \frac{1}{\sqrt{-\delta' D}}(v_2+v_4))$
$e_2 = \frac{1}{4}(\frac{1}{\sqrt{-\delta}}(v_1+v_3) - \frac{1}{\sqrt{-\delta'}}(v_2+v_4))$
$e_3 = \frac{1}{4\sqrt{D}}(v_1 - v_3 + v_2 - v_4)$
$e_4 = \frac{1}{4}(v_1 - v_3 - v_2 + v_4)$

and so:

$q(e_1) = (\frac{-1}{16D})(\frac{a}{\delta} + \frac{b}{\delta'})$
$q(e_2) = (\frac{-1}{16})(\frac{a}{\delta} + \frac{b}{\delta'})$
$q(e_3) = (\frac{-1}{16D})(a+b)$
$q(e_4) = (\frac{-1}{16})(a+b)$
$q(e_1+e_2) = (\frac{-1}{16D})(\frac{(1+\sqrt{D})^2}{\delta}a + \frac{(1-\sqrt{D})^2}{\delta'}b)$
$q(e_1+e_3) = (\frac{-1}{16D})(\frac{1+\delta}{\delta}a + \frac{1+\delta'}{\delta'}b)$
$q(e_1+e_4) = (\frac{-1}{16D})(\frac{1+\delta D}{\delta}a + \frac{1+\delta' D}{\delta'}b)$
$q(e_2+e_3) = (\frac{-1}{16D})(\frac{D+\delta}{\delta}a + \frac{D+\delta'}{\delta'}b)$
$q(e_2+e_4) = (\frac{-1}{16})(\frac{1+\delta}{\delta}a + \frac{1+\delta'}{\delta'}b)$
$q(e_3+e_4) = (\frac{-1}{16D})((1+\sqrt{D})^2a + (1-\sqrt{D})^2b))$

Thus:

$(e_1,e_2) = (\frac{-1}{16D})((\frac{(1+\sqrt{D})^2}{\delta}a + \frac{(1-\sqrt{D})^2}{\delta'}b) - (\frac{a}{\delta} + \frac{b}{\delta'}) - D(\frac{a}{\delta} + \frac{b}{\delta'})) = \frac{-1}{8\sqrt{D}}(\frac{a}{\delta} - \frac{b}{\delta'})$
$(e_1,e_3) = (\frac{-1}{16D})((\frac{1+\delta}{\delta}a + \frac{1+\delta'}{\delta'}b) - (\frac{a}{\delta} + \frac{b}{\delta'}) - (a+b)) = 0$
$(e_1,e_4) = (\frac{-1}{16D})((\frac{1+\delta D}{\delta}a + \frac{1+\delta' D}{\delta'}b) - (\frac{a}{\delta} + \frac{b}{\delta'}) - D(a+b)) = 0$
$(e_2,e_3) = (\frac{-1}{16D})((\frac{D+\delta}{\delta}a + \frac{D+\delta'}{\delta'}b) - D(\frac{a}{\delta} + \frac{b}{\delta'}) - (a+b)) = 0$
$(e_2,e_4) = (\frac{-1}{16})((\frac{1+\delta}{\delta}a + \frac{1+\delta'}{\delta'}b) - (\frac{a}{\delta} + \frac{b}{\delta'}) - (a+b) = 0$
$(e_3,e_4) = (\frac{-1}{16D})(((1+\sqrt{D})^2a + (1-\sqrt{D})^2b)) - (a+b) - D(a+b)) = \frac{-1}{8\sqrt{D}}(a-b)$

In particular we then have:

$$q(y_1,y_2,y_3,y_4) = \frac{-1}{16D}[(\frac{a}{\delta}+\frac{b}{\delta'})y_1^2 + D(\frac{a}{\delta}+\frac{b}{\delta'})y_2^+(a+b)y_3^3 + D(a+b)y_4^2 + 2(\frac{a}{\delta}-\frac{b}{\delta'})\sqrt{D}y_1y_2 + 2\sqrt{D}(a-b)y_3y_4]$$

Rescaling a, b allows us to write:

$$\begin{aligned} q(y_1,y_2,y_3,y_4) = {} & (a+b)y_1^2 + 2\sqrt{D}(a-b)y_1y_2 + D(a+b)y_2^2 \\ & + (a\delta + b\delta')y_3^3 + 2\sqrt{D}(a\delta - b\delta')y_3y_4 + D(a\delta + b\delta')y_4^2 \end{aligned}$$

but we require rational coefficients... so $(a+b), \sqrt{D}(a-b) \in \mathbb{Q}$, but then we get $a = b + r\sqrt{D}$ thus $2b + r\sqrt{D} \in \mathbb{Q}$, thus $b = s - \frac{r}{2}\sqrt{D}$ and $a = s + \frac{r}{2}\sqrt{D}$ for $s, r \in \mathbb{Q}$. In particular we have $b = a'$ and so we can rewrite this as:

$$\begin{aligned} q(y_1,y_2,y_3,y_4) = {} & (a+a')y_1^2 + 2\sqrt{D}(a-a')y_1y_2 + D(a+a')y_2^2 \\ & + (a\delta + a'\delta')y_3^3 + 2\sqrt{D}(a\delta - a'\delta')y_3y_4 + D(a\delta + a'\delta')y_4^2 \end{aligned}$$

The requirement that $(a\delta + a'\delta'), \sqrt{D}(a\delta - a'\delta')$ is then automatic.

We notice that this is up to multiples of 2 the form $Tr_{F/\mathbb{Q}}(ax\overline{x})$.

The forms $Tr_{K/\mathbb{Q}}(\lambda N_{F/K}(x))$

It is clear from its definition that T_F with the usual action of multiplication will preserve the norm form on F/K and thus the forms :

$$Tr_{K/\mathbb{Q}}(\lambda N_{F/K}(x))$$

This choosing basis F/Q of the form $a_1, a_2, a_3\sqrt{-\delta}, a_4\sqrt{-\delta}$ with $a_i \in K$ decomposes the form as:

$$Tr_{K/\mathbb{Q}}(\lambda N_{F/K}(x) \sim 2Tr_{K/\mathbb{Q}}(\lambda x^2) \oplus 2Tr_{K/\mathbb{Q}}(-\lambda\delta y^2)$$

Provided $Tr_{K/\mathbb{Q}}(\alpha) \neq 0$ we can use basis $1, \alpha'\sqrt{D}$ to express the form $Tr_{K/\mathbb{Q}}(\alpha y^2)$ as:

$$Tr_{K/\mathbb{Q}}(\alpha(y_1 + y_2\alpha'\sqrt{D})^2) = Tr_{K/\mathbb{Q}}(\alpha)y_1^2 + N_{K/\mathbb{Q}}(\alpha)DTr_{K/\mathbb{Q}}(\alpha)y_2^2$$

If $Tr_{K/\mathbb{Q}}(\alpha) = 0$ then the basis $1+\sqrt{D}, 1-\sqrt{D}$ gives us:

$$Tr_{K/\mathbb{Q}}(\alpha(y_1(1+\sqrt{D}) + y_2(1-\sqrt{D}))^2) = 2Tr_{K/\mathbb{Q}}(\alpha\sqrt{D})y_1^2 - 2Tr_{K/\mathbb{Q}}(\alpha\sqrt{D})y_2^2$$

In particular then:
Choose for $\lambda = \frac{1}{\sqrt{D}}$ then we get:

$$Tr_{K/\mathbb{Q}}(\lambda x^2) = 2x_1^2 - 2x_2^2$$

$$\begin{aligned} Tr_{K/\mathbb{Q}}(-\lambda\delta y^2) &= Tr_{K/\mathbb{Q}}(\frac{-\delta}{\sqrt{D}})y_1^2 + N_{K/\mathbb{Q}}(\frac{-\delta}{\sqrt{D}})DTr_{K/\mathbb{Q}}(\frac{-\delta}{\sqrt{D}})y_2^2 \\ &= (-2\delta_2)y_1^2 - (\delta_1^2 - D\delta_2^2)(-2\delta_2)y_2^2 \end{aligned}$$

So thus we get:

$$\begin{aligned} Tr_{K/\mathbb{Q}}(\lambda N_{F/K}(x)) &\sim 4x_1^2 - 4x_2^2 + 2(-2\delta_2)y_1^2 - 2(\delta_1^2 - D\delta_2^2)(-2\delta_2)y_2^2 \\ &\sim x_1^2 - x_2^2 - \delta_2 y_1^2 + N_{K/\mathbb{Q}}(\delta)\delta_2 y_2^2 \end{aligned}$$

This quadratic form has discriminant $N_{K/\mathbb{Q}}(\delta)$, Witt invariants $(-1,-1)_p(\delta_2, N(\delta))_p$ and signature $(2,2)$ If we had taken $\lambda = \frac{\delta_2}{\sqrt{D}}$ we would have had Witt invariants $(-1,-1)_p$.

If we instead consider the case $\lambda = \sqrt{D}\delta'$ we get the same thing essentially.

For all other cases we have $Tr_{K/\mathbb{Q}}(\alpha) \neq 0$ for $\alpha = \lambda, -\lambda\delta,$,rescale so $Tr_{K/\mathbb{Q}}(\lambda) = 1$ and thus we get:

$$Tr_{K/\mathbb{Q}}(\lambda N_{F/K}(x)) \sim Tr_{K/\mathbb{Q}}(\lambda)x_1^2 + N_{K/\mathbb{Q}}(\lambda)DTr_{K/\mathbb{Q}}(\lambda)x_2^2 + Tr_{K/\mathbb{Q}}(-\lambda\delta)y_1^2 + N_{K/\mathbb{Q}}(-\lambda\delta)DTr_{K/\mathbb{Q}}(-\lambda\delta)y_2^2$$

This quadratic form has discriminant $N_{K/\mathbb{Q}}(\delta)$ Witt invariants

$$(D,D)(D, N_{K/\mathbb{Q}}(\delta))(N_{K/\mathbb{Q}}(\delta), N_{K/\mathbb{Q}}(\lambda))(N_{K/\mathbb{Q}}(\lambda), N_{K/\mathbb{Q}}(\lambda))(-Tr_{K/\mathbb{Q}}(\lambda\delta), -N_{K/\mathbb{Q}}(\lambda\delta)D)$$

And signature determined by twice the number of positive/negative embeddings of λ.

The Hilbert Modular Surface

Consider the hilbert modular surface coming from the quadratic form $q_H(a,b,h) = ab - hh'$. With the choice of basis $(1,1,0),(1,-1,0),(0,0,1),(0,0,\sqrt{d})$ this quadratic form is equivalent to $q_H = [1,-1,-1,d]$, has discriminant $-d$, Witt invariants $(-1,-1)_p$ for each prime p and has signature $(2,2)$.

So what we have now, is that we have algebraic tori T_F which will be contained in the orthogonal group for the Hilbert modular space $\mathbb{H}_{\mathbb{Q}(\sqrt{N_{K/\mathbb{Q}}(\delta)})}$. Through the association of points of the space and abelian varieties we will then have that T_F will stabilize some point on $\mathbb{H}_{\mathbb{Q}(\sqrt{N(\delta)})}$ and thus will inject into the endomorphism group of the associated abelian variety. However, the abelian varieties associated to $\mathbb{H}_{\mathbb{Q}(\sqrt{N(\delta)})}$ should have CM through fields over $\mathbb{Q}(\sqrt{N_{K/\mathbb{Q}}(\delta)})$ and not over K.

We are thus driven to the questions:

- When can $N_{K/\mathbb{Q}}(\delta) = d$?

  The answer is iff $(d,D)_p = 1$ for all primes. This is a local class field theory question.

- How does an abelian variety end up with CM by two seemingly unassociated fields?

  The answer (in part) is in the next section and amounts to showing how the fields are associated

Field towers

Make the following definitions (compatible with previous notation)

$K = \mathbb{Q}(\sqrt{D})$

$F = \mathbb{Q}(\sqrt{D}, \sqrt{-\delta})$ $F' = \mathbb{Q}(\sqrt{D}, \sqrt{-\delta'})$

$L = \mathbb{Q}(\sqrt{\delta\delta'})$

$G = \mathbb{Q}(\sqrt{\delta\delta'}, \sqrt{-\delta} + \sqrt{-\delta}')$ $G' = \mathbb{Q}(\sqrt{\delta\delta'}, \sqrt{-\delta} - \sqrt{-\delta}')$

$\overline{F} = FF' = GG' = \mathbb{Q}(\sqrt{D}, \sqrt{-\delta}, \sqrt{-\delta'})$

We have the following diagram:

$$\begin{tikzcd}
 & & \overline{F} \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
F \arrow[dr, dash] & F' \arrow[d, dash] & KL \arrow[dl, dash] \arrow[dr, dash] & G \arrow[d, dash] & G' \arrow[dl, dash] \\
 & K \arrow[dr, dash] & & L \arrow[dl, dash] & \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

Which is possibly incomplete and with redundancy.

Note that:

$\sqrt{-\delta} + \sqrt{-\delta'} = \sqrt{-2(\delta_1 - \sqrt{\delta\delta'})}$

$\sqrt{-\delta} - \sqrt{-\delta'} = \sqrt{-2(\delta_1 + \sqrt{\delta\delta'})}$

and $(-2(\delta_1 - \sqrt{\delta\delta'}))(-2(\delta_1 + \sqrt{\delta\delta'})) = 4(\delta_1^2 - (\delta_1^2 - D\delta_2^2)) = 4D\delta_2^2$
and so we have that G, G' are quadratic imaginary (over L) when δ is not in $\mathbb{Q}$.

We have 3 cases to consider

F cyclic Galois

We immediately have $\overline{F} = F = F'$. Since KL is totally real this implies $KL \subset K$ implies $L \subset K$. We note that in the case of cyclic Galois $L \not\subseteq \mathbb{Q}$ since if it were, then at least one of G, G' would be an imaginary quadratic extension of $\mathbb{Q}$ contained in F, but F being cyclic galois CM, means this does not happen.

Therefor we have in this case that: $\overline{F} = F = F' = G = G'$
$K = KL = L$ in particular $D = \delta\delta'$ mod squares.

Conversely, one can has that if $D = \delta\delta'$ mod squares so that $K = KL = L$, one gets then that: $\overline{F} = F = F' = G = G'$ and so F is at least galois. ($F = G = G'$ follows from δ not being in $\mathbb{Q}$ with remarks above)

We have an element of $Gal(K/\mathbb{Q})$ which takes $\delta \mapsto \delta'$. It must extend to an element of $Gal(F/\mathbb{Q})$ in 2 ways and each extension can act on $\sqrt{-\delta}, \sqrt{-\delta'}$ in one of the following ways:

- $\sqrt{-\delta} \mapsto \sqrt{-\delta}$

  would imply $\delta \mapsto \delta$ so not allowed as an extension

- $\sqrt{-\delta} \mapsto -\sqrt{-\delta}$

  would imply $\delta \mapsto \delta$ so not allowed as an extension

- $\sqrt{-\delta} \mapsto \sqrt{-\delta'}$

  - $:\sqrt{-\delta'} \mapsto \pm\sqrt{-\delta'}$
    now allowed as above
  - $:\sqrt{-\delta'} \mapsto \sqrt{-\delta}$
    We have that for such an automorphism G will be contained in its fixed field, but $G = F$
  - $:\sqrt{-\delta'} \mapsto -\sqrt{-\delta}$ such an element has order 4 so the galois group is C_4.

- $\sqrt{-\delta} \mapsto -\sqrt{-\delta'}$

  - $:\sqrt{-\delta'} \mapsto \pm\sqrt{-\delta'}$
    now allowed as above
  - $:\sqrt{-\delta'} \mapsto \sqrt{-\delta}$
    such an element has order 4 so the galois group is C_4.
  - $:\sqrt{-\delta'} \mapsto -\sqrt{-\delta}$
    We have that for such an automorphism G' will be contained in its fixed field, but $G' = F$

F is bi-quadratic

We again have $\overline{F} = F = F'$ but by the above work we know that $L \neq K$, but $L \subset G \subset \overline{F} = F$ is totally real, so we conclude $L = \mathbb{Q}$. We then have that at least one of G, G' is quadratic imaginary over $\mathbb{Q}$, without loss of generality suppose it is G (this amounts to fixing an embedding of everthing in $\mathbb{C}$ and supposing that $\sqrt{-\delta}, \sqrt{-\delta'}$ where both taken to be the square roots in the upper half plane, we might as well have made this assumption earlier). We then arrive at the following diagram of fields:

$$\begin{tikzcd}
 & F = \mathbb{Q}(\sqrt{D}, \sqrt{-\delta}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K = \mathbb{Q}(\sqrt{D}) \arrow[dr, dash] & G = \mathbb{Q}(\sqrt{-\delta} + \sqrt{-\delta'}) \arrow[d, dash] & \mathbb{Q}(\sqrt{D}(\sqrt{-\delta} + \sqrt{-\delta'})) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Note that for δ not in $\mathbb{Q}$ then $\mathbb{Q}(\sqrt{D}(\sqrt{-\delta} + \sqrt{-\delta'})) = G'$.

F is not galois

Then nothing in the top diagram collapses, we should though note that we do in fact have: $F \simeq F'$ and $G \simeq G'$ coming from galois automorphisms of $\overline{F}$, moreover the map taking $G' \to G$ can be taken to be the galois element σ whose fixed field is F. We have that for an element $f \in F$, $N_{\overline{F}/G}(F) = ff'$, $N_{\overline{F}/G'}(F) = f\overline{f'} = \sigma(ff')$.

Moreover we should generally expect to have the maps of norm 1 elements:

$$\pm 1 \longrightarrow F^{(1)} \times F'^{(1)} \xrightarrow{(f_1,f_2) \mapsto f_1 f_2} \overline{F}^{(1)}$$

$$\pm 1 \longrightarrow \overline{F}^{(1)} \xrightarrow{N_{\overline{F}/F} \times N_{\overline{F}/F'}} F^{(1)} \times F'^{(1)}$$

$$\pm 1 \longrightarrow F^{(1)} \xrightarrow{N_{\overline{F}/G} \times N_{\overline{F}/G'}} G^{(1)} \times G'^{(1)}$$

However in the last map, the images are of the form $(ff', \sigma(ff'))$ and so we lose no information by further projecting to $G^{(1)}$. In particular we have maps:

$$\pm 1 \longrightarrow F^{(1)} \xrightarrow{\alpha : f \mapsto ff'} G^{(1)}$$

$$\pm 1 \longrightarrow G^{(1)} \xrightarrow{\beta : g \mapsto g\sigma(g)} F^{(1)}$$

Moreover, the compostion $\beta \circ \alpha : f \mapsto f^2$ and $\alpha \circ \beta : g \mapsto g^2$.

And so what we see is that $F^{(1)} \sim G^{(1)}$.

What this says about abelian varieties with CM

Basically it says that essentially, all is well.

In the cyclic galois case we have in fact no surprises, we end up with CM by a field that is in fact an extension of a field that should be associated to the hilbert modular space.

In the bi-quadratic case, the hilbert modular space we thought we had turned out not to really be one.

In the non-galois case we see that our tori is isogenous to one coming from a CM field of a field associated to the hilbert modular surface, so we ought not be all that surprised that something might have CM coming from both.

Moreover, if you look at all the cases, you should see at least a hint of how to get our tori we started with, to preserve a quadratic form of discriminant D (and with the desired other invariants), The only issue with the construction, is trying to deal with the isogeny and make it into an isomorphism.

TODO

- What other faithful actions can $R_{K/\mathbb{Q}}(R^{(1)}_{F/K}(\mathbb{C}_m))$ have on a vector space s.t. it can preserve a quadratic form on it? We want a character that acts like $R^{(1)}_{K/\mathbb{Q}}(\mathbb{C}_m)$ and one that acts like $R^{(1)}_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{C}_m)$?