Why We Can't Have More Cross Products

Andrew Fiori

University of Calgary

February 2017

What is a Cross Product?

Starting with a vector space over $\mathbb{R}$, so $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

What is a Cross Product?

Starting with a vector space over $\mathbb{R}$, so $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

What is a Cross Product?

Starting with a vector space over $\mathbb{R}$, so $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

2. (Parollellogram)

$$(\vec{x} \times \vec{y}) \cdot (\vec{x} \times \vec{y}) = (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{y})^2$$

What do we want to say about them?

The main result of this talk is that cross products only exist for $\mathbb{R}^n$ if $n = 0, 1, 3, 7$.

The main result of this talk is that cross products only exist for $\mathbb{R}^n$ if $n = 0, 1, 3, 7$.

This result is true in more generality than I will show. In particular it is true if we change what we mean by a cross product a little:

What is a Cross Product?

Starting with a vector space over $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

which satisfies:

What is a Cross Product?

Starting with a vector space over $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

What is a Cross Product?

Starting with a vector space over $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

2. (Parollellogram)

$$(\vec{x} \times \vec{y}) \cdot (\vec{x} \times \vec{y}) = (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{y})^2$$

What is a Cross Product?

Starting with a vector space over $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

Continuous Map

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

2. (Parollellogram)

$$(\vec{x} \times \vec{y}) \cdot (\vec{x} \times \vec{y}) = (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{y})^2$$

What is a Cross Product?

Starting with a vector space over $V = \mathbb{R}^n$, with the standard dot product:

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_y.$$

the key features that characterize the cross product are that it is a bilinear map:

Continuous Map

$$V \times V \to V$$

which satisfies:

1. (Orthogonality)

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{y}) \cdot \vec{y} = 0$$

2. (Parollellogram)

$$(\vec{x} \times \vec{y}) \cdot (\vec{x} \times \vec{y}) = (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{y})^2$$

$\vec{x} \times \vec{y} = 0$ **if and only if they are Linearly Dependent.**

Why not prove the more general result?

The proof of the more general result requires a hard result in topology/differential geometry. Namely:

"Non-parallelizability of the N-Sphere" (sphere in $\mathbb{R}^{N+1}$) except for $N = 0, 1, 3, 7$.

which can be viewed as a generalization of the Hairy Ball Theorem, which says you can't properly comb a hairly ball in $\mathbb{R}^3$ (or $\mathbb{R}^n$ for $n > 1$ odd).

Why not prove the more general result?

The proof of the more general result requires a hard result in topology/differential geometry. Namely:

"Non-parallelizability of the N-Sphere"
(sphere in $\mathbb{R}^{N+1}$) except for $N = 0, 1, 3, 7$.

which can be viewed as a generalization of the Hairy Ball Theorem, which says you can't properly comb a hairly ball in $\mathbb{R}^3$ (or $\mathbb{R}^n$ for $n > 1$ odd).

The result I am showing uses instead that:

"$N + 1$-dimensional Hurwitz algebras exist only for $N = 0, 1, 3, 7$."

(Note: there is a more elementary proof than the one I am giving.)

The Algebra associated to a Cross Product

We shall use the cross product on V to define an $\mathbb{R}$-algebra structure on

$$A = \mathbb{R} \oplus V = \mathbb{R}^{n+1}.$$

by:

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

We shall use the cross product on V to define an $\mathbb{R}$-algebra structure on

$$A = \mathbb{R} \oplus V = \mathbb{R}^{n+1}.$$

by:

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Recall: An $\mathbb{R}$-*algebra* is just a $\mathbb{R}$-vector space with an $\mathbb{R}$-bilinear multiplication:

$$A \times A \to A$$

which is left and right distributive:

$$a(b+c) = ab + ac \qquad (a+b)c = ac + bc$$

but not necessarily commutative, or associative, so **can't** typically assume:

$$ab = ba \qquad \text{or} \qquad a(bc) = (ab)c$$

and it may not have an identity.

The algebra A we we just defined will have some special properties that can't possibly be satisfied unless n is one of $0, 1, 3, 7$.

Where are we going with this?

The algebra A we we just defined will have some special properties that can't possibly be satisfied unless n is one of $0, 1, 3, 7$.

An an aside, I will point out that the thing I just defined would (with a bit of work) allow you to construct a parallelization of the N-sphere (even if we had dropped to the weaker hypothesis.) Which Adams tells us typically does not exist.

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra is unital

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra is unital

That is, there is an identity element namely $1_A = (1, 0)$.
In particular $1_A = (1, 0)$ satisfies:

$$(1, 0)(x_0, \vec{x}) = (x_0, \vec{x}) = (x_0, \vec{x})(1, 0).$$

Properties of the Algebras

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra has an inner product (positive definite):

Properties of the Algebras

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra has an inner product (positive definite):

We can extend the dot product on V to A by:

$$(x_0, \vec{x}) \cdot (y_0, \vec{y}) = x_0 y_0 + \vec{x} \cdot \vec{y}.$$

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra has an inner product (positive definite):
We can extend the dot product on V to A by:

$$(x_0, \vec{x}) \cdot (y_0, \vec{y}) = x_0 y_0 + \vec{x} \cdot \vec{y}.$$

Claim: The algebra has an (anti-)involution:
We may define an (anti-)involution on A:

$$a^* = 2(1_A \cdot a)1_A - a.$$

Properties of the Algebras

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x}\cdot\vec{y}, x_0\vec{y} + y_0\vec{x} + \vec{x}\times\vec{y}).$$

Claim: The algebra has an inner product (positive definite):
We can extend the dot product on V to A by:

$$(x_0, \vec{x})\cdot(y_0, \vec{y}) = x_0 y_0 + \vec{x}\cdot\vec{y}.$$

Claim: The algebra has an (anti-)involution:
We may define an (anti-)involution on A:

$$a^* = 2(1_A \cdot a)1_A - a.$$

These are related by the observation:
With $a = (x_0, \vec{x})$ we have $a^* = (x_0, -\vec{x})$ and:

$$ab^* + ba^* = 2(a\cdot b)1_A.$$

The above are all immediate to check.
This essentially shows that the algebra is a nicely normed $*$-algebra.

Properties of the Algebras

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x}\cdot\vec{y}, x_0\vec{y} + y_0\vec{x} + \vec{x}\times\vec{y}).$$

Claim: The algebra is a (multiplicatively) normed algebra

In the sense that:

For all $a, b \in A$ we have:

$$(a\cdot a)(b\cdot b) = (ab)\cdot(ab)$$

or equivalently that:

For all $a, b, c, d \in A$ we have:

$$2(a\cdot b)(c\cdot d) = ((ac)\cdot(bd)) + ((ad)\cdot(bc)).$$

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra is a (multiplicatively) normed algebra

In the sense that:

For all $a, b \in A$ we have:

$$(a \cdot a)(b \cdot b) = (ab) \cdot (ab)$$

or equivalently that:

For all $a, b, c, d \in A$ we have:

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc)).$$

This is an easy (though tedious) check using both the orthogonality and parallelogram properties of the cross product.

Recall

$$(x_0, \vec{x})(y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y}, x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}).$$

Claim: The algebra is a (multiplicatively) normed algebra

In the sense that:

For all $a, b \in A$ we have:

$$(a \cdot a)(b \cdot b) = (ab) \cdot (ab)$$

or equivalently that:

For all $a, b, c, d \in A$ we have:

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc)).$$

This is an easy (though tedious) check using both the orthogonality and parallelogram properties of the cross product. The equivalence of the two statements uses only the linearity of the dot product and the distributivity of multiplication.

What have we shown so far?

We have shown that our algebra A is a finite dimensional unital multiplicatively normed algebra (with a positive definite norm).
These are called (Euclidean) Hurwitz algebras, these are the things that are rare, that is what we will now prove.
For the next little while, we will be taking A to be a Hurwitz algebra, and proving some things about it.

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: The involution $a \mapsto a^*$ is actually the 'adjoint'

That is, multiplication by a is an endomorphism of a vector space with an inner product, and multiplication by a^* is the adjoint to that endomorphism.

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: The involution $a \mapsto a^*$ is actually the 'adjoint'

That is, multiplication by a is an endomorphism of a vector space with an inner product, and multiplication by a^* is the adjoint to that endomorphism.

Specializing $b = 1_A$ and rearranging slightly we obtain:

$$2(a \cdot 1_A)(d \cdot c) - ((ad) \cdot c) = ((ac) \cdot d)$$

Applying linearity of dot product and distributivity of products in A we see:

$$((ac) \cdot d) = (c \cdot (a^* d))$$

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: The involution $a \mapsto a^*$ is actually the 'adjoint'

That is, multiplication by a is an endomorphism of a vector space with an inner product, and multiplication by a^* is the adjoint to that endomorphism.

Specializing $b = 1_A$ and rearranging slightly we obtain:

$$2(a \cdot 1_A)(d \cdot c) - ((ad) \cdot c) = ((ac) \cdot d)$$

Applying linearity of dot product and distributivity of products in A we see:

$$((ac) \cdot d) = (c \cdot (a^* d))$$

So that a^* is the "adjoint" under the bilinear pairing.

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: The involution $a \mapsto a^*$ is actually the 'adjoint'

That is, multiplication by a is an endomorphism of a vector space with an inner product, and multiplication by a^* is the adjoint to that endomorphism.

Specializing $b = 1_A$ and rearranging slightly we obtain:

$$2(a \cdot 1_A)(d \cdot c) - ((ad) \cdot c) = ((ac) \cdot d)$$

Applying linearity of dot product and distributivity of products in A we see:

$$((ac) \cdot d) = (c \cdot (a^* d))$$

So that a^* is the "adjoint" under the bilinear pairing.

Similarly one can show:

$$((ac) \cdot d) = (a \cdot (dc^*))$$

so it is also the adjoint for multiplication on the right!!

Recall

$$ab^* + ba^* = 2(a \cdot b)1_A.$$

Claim: We can explicitly describe the failure of commutativity:

Recall

$$ab^* + ba^* = 2(a \cdot b)1_A.$$

Claim: We can explicitly describe the failure of commutativity:

If $a, b \in A$ and $1_A \cdot a = 1_A \cdot b = a \cdot b = 0$ then:

$$ab = -ba = b^* a.$$

So that the algebra is quite possibly not commutative.

Recall

$$ab^* + ba^* = 2(a \cdot b)1_A.$$

Claim: We can explicitly describe the failure of commutativity:
If $a, b \in A$ and $1_A \cdot a = 1_A \cdot b = a \cdot b = 0$ then:

$$ab = -ba = b^* a.$$

So that the algebra is quite possibly not commutative.
Note that under the conditions $1_A \cdot a = 1_A \cdot b = 0$ we have:

$$a^* = -a \qquad b^* = -b.$$

Combining this with the above we have:

$$ab = -ab^* = ba^* = -ba = b^* a.$$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$(-a(bc)) \cdot d =$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$(-a(bc)) \cdot d = a(cb) \cdot d =$

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) =$$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

$$-(cd) \cdot (a^* b) =$$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

$$-(cd) \cdot (a^* b) = -d \cdot (c^*(a^* b)) =$$

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

$$-(cd) \cdot (a^* b) = -d \cdot (c^*(a^* b)) = -((a^* b)c) \cdot d$$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

$$-(cd) \cdot (a^* b) = -d \cdot (c^*(a^* b)) = -((a^* b)c) \cdot d = ((ab)c \cdot d).$$

Properties of Hurwitz Algebras

Recall

$$2(a \cdot b)(c \cdot d) = ((ac) \cdot (bd)) + ((ad) \cdot (bc))$$

Claim: We can explicitly describe the failure of associativity:
If $a, b, c \in A$ and
$1_A \cdot a = 1_A \cdot b = 1_A \cdot c = a \cdot b = a \cdot c = b \cdot c = (ab) \cdot c = 0$ then:

$$(ab)c = -a(bc).$$

So that the algebra is quite possibly not associative.
For all d we have:

$$(-a(bc)) \cdot d = a(cb) \cdot d = (cb) \cdot (a^* d) = 2(c \cdot a^*)(b \cdot d) - (cd) \cdot (a^* b)$$

By perpendicularity of a and c we get $2(c \cdot a^*)(b \cdot d) = 0$ so the above equals:

$$-(cd) \cdot (a^* b) = -d \cdot (c^*(a^* b)) = -((a^* b)c) \cdot d = ((ab)c \cdot d).$$

And since this holds for all d and the dot product is a perfect pairing... done!

Recall

$$((ac)\cdot d)) = (c\cdot(a^*d))$$

Claim: Hurwitz Algebras are Alternating

Properties of Hurwitz Algebras

Recall

$$((ac) \cdot d)) = (c \cdot (a^* d))$$

Claim: Hurwitz Algebras are Alternating

That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

Recall

$$((ac)\cdot d)) = (c\cdot(a^*d))$$

Claim: Hurwitz Algebras are Alternating
That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$((aa^*)b\cdot c) =$

Recall

$$((ac)\cdot d)) = (c\cdot(a^* d))$$

Claim: Hurwitz Algebras are Alternating

That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b\cdot c) = (((a\cdot a)b)\cdot c) =$$

Recall

$$((ac)\cdot d)) = (c \cdot (a^* d))$$

Claim: Hurwitz Algebras are Alternating
That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b \cdot c) = (((a \cdot a)b) \cdot c) = (a \cdot a)(b \cdot c) =$$

Properties of Hurwitz Algebras

Recall

$$((ac)\cdot d)) = (c\cdot(a^* d))$$

Claim: Hurwitz Algebras are Alternating
That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b\cdot c) = (((a\cdot a)b)\cdot c) = (a\cdot a)(b\cdot c) = (ab)\cdot(ac) =$$

Properties of Hurwitz Algebras

Recall

$$((ac)\cdot d)) = (c\cdot(a^* d))$$

Claim: Hurwitz Algebras are Alternating

That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b\cdot c) = (((a\cdot a)b)\cdot c) = (a\cdot a)(b\cdot c) = (ab)\cdot(ac) = (a^*(ab))\cdot c$$

Properties of Hurwitz Algebras

Recall

$$((ac)\cdot d)) = (c\cdot(a^* d))$$

Claim: Hurwitz Algebras are Alternating
That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b\cdot c) = (((a\cdot a)b)\cdot c) = (a\cdot a)(b\cdot c) = (ab)\cdot(ac) = (a^*(ab))\cdot c$$

Replacing $a^* = (2(a, 1_A)1_A - a)$ we can solve to obtain:

$$((aa)b)\cdot c = (a(ab))\cdot c$$

for all a, b, c.

Properties of Hurwitz Algebras

Recall

$$((ac) \cdot d)) = (c \cdot (a^* d))$$

Claim: Hurwitz Algebras are Alternating
That is, for all $a, b \in A$ we have:

$$(aa)b = a(ab)$$

For all a, b, c we have:

$$((aa^*)b \cdot c) = (((a \cdot a)b) \cdot c) = (a \cdot a)(b \cdot c) = (ab) \cdot (ac) = (a^*(ab)) \cdot c$$

Replacing $a^* = (2(a, 1_A)1_A - a)$ we can solve to obtain:

$$((aa)b) \cdot c = (a(ab)) \cdot c$$

for all a, b, c. Again exploiting the perfect pairing we obtain the result.

Aside from the cross product, how can we construct algebras like this?

Given an algebra A with an anti-involution, we can construct a new product structure on:

$$\mathcal{C}(A) = A \oplus A$$

Aside from the cross product, how can we construct algebras like this?

Given an algebra A with an anti-involution, we can construct a new product structure on:

$$\mathcal{C}(A) = A \oplus A$$

by the rule:

$$(a, b) \cdot (c, d) = (ac - d^* b, bc^* + da)$$

The involution is:

$$(a, b)^* = (a^*, -b)$$

Aside from the cross product, how can we construct algebras like this?

Given an algebra A with an anti-involution, we can construct a new product structure on:

$$\mathcal{C}(A) = A \oplus A$$

by the rule:

$$(a, b) \cdot (c, d) = (ac - d^* b, bc^* + da)$$

The involution is:

$$(a, b)^* = (a^*, -b)$$

We call this process the Cayley-Dickson process $\mathcal{C}(A)$.

How to get cross products

- $\mathbb{R}$ has the trivial involution
- $\mathcal{C}(\mathbb{R}) \simeq \mathbb{C}$ with standard involution.
- $\mathcal{C}(\mathbb{C}) \simeq \mathbb{H}$ (Hamilton Quaternions) with standard involution.
- $\mathcal{C}(\mathbb{H}) \simeq \mathbb{O}$ (Octonions) with standard involution.

Each of these 4 algebras $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ is a Hurwitz algebra. Consequently we definitely have $0, 1, 3, 7$ dimensional cross products!!!

How to get cross products

- $\mathbb{R}$ has the trivial involution
- $\mathcal{C}(\mathbb{R}) \simeq \mathbb{C}$ with standard involution.
- $\mathcal{C}(\mathbb{C}) \simeq \mathbb{H}$ (Hamilton Quaternions) with standard involution.
- $\mathcal{C}(\mathbb{H}) \simeq \mathbb{O}$ (Octonions) with standard involution.

Each of these 4 algebras $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ is a Hurwitz algebra.
Consequently we definitely have $0, 1, 3, 7$ dimensional cross products!!!
By projecting onto the orthogonal complement of the unit element we can get a cross product:

$$(0, \vec{x})(0, \vec{y}) = (\vec{x} \cdot \vec{y}, \vec{x} \times \vec{y}).$$

Low Dimension Classification

We claim that if A is any (Euclidean) Hurwitz algebra over $\mathbb{R}$ and has dimension at least:

- 1 then A has $\mathbb{R}$ as a subalgebra.
- 2 then A has $\mathbb{C}$ as a subalgebra.
- 3 then A has $\mathbb{H}$ as a subalgebra, so is dimension at least 4.
- 5 then A has $\mathbb{O}$ as a subalgebra, so is dimension at least 8.
- 9 then A has $\mathcal{C}(\mathbb{O})$ as a subalgebra.

The claim will follow from the following:
Let A be a Hurwitz algebra, let B be a subalgebra, and let $i \in A$ be an element such that $B \cdot i = 0$ and $i \cdot i = 1$. Then the algebra generate by B and i is isomorphic to:

$$\mathcal{C}(B)$$

under the natural map:

$$(B \oplus B) \to A$$

given by:

$$(a, b) \mapsto a + bi.$$

Low Dimension Classification - Proof Sketch

Proof idea:
By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a, 0)(b, 0), (a, 0)(0, b), (0, a)(b, 0), (0, a)(0, b).$$

Proof idea:

By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a, 0)(b, 0), (a, 0)(0, b), (0, a)(b, 0), (0, a)(0, b).$$

We can simply further by assuming that $a = (x_0, \vec{x})$ and $b = (y_0, y'\vec{x} + \vec{y})$ with $\vec{x} \cdot \vec{y} = 0$.

Low Dimension Classification - Proof Sketch

Proof idea:
By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a,0)(b,0),(a,0)(0,b),(0,a)(b,0),(0,a)(0,b).$$

We can simply further by assuming that $a=(x_0,\vec{x})$ and $b=(y_0,y'\vec{x}+\vec{y})$ with $\vec{x}\cdot\vec{y}=0$.

If we can express the non-commutativity and non-associativity of a,b and i we can describe the algebra they generate.

Proof idea:
By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a, 0)(b, 0), (a, 0)(0, b), (0, a)(b, 0), (0, a)(0, b).$$

We can simply further by assuming that $a = (x_0, \vec{x})$ and $b = (y_0, y'\vec{x} + \vec{y})$ with $\vec{x} \cdot \vec{y} = 0$.

If we can express the non-commutativity and non-associativity of a, b and i we can describe the algebra they generate.

On the A side they fall into the cases I already showed you!!!

Proof idea:
By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a,0)(b,0),(a,0)(0,b),(0,a)(b,0),(0,a)(0,b).$$

We can simply further by assuming that $a = (x_0, \vec{x})$ and $b = (y_0, y'\vec{x} + \vec{y})$ with $\vec{x} \cdot \vec{y} = 0$.

If we can express the non-commutativity and non-associativity of a, b and i we can describe the algebra they generate.

On the A side they fall into the cases I already showed you!!!

On the $\mathcal{C}(B)$ side it is easy to see these relations are the same.

Proof idea:

By linearity it will suffice to check this is a homomorphism on products of the form:

$$(a, 0)(b, 0), (a, 0)(0, b), (0, a)(b, 0), (0, a)(0, b).$$

We can simply further by assuming that $a = (x_0, \vec{x})$ and $b = (y_0, y'\vec{x} + \vec{y})$ with $\vec{x} \cdot \vec{y} = 0$.

If we can express the non-commutativity and non-associativity of a, b and i we can describe the algebra they generate.

On the A side they fall into the cases I already showed you!!!

On the $\mathcal{C}(B)$ side it is easy to see these relations are the same.

The details of the proof are a tedius case analysis on the above cases.

Non-Existence in Large Dimensions

Non-Existence in Large Dimensions

The algebra $\mathcal{C}(\mathbb{O})$ is called the sedonions.

Non-Existence in Large Dimensions

The algebra $\mathcal{C}(\mathbb{O})$ is called the sedonions.
The algebra $\mathcal{C}(\mathbb{O})$ is not a Hurwitz algebra.

The algebra $\mathcal{C}(\mathbb{O})$ is called the sedonions.
The algebra $\mathcal{C}(\mathbb{O})$ is not a Hurwitz algebra.
In particular it has elements $i + j$ which are zero divisors, but where $(i+j)^2 = -2$. eg:

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0)$$

is zero and

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)$$

is -2

Non-Existence in Large Dimensions

The algebra $\mathcal{C}(\mathbb{O})$ is called the sedonions.
The algebra $\mathcal{C}(\mathbb{O})$ is not a Hurwitz algebra.
In particular it has elements $i + j$ which are zero divisors, but where $(i + j)^2 = -2$. eg:

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0)$$

is zero and

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)$$

is -2
Therefore, there is no cross product in dimensions larger than 7!!!

Non-Existence in Large Dimensions

The algebra $\mathcal{C}(\mathbb{O})$ is called the sedonions.
The algebra $\mathcal{C}(\mathbb{O})$ is not a Hurwitz algebra.
In particular it has elements $i+j$ which are zero divisors, but where $(i+j)^2 = -2$. eg:

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0)$$

is zero and

$$(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)$$

is -2
Therefore, there is no cross product in dimensions larger than 7!!!
So cross products only exist in dimensions:

$$0, 1, 3, \text{and } 7.$$

Where did I actually really need that I was working over $\mathbb{R}$?

What do you need to change/keep the same to do this for other fields like $\mathbb{C}$ or $\mathbb{Q}_p$ or $\mathbb{Q}$?

What can you say about the "Hermitian" inner products on $\mathbb{C}^n$?

The End

Thank You.