Dimensions of Spaces of Modular Forms

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The Goal of this talk to is explain the status of my 'thesis project'. That is, the project that motivated the tangentially related projects that actually became my thesis.

Dimensions of Spaces of Modular Forms on Orthogonal Locally Symmetric Spaces

The project was originally suggested to me by my PhD supervisor Eyal Goren.

I have been working on this with the help of Mike Roth.

Talk Plan:

- Explain the problem.
- Explain the main ideas in the approach, highlighting remaining difficulties.
- Explain what the final result will look like.

The Symmetric Spaces

Let G be a reductive group over $\mathbb{Q}$ and $K \subset G(\mathbb{R})$ a maximal compact subgroup such that the space:

$$D = G(\mathbb{R})/K$$

is a Hermitian symmetric space (by Cartan's classification this assumption places strong restrictions on G).

By a theorem of Harish-Chandra there exists a parabolic subgroup P of $G_{\mathbb{C}}$ such that $P(\mathbb{C}) \cap G(\mathbb{R}) = K$ and hence:

$$D = G(\mathbb{R})/K \hookrightarrow G(\mathbb{C})/P(\mathbb{C}) = \check{D}$$

The space $\check{D}$ is a projective algebraic variety. The inclusion of D is as a complex analytic manifold (though you can view this as defining the $\mathbb{C}$-structure on D).

I am mostly interested in the case of G an orthogonal group attached to a quadratic form of signature $(2, n)$

The Locally Symmetric Spaces

Suppose $\Gamma \subset G(\mathbb{Q})$ is a (neat) arithmetic subgroup. By a theorem of Baily-Borel the space:

$$X = \Gamma \backslash D$$

has the (canonical) structure of a quasi-projective algebraic variety (over $\mathbb{C}$).
The projective space into which X embeds is precisely:

$$\overline{X} = Proj(\tilde{H}^0(D, \mathcal{O}_{\check{D}}(k)|_D)^{\Gamma})$$

where the grading is provided naturally by k.
This definition gives X a line bundle the sections of whose powers are modular forms.

(The $\tilde{\cdot}$ indicates growth conditions which I am glossing over and can be ignored completely in the orthogonal case whenever $n \geq 3$, and in most cases when $n = 2$).

Some Remarks

That the above works is actually a combination of big theorems of Baily-Borel and Satake.

- The spaces $\overline{X}$, were constructed analytically over $\mathbb{C}$ by Satake by taking a quotient of a subset of $\check{D}$ with a very specific topology.
- The bundle we are looking at can also be constructed directly on Satake's space by adding growth conditions near the boundary (by way of pull-back and the boundary of D in $\check{D}$). Baily-Borel proved that the sections of this bundle has a *Proj* whose associated analytic space is Satake's space.

The Question

What are the dimensions of the vector spaces $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(k))$?

Given any variety, this is something we might naturally want to know.

The biggest problem is that our explicit understanding of the variety $\overline{X}$ and its boundary is actually as a complex analytic variety from Satake's construction.
In order to get around this, it is helpful to be able to invoke theorems that let us sidestep actually understanding the geometry of this variety.

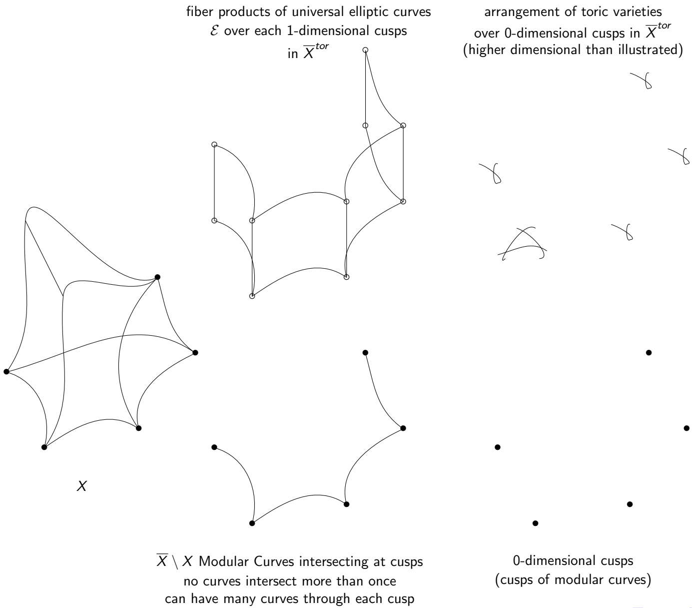
Step 0: Desingularizing $\overline{X}$

The variety X is regular, but the variety $\overline{X}$ almost never is, in particular, $\Delta = \overline{X} \setminus X$ may have very high codimension. For $O(2, n)$ varieties the codimension is at least $n - 1$.

Ash-Mumford-Rapaport-Tai constructed non-canonical desingularizations using families of admissible locally finite regular rational partial polyhedral cone decompositions of certain homogeneous self adjoint cones.
An example is the Hirzebruch resolution of singularities for Hilbert modular surfaces.

With their construction we obtain a variety $\overline{X}^{tor}$ which is a blowup of $\overline{X}$ along Δ.
By abuse of notation we will call $\Delta = \overline{X}^{tor} \setminus X$. The boundary consists of normal crossing divisors.

Picture of $\overline{X}^{tor}$ over $\overline{X}$ for $O(2, n)$



Step 1: Computing the correct thing is hard, so lets compute something different

Let Q_n denote the Riemann-Roch polynomial for line bundles L on Y of dimension n, that is the polynomial such that:

$$\chi(L) = Q_n(c_1(L); c_1(\Omega_Y), \cdots, c_n(\Omega_Y))$$

Theorem (Hirzebruch-Mumford)

Let C_Γ be the ratio of the normalized volumes of $\check{D}$ and $\overline{X}$. Then:

$$(-1)^n C_\Gamma Q_n(c_1(\mathcal{O}_{\check{D}}(k)); c_1(\Omega_{\check{D}}), \cdots, c_n(\Omega_{\check{D}})) =$$
$$Q_n(kc_1(\Omega^n_{\overline{X}^{tor}}(\log \Delta)); c_1(\Omega_{\overline{X}^{tor}}(\log \Delta)), \cdots, c_n(\Omega_{\overline{X}^{tor}}(\log \Delta)))$$

Note that the Euler characteristic of the bundles on $\check{D}$ are known. In the $O(2, n)$ case, $\check{D}$ is defined by a quadratic in $\mathbb{P}(n+1)$ and so in the left hand side of the above we obtain by the adjunction formula:

$$\binom{n+1-nk}{n} - \binom{n-1-nk}{n}$$

The Good and the Bad in this?

- Mumford showed $\Omega^n_{\overline{X}^{tor}}(\log\Delta)$ is the bundle of modular forms (also that $\Omega^n_{\overline{X}^{tor}}(\log\Delta)^{\otimes k-1}\otimes\Omega^n_{\overline{X}^{tor}}$ is the bundle of weight k cusp forms).
- $\Omega^n_{\overline{X}^{tor}}(\log\Delta)$ satisfies the conditions for Kodaira vanishing and so the Euler characteristic is useful. (At least for cusp forms by evaluating at $1-k$ (Serre duality also comes in... this also fixes that $(-1)^n$). For the remainder of this talk I will pretend this Euler characteristic is exactly what I want to compute).
- Sadly, the right hand side of the equation:

$$Q_n\left(kc_1(\Omega^n_{\overline{X}^{tor}}(\log\Delta)); c_1(\Omega_{\overline{X}^{tor}}(\log\Delta)),\cdots,c_n(\Omega_{\overline{X}^{tor}}(\log\Delta))\right)$$

isn't actually computing Euler characteristics, or anything obviously meaningful to us as:

$$\Omega_{\overline{X}^{tor}}(\log\Delta)\neq\Omega_{\overline{X}^{tor}}.$$

Step 2: So now that we have the wrong answer, what now?

The obvious question to ask is: what is the difference between:

$$Q_n(c_1(L); c_1(\Omega_{\overline{X}^{tor}}(\log \Delta)), \cdots, c_n(\Omega_{\overline{X}^{tor}}(\log \Delta)))$$

and

$$Q_n(c_1(L); c_1(\Omega_{\overline{X}^{tor}}), \cdots, c_n(\Omega_{\overline{X}^{tor}}))?$$

So, the second part of our strategy is to find 'universal' formulas to describe this difference. Note that in this talk I will focus on the case where L and X are the objects we are considering, even though my strategy does generalize.

Aside:

- Its not clear to me if anyone has worked on this in any generality, but this may be because I don't know the correct search terms. I would be curious to know of anywhere this is done.
- I suspect doing the above in general leads to Riemann-Hurwitz type results (presumably of the sort that are already well-known, but possibly 'stronger').

What makes us think we can compute this difference?

Some useful facts:

- $c_1(\Omega^n_{\overline{X}^{tor}}(\log \Delta))^\ell$ is supported off of cusps whose image in $\overline{X}$ has dimension less than ℓ.

  This bounds the degree of the error term as a polynomial in k.

- $c_i(\Omega_{\overline{X}^{tor}}) = \sum (-1)^j \Delta_j c_{i-j}(\Omega_{\overline{X}^{tor}}(\log \Delta))$ where Δ_j is the jth symmetric polynomial in the irreducible components of Δ.

  This tells us that there is a natural algebraic manipulation that makes it look like we are doing something useful.

- Logarithmic Chern classes restrict to the boundary divisors. $(c_i(\Omega_X(\log \Delta)) \cdot D = c_i(\Omega_D(\log \Delta')))$

  This means we can understand most of the error terms by studying only the boundary.

- Logarithmic Chern classes of toric varieties are trivial.

  Which is helpful as $\overline{X}^{tor}$ is constucted using toric varieties.

What is even better in the O(2,n) case?

What makes this really good in the $O(2,n)$ case:

- In the $O(2,n)$ case, the boundary of $\overline{X} \setminus X$ has dimension at most 1.
- In the $O(2,n)$ case, the intersection of any two non-equal boundary components of $\overline{X}^{tor} \setminus X$ is either empty, or a toric variety.
- In fact, the pieces of $\overline{X}^{tor} \setminus X$ over the 0-dimensional cusps of $\overline{X}$ are toric varieties.

The effect is that aside from self intersection terms over 1-dimensional cusps, all that remains is in the degree 0 (with respect to k) part.

What we obtain in $O(2,n)$ case

The error term when trying to 'drop the logs' consists of a symmetric expression in the following types of terms:

- Self intersection terms over the 1-dimensional boundary components of $\overline{X}$:

  $$D^{\ell} P_{\ell}\left(c_1(\Omega^n_{\overline{X}^{tor}}(\log\Delta)); c_1(\Omega_{\overline{X}^{tor}}(\log\Delta)), \cdots, c_n(\Omega_{\overline{X}^{tor}}(\log\Delta))\right)$$

  where D is an irreducible component of the boundary, and P_ℓ is a polynomial depending only on ℓ. (Only need to consider D over the 1-dimensional cusps).
- A purely combinatorial term:

  $$Q_n(0; \Delta_1, \cdots, \Delta_n)$$

We will first look at what can be done about P_ℓ.

Computing P_ℓ

Pretend there is a variety V of dimension ℓ such that:

$$ch(\Omega_V) = 1 - D,$$

and that Y is ANY variety of dimension $n - \ell$, and that L is a line bundle on Y, then a mental excerise reveals:

$$Q_n(c_1(L); c_i(\Omega_{V\times Y})) = D^\ell P_\ell(c_1(L); c_i(\Omega_Y))$$

The point is that all the terms containing D^ℓ are the same in both calculations. Moreover, as $V \times Y$ is a product and we know the dimensions... all the terms which do not vanish, contain D^ℓ.

But, properties of Todd classes then immediately reveal that

$$P_\ell = C_\ell Q_{n-\ell}$$

where C_ℓ is the Euler characteristic of the trivial bundle on V (with $D^\ell = 1$). This is true even if V does not exist as this is really all just a formal calculation in universal polynomials.

Some further hurdles

If it were true that $D^\ell \cdot c_i(\Omega_{\overline{X}^{tor}}(\log \Delta)) = c_i(\Omega_{D^\ell}(\log \Delta'))$, we would have something great. As then our error terms would be almost like (ignoring that we still have log terms) computing the Euler characteristic of our line bundle on D^ℓ (or at least its irreducible components).

Some 'complications':

- Logarithmic chern classes do not (necessarily) restrict cleanly to the self intersections of the boundary.
  This only works nicely when $\ell = 1$.
- D^ℓ is not likely to be irreducible.
  This means we need to deal with its irreducible components.
- We still have $(\log \Delta')$ on the boundary.
  Removing the $(\log \Delta')$ then becomes recursive, but now the entire boundary is toric varieties, so the recursion stops.

The first of these is the only serious issue.

How to Overcome the Problems

- There is a 'universal' polynomial relating $D^\ell \cdot c_i(\Omega_X)$ to $c_j(\Omega_{D^\ell})$ and symmetric polynomials in the divisors B_i equivalent to D.

$$(\prod_i B_i) ch(\Omega_X) = (\prod_i B_i) ch(\Omega_X(\log B_i)) \prod (1 - B_i),$$

  so by adding additional log terms we can restrict Chern classes.
- The technique that we used to relate P_ℓ and $Q_{n-\ell}$ is more general, and we can inductively apply a similar procedure using the above substitutions.

In the end we will have a weighted sum of terms:

$$Q_m(c_1(L); c_i(\Omega_{D^m}(\log \Delta')))$$

where D^m represents its various irreducible components. The weights depend on the intersection theory and the recursion.

Step 3: Some Intersection Theory on these Varieties

We have reduced the problem to two main things:

- Compute the appropriate weights for each irreducible component in the self intersection.

  Given $D \sim \sum_j a_{ij} E_{ij}$ for $i = 1, \ldots, n$ (where all E_{ij} intersect transversely) the formula arising from D^ℓ is roughly:

$$C_\ell \sum_{\underline{b}} \left(-\frac{1}{2}\right)^{|b|-\ell} \binom{|b|}{\ell} \prod_{i=1}^{\#b} \left(\sum_j a_{ij} b_{ij} \right) \prod_{i,j} E_{ij}^{b_{ij}} Q_{n-|b|}(\ldots)$$

  where $\underline{b}$ is a maxrix whose entries are all 0 or 1, $\#b$ is the number of non-zero rows of $\underline{b}$, $|b|$ is the number of non-zero entries of $\underline{b}$,

- So we need to be able to find the relations, and then compute the actual Euler Characteristics on these components:

$$\prod_{i,j} E_{ij}^{b_{ij}}.$$

What do the relevant pieces of the boundary look like?

Why is this easier than the original problem?

- There is a class of functions on $O(2,n)$ varieties called Borcherds forms. These have known divisors off the boundary (Heegner divisors), and we should be able to compute the divisors on the boundary. The divisors off the boundary are supported on $O(2,n-1)$-subvarieties, their interesection with the boundary, is a piece of their own boundary.
- Thus, for D an irreducible boundary component over a one dimensional cusp, the irreducible components of D^ℓ (for $\ell < n$) can be represented by cycles of the form $\overline{\mathcal{E}^{(n-\ell-1)}}$, compactifications of the $(n-1-\ell)$-fold fiber products of the universal elliptic curve over the modular curve (with some level structure).
- We still need to find some Borcherds forms which have zeros or poles along the boundary. (This is done in the $O(2,1)$,$O(2,2)$ (Heegner divisors $=$ Hirzebruch-Zagier-divisers),and $O(2,3)$ (Heegner divisors $=$ Humbert Surfaces) cases where formulas already exist!)

The facts and theorems we need:

- The bundle $\Omega^n_{\overline{X}^{tor}}(\log \Delta)$ is the pull back of the bundle of modular forms on $\overline{X}$.
- Restricting to the boundary components we thus find:

$$\Omega^n_{\overline{X}^{tor}}(\log \Delta)|_{\overline{\mathcal{E}(m)}} = \pi^*(M_k(\Gamma'))$$

  where $M_k(\Gamma')$ is the bundle of modular forms on the usual modular curve $\overline{Y}(\Gamma')$ of some level.
- The Leray spectral sequence and projection formulas tell us:

$$\chi(\Omega^n_{\overline{X}}(\log)|_{\overline{\mathcal{E}(m)}}) = \sum_i (-1)^i \chi(M_k(\Gamma') \otimes R^i \pi_* \mathcal{O}_{\overline{\mathcal{E}(m)}})$$

- If we assume $\overline{\mathcal{E}^{(n-\ell-1)}}$ can be deformed to $\overline{\mathcal{E}}^{n-\ell-1}$ by a series of blow-ups and blow-downs then we can work with $\overline{\mathcal{E}}^{n-\ell-1}$. Then cohomology and base change allows us to conclude that:

$$R^i \pi_* \mathcal{O}_{\overline{\mathcal{E}}^{(m)}} = \left((R^1 \pi_* \mathcal{O}_{\overline{\mathcal{E}}})^{\otimes i}\right)^{\oplus \binom{m}{i}}$$

Putting all of this together, plus the fact that the line bundle of modular forms is defined by pushforward of the canonical dualizing sheaf from the universal elliptic curve then:

$$\chi(\Omega^n_X(\log)|_{D^\ell}) = \sum_{i=0}^{n-1-\ell} (-1)^i \binom{n-1-\ell}{i} \chi(M_{k-i}(\Gamma')).$$

Now as $\chi(M_{k-i}(\Gamma'))$ is a degree 1 polynomial in i. This sum is 0 unless $m = n-1-\ell = 0, 1$. Otherwise it depends on the formula for $\chi(M_{k-i}(\Gamma'))$.

A Note on our Assumption:

Why should we be able to assume $\overline{\mathcal{E}^{(n-\ell-1)}}$ can be deformed to $\overline{\mathcal{E}}^{n-\ell-1}$ by a series of blow-ups and blow-downs?

Both are defined by certain cone decompositions and one expects a sequence of refinements to transform one to the other.

Step 5: The Combinatorial Terms

What about the:

$$Q_n(0; \Delta_1, \cdots, \Delta_n)$$

type expressions that will arise?

If $\overline{X}^{tor}$ where a toric variety, $Q_n(0; \Delta_1, \cdots, \Delta_n)$ would be computing the Euler characterstic of the trivial bundle, which is more or less the arithmetic genus.

- $\overline{X}^{tor}$ is not a toric variety, but virtually all of the intersection theory (except for the Δ_1^n term), is computed in toric subvarieties.
- The arithmetic genus is a birational invariant, which the term we are computing should be.
- The arithmetic genus for toric varieties is computed by way of the combinatorics of incidence relations.

Question: How to interpret $Q_n(0; \Delta_1, \cdots, \Delta_n)$ for an arrangement of toric varieties, and what is the significance of the pure self intersection terms?

Summary: What I still don't have a handle on.

- The combinatorics for the boundary, the construction of $\overline{X}^{tor}$ uses a not strictly effective existential proof.
  (Giving a computationally useful description is potentially a thesis problem on its own.)

  Consequently I can't compute the $Q_m(0; \Delta_1, \cdots, \Delta_n)$-type terms. Even though, the final answer **can't** actually depend on the non-canonical choice of boundary.
- A good way to construct a family of Borcherds forms for which the collection of all divisors intersect transversely.

  Again, using different collections of Borcherds forms must give the same answer, even though such choices are not canonical.
- A way to check the assumption regarding: $\overline{\mathcal{E}^{n-\ell-1}}$ and $\overline{\mathcal{E}}^{n-\ell-1}$, or a work around if the assumption does not hold.
  To formally check in any case requires knowing solving problem 1.

Summary: What the formula should look like.

The main contribution:

$$C_\Gamma\left(\binom{n+1-nk}{n}-\binom{n-1-nk}{n}\right)$$

Error terms for 1-dimensional boundary component $D \subset \overline{X}$:

$$\sum_D A(D)\chi(M_k(\Gamma(D))) + B(D)$$

- $A(D)$ depends only on a collection of Borcherds forms with poles at D.
- $B(D)$ depends only on the genus of D, the structure of cusps of D, and a collection of Borcherds forms.

Error terms for 0-dimensional boundary components $D \subset \overline{X}$:

$$\sum_D B(D)$$

- $B(D)$ depends only on $Q_n(0;\Delta_1,\cdots,\Delta_n)$ for Δ restricted to the arrangement of toric varieties in the fiber over D.

Oh yeah... and replace k by $1-k$ to actually get the dimension of cusp forms... (which works on $M_k(\Gamma(D))$ also).

The Future

I would like to actually find at least one case (which ideally isn't already solved) where I can describe the combinatorics well enough to actually write down a meaningful answer.

More ambitiously I would like to be able to completely express the 'answer' in terms in terms of a clean general purpose procedure. (This would include describing how to find the collection of Borcherds forms).

Even better, I would like the formula to express the fact that non-canonical choices don't matter.

The End

Thank you.