Kuga Varieties Applications

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Two applications

The goal of this talk is to describe two applications of Kuga varieties. These are:

1. Applications to the Hodge conjecture.
   (Abdulali - Abelian Varieties and the General Hodge Conjecture and Hodge Structures on Abelian Varieties of Type III)
2. Relating modular forms to Galois representations and deducing properties of these.
   (Deligne - Forme Modulaires et Representation l-adic)

The standard Hodge Conjecture tells us that via the cycle class map the space of algebraic cycles of co-dimension k generate the space of rational hodge cycles. More precisely

Conjecture (Hodge Conjecture)

Let X be a smooth projective complex variety then

$$H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X) = c\ell(Z^k(X))$$

where $c\ell$ associates to each cycle its fundamental class.

General Hodge Conjecture

For a more general formulation we need to define several filtrations:

- The arithmetic filtration: $F_a^r H^n(X, \mathbb{Q})$ is the span of the cohomology classes supported on codimension r subvarieties. (The pushforwards of cohomology classes of subvarieties of codimension r)
- The Hodge filtration: $F^r H^n(X, \mathbb{C}) = \oplus_{p+q=n, p\geq r} H^{p,q}(X)$.
- Grothendiecks corrected filtration: $F_{\mathbb{Q}}^r H^n(X, \mathbb{Q})$ is the largest $\mathbb{Q}$-hodge sub-structure of $F^r H^n(X, \mathbb{C}) \cap H^n(X, \mathbb{Q})$.

Conjecture (General Hodge Conjecture)

Let X be a smooth projective complex variety then in the notation above:

$$F_a^r H^n(X, \mathbb{Q}) = F_{\mathbb{Q}}^r H^n(X, \mathbb{Q})$$

We remark that for $r = n/2$ this gives the usual Hodge conjecture.

Dominating Families

Definition

We say a class $\mathcal{Y}$ of varieties **dominates** X if for every irreducible hodge structure M in $H^n(X, \mathbb{Q})$ there exists a $Y \in \mathcal{Y}$ and $W \subset H^s(Y, \mathbb{Q})$ such that W is equivalent to M and $h(M) := \min\{p | W^{p,\dim(Y)} \neq 0\} = 0$.

Proposition (Grothendieck)

Suppose X is dominated by $\mathcal{Y}$ and for all $Y \in \mathcal{Y}$ the Hodge conjecture holds for $X \times Y$ then the general Hodge conjecture holds for X.

($\Leftarrow$) If M is supported on Z then $f_* : H^{n-2h(V)}(Z, \mathbb{Q}) \to H^n(X, \mathbb{Q})$ is a morphism of hodge structures whose image contains V, hence $H^{n-2h(V)}(Z, \mathbb{Q})$ contains W.
($\Rightarrow$) The equivalence of W and M is induced by an element $\xi \in H^{2\dim(Y)+2h(M)}(Y \times X, \mathbb{Q})$. By the Hodge conjecture $\xi = c\ell(Z)$. That M is supported on $p_{2*}(Z)$ implies $V \subset F_a^{h(V)} H^n(X, \mathbb{Q})$.

What is known?

Very few cases of the Hodge conjectures are actually known. There are special cases by

- Type I, II abelian varieties whose Hodge rings generated by divisors (Hazama, Tankeev, Mattuck, Gordon)
- Powers of CM elliptic curves (Shioda)
- Powers of certain CM abelian varieties (Tankeev)
- Certain 4 dimensional abelian varieties with CM by $\mathbb{Q}(i)$ (Schoen).

Based on the above, it seems that we might have better luck proving results in special cases where we have extra structure. (It actually turns out that less special abelian varieties are easier to handle.)

Hodge Group

Definition

Let A be a (polarized) abelian variety, the **Hodge group** $G(A)$ of A is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(H^1(A,\mathbb{Q}))$ through which the morphism defining the Hodge structure on $H^1(A,\mathbb{Q})$ factors. The **Lefschetz group** $L(A)$ of A is the centralizer of $\mathrm{End}(A)\otimes\mathbb{Q}$ in $\mathrm{Sp}(H^1(A,\mathbb{Q}))$.

We have natural inclusions $G(A) \subset L(A) \subset \mathrm{Sp}(H^1(A,\mathbb{Q}))$ and by the constructions we have seen previously these inclusions give rise to Kuga varieties $\mathcal{H}(A) \subset \mathcal{L}(A) \subset \mathcal{A}$ having A as a fiber.

Definition

A is said to be of **PEL-type** if its Hodge group is that of a general fibre of a PEL-family (Sp_n,SO^*_{2n},$\mathrm{SU}_{p,q}$). Equivalently if the Lefschetz group equals the Hodge group.

"The smallest $\mathbb{Q}$-subvariety of the moduli space to which A belongs is a PEL-family"

Endomorphisms of Abelian Varieties

We categorize simple abelian varieties by their endomorphism rings $\mathrm{End}(A) \otimes \mathbb{Q}$.

I A totally real number field.

II A totally indefinite quaternion algebra over a totally real field.

III A totally definite quaternion algebra over a totally real field.

IV A division algebra whose center is a CM-field.

Having a complete list of the possibilities allows us to classify the possible Lefshetz groups. (Unfortunately the standard numbering for these groups doesn't match that for endomorphism rings (I,II are Sp_n,III is SO^*_{2n},IV is $\mathrm{SU}_{p,q}$).)

Some Positive Results

Theorem (Abdulali)

Let A be an abelian variety of PEL-type with semi-simple Hodge group and such that for all type III simple factors B of A the cohomology $H^1(B, \mathbb{Q})$ has odd dimension over $\mathrm{End}(B)$. Then A is dominated by powers of itself.

(By the work of Shimura type III factors as above do exist in characteristic 0. Though the dimension of $H^1(B, \mathbb{Q})$ can never be 1 (over $\mathrm{End}(B)$) thus B has dimension at least 6)

In some cases it is known that the Hodge conjectures for A would imply them for their powers, combining this with explicit cases where the Hodge conjecture is known gives us the general Hodge conjectures for these varieties. As an example, for type I and II this holds when the Hodge rings of A^n are generated by divisors.

Some Negative results

Theorem (Abdulali)

Let A be a simple abelian variety of type III which is a general member of a PEL-family. Let d be the discriminant of the skew-hermitian form for the polarization of A. Then A is dominated by its powers if and only if d is not a square.

Theorem (Abdulali)

If d as above is a square and $\dim(H^1(A,\mathbb{Q})) > 4$ (dimension over $\mathrm{End}(A)$) then A is not dominated by the class of all abelian varieties.

(If d as above is a square and $\dim(H^1(A,\mathbb{Q})) = 4$ then A dominate by $B^n \times A^m$ for some abelian variety B.)

Ideas of Proofs

In both the affirmative and negative cases, the knowledge of the Hodge groups for general members of PEL families together with Satake's classification of the possible Kuga varieties plays a crucial role.

Using this classification one is able to reduce the problem to that of either explicitly showing a highest weight vector can or can not be found in the powers of some other representation. Note:

- The problem reduces to considering simple factors in an isogenous abelian variety.
- To show something dominates one needs to show that for each irreducible Hodge substructure U in the cohomology of A^a we can find one in $H^b(A^c, \mathbb{R})$ containing $(b, 0)$ forms which is $G(A^a)$ equivalent to U.

More details Positive

1. By results of Satake decompose $H^1(A, \mathbb{R}) = \oplus V_\alpha$ where each V_α is acted upon by a unique simple factor of $G(A)$. It suffices to treat simple factors.
2. Via case by case analysis on the (G, V) look at the conditions under which $\Lambda^c V^b$ contains a given representation.
3. For type III the interesting case is the top symmetric power. Over $\overline{\mathbb{Q}}$ this splits as two irreducibles only one of which contains a $(b, 0)$-form.
   When d is not a square the Galois action on roots implies this splitting does not happen so the two representations always appear together.
4. Conclude that large values of c allow us to find the Hodge structure we are looking for.

More details Negative

1. Conversely to the above, we find there exists an irreducible representation of $G(A)$ which is defined over $\mathbb{Q}$ and contains no $(b, 0)$ forms.
   This gives us an effective Hodge structure which can never appear in the cohomology of the powers of A.
2. Show that Kuga varieties arising from equivalent H_2 representations give isogenous abelian varieties.
3. Supposing A is dominated by B we show that $G(A) \times G(B) \supset G(A \times B)$ is isogenous to $G(A)$.
4. Via Satake's classification we conclude $G(A), G(B)$ are equivalent H_2 representations so that B is isogenous to a power of A.
5. The dimension 4 case is a consequence of exceptional isomorphisms in the classification.

The second applications is a way to construct sheaves for modular forms and Galois representations. First a few definitions:
Denote by $Y(\Gamma)$ the open modular curve, $X(\Gamma)$ its usual compactification and $\pi : \mathcal{E}(\Gamma) \to X(\Gamma)$ the universal elliptic curve having zero section e.
Define $\omega = e^*\Omega^1_{\mathcal{E}/X}$, $T_{\mathbb{Z}}(\mathcal{E}) = ((R^1\pi_*)\underline{\mathbb{Z}})^\vee$.
There are exact sequence of sheaves of sections (over X):

$$1 \to T_{\mathbb{Z}}(\mathcal{E}) \to \omega^{-1} \to \mathcal{E} \to 1$$

$$1 \to \omega \to (R^1\pi_*\underline{\mathbb{R}}) \otimes_{\mathbb{R}} \mathcal{O}_X \to \omega^{-1} \to 1$$

We can view these fiberwise as being the realization of E as a quotient of its tangeant space by its homology and the Hodge splitting of de Rham cohomology.

Definition of Modular Forms

One checks that $\Omega^1_Y = \omega^2|_Y$, and that the inclusion $\Omega^1_X \to \omega^2$ amounts to requiring simple zeros at the cusps.

Definition

We define the space of cusp forms of weight $k+2$ for Γ to be:

$$H^0(X(\Gamma), \Omega^1_X \otimes \omega^k)$$

Proposition

The de Rham resolution of the map $\Omega^1 \otimes \omega^k \to \Omega(\mathrm{Sym}^k(R^1\pi_*)\underline{\mathbb{R}})$ yields an isomorphism:

$$sh : H^0(X, \Omega^1 \otimes \omega^k) \oplus \overline{H^0(X, \Omega^1 \otimes \omega^k)} \to \tilde{H}^1(X, \mathrm{Sym}^k(R^1\pi_*)\underline{\mathbb{R}} \otimes \mathbb{C})$$

($\tilde{H}$ indicates the image of cohomology with compact support.)

Galois Representation

In order to actually construct Galois representations and compare them to the Hecke modules we need a further construction: Consider the $\mathbb{Q}$-vector spaces:

$$_nW = \tilde{H}^1(X(\Gamma(n)), \mathrm{Sym}^k(R^1\pi_{n*}\underline{\mathbb{Q}}))$$

And the $\mathbb{Q}_\ell$-vector space (and Galois module):

$$_nW_\ell = {}_nW \otimes \mathbb{Q}_\ell = \tilde{H}^1(X(\Gamma(n)) \otimes \overline{\mathbb{Q}}, \mathrm{Sym}^k(R^1\pi_{n*}\underline{\mathbb{Q}_\ell}))$$

Let $s : X(\Gamma(n)) \to Spec(\mathbb{Z})$ then:

Proposition

$_nW_\ell$ is the fibre over $\mathbb{Z}[1/n, 1/\ell]$ at $\overline{\mathbb{Q}}$ of $(R^i\tilde{s})(\mathrm{Sym}^k(R^1\pi_{n*}\underline{\mathbb{Z}_\ell})) \otimes \mathbb{Q}_\ell$ (image with proper support)

We have seen that:

$$ {}_nW \otimes \mathbb{C} = H^0(X, \Omega^1 \otimes \omega^k) \oplus \overline{H^0(X, \Omega^1 \otimes \omega^k)} $$

The above construction thus allows us to relate a space of modular forms as a Hecke module to a Galois module.
One can show that the construction of Hecke operators is compatible with base change to $\mathbb{F}_p$, we may thus compute and compare T_p,F_p,V_p on this special fibre over p. Doing this we obtain the usual results:

$$ T_p = F + I_p^* V \qquad FV = p^{k+1} $$

$$ (1 - T_p X + p^{k+1} I_p^* X^2) = (1 - FX)(1 - I_p^* VX) $$

(I_p is the map $(E, \alpha) \mapsto (E, \alpha/p)$)

Relation to Universal Elliptic Curve

Consider $\pi_k : \mathcal{E}^k \to X$, the k-fold product of the universal elliptic curve over X. (It is worth pointing out that this is a Kuga variety.) We have that $\mathrm{Sym}^k R^1\pi_*\underline{\mathbb{Z}}$ is a direct factor of $(R^1\pi_*\underline{\mathbb{Z}})^k$, which in turn by the Kunneth formula is a direct factor of $R^k\pi_{k*}\underline{\mathbb{Z}}$. Consequently, the geometry of $\mathcal{E}^k$ and properties of its cohomology give us properties of associated Galois representations.

Theorem (Deligne)

The Weil conjectures for $\mathcal{E}^k$ imply the Ramanujan-Peterson conjectures on Hecke eigenvalues.

Key arguement is that if S is smooth over $\mathbb{F}_p$, $f : A \to S$ an abelian scheme such that $A \subset A^*$ is an open in a smooth projective scheme over $\mathbb{F}_p$ then the eigenvalues of F on $\tilde{H}^i(\overline{S}, R^j f_*\underline{\mathbb{Q}_l})$ are algebraic integers of absolute value $p^{i+j/2}$. Then construct such a model of $\mathcal{E}^k$. Finally use $\det(1 - T_pX + p^{k+1}X^2) = \det(1 - FX)$.

The End

Thank you.