Characterization of Special Points on Orthogonal Shimura Varieties

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The Goal

The goal of this talk to explain some of the results which are in my thesis. Most of these results can also be found in my paper:

Andrew Fiori *The Characterization of Special Points on Orthogonal Symmetric Spaces*, Journal of Algebra, Volume 372, 15 December 2012, Pages 397-419.

which the title suggests gives a characterization of the special fields associated to special point on orthogonal Shimura varieties.

So the first thing I will do, is explain how the problem I solve relates to this.

Motivation - Special Points on Shimura Varieties

The (complex) points on the Shimura variety associated to the group $G_{\mathbb{Q}}$ correspond to maps:

$$h : \mathbb{S}_{\mathbb{R}} = Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}$$

satisfying several axioms.

These points are called **special** if they factor through an algebraic torus, T defined over $\mathbb{Q}$.

$$h : \mathbb{S}_{\mathbb{R}} \to T_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$$

Why are Special Points Special?

Given a special point:

$$h : \mathbb{S}_{\mathbb{R}} \to T_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$$

there is an associated **special field** $E/\mathbb{Q}$ which will be a subfield of the splitting field of $T_{\mathbb{Q}}$.

Part of the importance is the following:
For the canonical model of the Shimura variety, the the field of definition of a special point is a subfield of the Hilbert class field of (a field closely related to) E.

This fact in the case of the upper half plane is part of the theory of complex multiplication and gives us an explicit class field theory for quadratic imaginary fields.

In Which Groups Are We Interested?

As the title suggested, we are interested in orthogonal groups.

For the remainder of this talk k is either a number field or a completion (p-adic, real or complex) of a number field.

Definition

A **quadratic space** over k is a vector space V and a map $q : V \to k$ where $q(x+y) - q(x) - q(y)$ is bilinear.

Definition

Let (V, q) be a quadratic space over k. The **orthogonal group** is the algebraic group whose points over a field K/k are:

$$O_q(K) := \{g \in \mathrm{GL}(V \otimes_k K) \mid q(gx) = q(x)\ \forall x \in V \otimes_k K\}$$

Invariants of Quadratic Forms

Understanding orthogonal groups involves understanding the underlying quadratic form, so we will need to make use of the invariants of these.

Given a quadratic form q we may fix a basis $e_1, \dots e_n$ in which we may express the form as:

$$q(x_1 e_1 + \dots x_n e_n) = \sum a_i x_i^2.$$

The following invariants determine q up to isomorphism over k:

- $D(q) = \prod_i a_i$ the **discriminant**.
- For ν a place of k, $H_\nu(q) = \prod_{i<j}(a_i, a_j)_\nu$ the **Hasse invariant**.
- For each $\rho : k \to \mathbb{R}$, the **signature**, $(r^+, r^-)_\rho = (|\{i \mid \rho(a_i) > 0\}|, |\{i \mid \rho(a_i) < 0\}|)_\rho$.

Orthogonal Groups with Shimura Varieties

There are Shimura varieties associated to orthogonal groups of signature $(2, n)$ over $\mathbb{Q}$.

Example

The following are well known examples (note that the statements about special fields may be misleading):

- The $(2, 1)$ case includes the modular curve and Shimura Curves. Here special fields are the quadratic CM-fields which split an associated quaternion algebra.
- The $(2, 2)$ case includes Hilbert modular surfaces. Here the special fields are quartic CM-fields containing $\mathbb{Q}(\sqrt{D})$.
- The $(2, 3)$ case includes the Siegel space of dimension 3. Here the special fields are quartic CM-fields with no conditions.

The classifications of special points are reasonably well known in these lower dimensional examples, but where do they come from?

We can now state our goal as the following:
Understand the special fields that can be associated to the special points for the Shimura varieties associated to O_q *where* q *is a quadratic form of signature* $(2, n)$ *over* $\mathbb{Q}$*.*

This problem relates to a special case of the problem:

Characterize all the maximal tori:

$$T_k \hookrightarrow O_q$$

where q *is any quadratic form over a number field* k*.*

This is the problem I will now focus on. First we will discuss the basic objects that are at play.

Algebraic Tori and Galois Descent

An **algebraic torus** T over k is an algebraic group over k such that $T_{\overline{k}} \simeq \mathbb{G}_m^n$ for some n.

Theorem

There is a (contravariant) equivalence of categories between the category of algebraic tori over k and the category of finitely generated $\mathbb{Z}$-torsion free continuous $\mathbb{Z}[\Gamma]$-modules.

The above correspondence takes:

$$T \mapsto \mathrm{Hom}_{\overline{k}}(T, \mathbb{G}_m)$$

and takes the (finitely generated $\mathbb{Z}$-torsion free continuous) $\mathbb{Z}[\Gamma]$-module M to:

$$M \mapsto \mathrm{Spec}((\overline{k}[M])^{\Gamma}).$$

Étale Algebras and Galois Descent

By an **étale algebra** E over k we mean a finite product $E = \prod_i E_i$ of separable field extensions E_i over k.

Theorem

There is an equivalence of categories between the category of étale algebras over k and the category of finite $\Gamma = \mathrm{Gal}(\overline{k}/k)$-sets.

The above correspondence takes:

$$E \mapsto \mathrm{Hom}_k(E, \overline{k})$$

and takes the Galois set $X = \{\rho_1, \ldots \rho_n\}$ to:

$$X \mapsto \left(\bigtimes_{\rho \in X} \overline{k}_\rho \right)^\Gamma.$$

Étale Algebras and Algebraic Tori

Given an étale algebra E we define the algebraic torus T_E whose points over a field K/k are:

$$T_E(K) := (E \otimes_k K)^\times$$

This corresponds to taking the $\mathbb{Z}[\Gamma]$-module $\bigoplus_{\rho \in \mathrm{Hom}_k(E,\overline{k})} \mathbb{Z}e_\rho$ where the Galois action is by permutation of the e_ρ via the natural action of Γ on $\mathrm{Hom}_k(E,\overline{k})$.

Theorem

For every algebraic torus T over k there exists an étale algebra E over k and a morphism $\tau : T \hookrightarrow T_E$.

Suppose X is any Galois stable spanning set for the $\mathbb{Z}[\Gamma]$-module $\mathrm{Hom}_{\overline{k}}(T, \mathbb{G}_m)$. Then we can take E to be the étale algebra associated to X. This is not unique.

If done properly one of the field factors of E should be the special field.

Étale Algebras with Involution and Tori

By an **étale algebra with involution** (E, σ) over k we mean an étale algebra E over k together with an involution $\sigma : E \to E$ such that $dim_k(E) = 2dim_k(E^\sigma)$.

Given an étale algebra with involution $(E, \sigma)/k$ we define the algebraic torus $T_{E,\sigma} \subset T_E$ by specifying that its points over a field K/k are:

$$T_{E,\sigma}(K) := \{x \in (E \otimes_k K)^\times \mid x\sigma(x) = 1\}$$

As an example taking with $(E, \sigma) = (\mathbb{C}, \bar{\cdot})$ we obtain:

$$T_{E,\sigma} = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) | a^2 + b^2 = 1\} = \mathrm{SO}_{2,\mathbb{R}} .$$

So we have found an orthogonal group, can this be generalized?

Étale Algebras with Involution and Quadratic Forms

Given an étale algebra with involution $(E,\sigma)/k$, and $\lambda \in (E^\sigma)^\times$. We define a quadratic form $q : E \to k$ by:

$$q_{E,\sigma,\lambda}(x) = \frac{1}{2} Tr_{E/k}(\lambda x \sigma(x)).$$

This makes $(E, q_{E,\sigma,\lambda})$ a quadratic space over k of dimension $dim(E)$.

Proposition

For all $\lambda \in (E^\sigma)^\times$ there is a natural injective map:

$$T_{E,\sigma} \hookrightarrow O_{q_{E,\sigma,\lambda}}$$

induced by the action of multiplication of $E^\times$ on E.

This is a nice construction, but does it give us everything?

The First Characterization of Tori in Orthogonal Groups

Proposition

Let (V, q) be a quadratic space over k of even dimension. If a torus

$$T \hookrightarrow O_q$$

as a maximal torus then:

- *There exists (E, σ) over k with $T \simeq T_{E,\sigma}$.*
- *There exists $\lambda \in (E^\sigma)^\times$ with $q \simeq q_{E,\sigma,\lambda}$.*

Taking E to be the span of $T(k)$ in $\mathrm{GL}(V)$ and σ the adjoint with respect to q gives us (E, σ) concretely when V is even dimensional. Finding λ is more subtle.

This gives us our first characterization of the maximal tori in O_q.

The Invariants of $q_{E,\sigma,\lambda}$

Given q and (E,σ) how would we know if this is possible? Does there exist λ such that $q \simeq q_{E,\sigma,\lambda}$?

The invariants of $q_{E,\sigma,\lambda}$ are:

- $D(q_{E,\sigma,\lambda}) = (-1)^{n/2}\delta_{E/k}$.
- $H(q_{E,\sigma,\lambda}) = Cor_{E^\sigma/k}((((-1)^{n/2}\lambda f_z'(z), z)_{E^\sigma})(-1,-1)_k^{n(n-2)/8}$ where $z \in E^\sigma$ is such that $\sqrt{z}$ primitively generates E and f_z is its minimal polynomial.
- Signature is $(2r+2s+t, n-(2r+2s+t))_\rho$ where
  - s is the number of real places of E^σ over ρ that become complex in E having $\lambda > 0$,
  - r is the number of complex places of E^σ over ρ, and
  - t is the number of real places of E^σ over ρ that stay real in E.

(Hasse invariant uses results of Brusamarello, Chuard-Koulmann, Morales)

Second Characterization

We conclude $T_{E,\sigma} \hookrightarrow O_q$ if $\delta_{E/k} = (-1)^{n/2} D(q)$ and there exists $\lambda \in E^\sigma$ such that:

- $Cor_{E^\sigma/k}(((-1)^{n/2} f_z'(z)\lambda, z)) = H(q)(-1,-1)_k^{n(n-2)/8}$.
- It gives the right signature.

It is easy to check the discriminant and signature conditions.
It is not totally apparent when a λ satisfying the Hasse invariant conditions exists.

Our next goal will be to rephrase the conditions from an abstract existence statement in λ to a condition more intrinsic in (E, σ).

σ-types for Étale Algebras with Involutions

This definition imitates the definition of CM-types and CM-reflex fields which is prevalent in the theory of Complex multiplication for abelian varieties and consequently in the theory of canonical models of Shimura varieties.

Definition

Given (E, σ) a σ-**type** is $\phi \subset Hom(E, \overline{k})$ a subset such that:

$$\phi \cap \sigma(\phi) = \emptyset \text{ and } \phi \cup \sigma(\phi) = Hom(E, \overline{k}).$$

For example for CM-fields, a σ-type amounts to choosing precisely one complex embedding of E for each pair of complex conjugate embeddings.

Given (E, σ) we can consider Φ the collection of all of its σ-types. The set Φ comes with a natural action of both Γ and σ, these actions commute. For a σ-type ϕ we shall denote by $\Gamma\phi$ the Galois orbit of ϕ in Φ and by $\Gamma\phi\sigma$ the Galois orbit of $\sigma(\phi)$.

Definition

To each σ-type ϕ of (E, σ) we associate a σ-**reflex algebra** (E^{ϕ}, σ) by way of the Galois set:

$$\Gamma\phi \cup \Gamma\phi\sigma \subset \Phi.$$

We also may consider the **complete reflex algebra** (E^{Φ}, σ) by way of the Galois set Φ.

Computing Reflex Algebras

The group $\Gamma = Gal(\overline{k}/k)$ acts on the set Φ of all σ-types. Set Γ_ϕ to be the stabilizer of ϕ. We may check that the reflex algebra is the following:

$$E^\phi = \begin{cases} \overline{k}^{\Gamma_\phi} & \Gamma\phi\sigma = \Gamma\phi \\ \overline{k}^{\Gamma_\phi} \oplus \overline{k}^{\Gamma_\phi} & \text{otherwise} \end{cases}$$

This characterization allows us to show:

Theorem

The complete reflex algebra E^Φ has trivial factors if and only if $E = E_1 \times E_2$ where $E_1 \simeq E_2$ are interchanged by σ. If E^Φ has no trivial factors then all its factors have even degree.

Associated to a quadratic space (V, q) is the even Clifford algebra:

$$Cliff^+(V,q) = \oplus_n V^{\otimes 2n}/(x \otimes x - q(x))$$

There exists a canonical involution τ on $Cliff^+(V,q)$.

Theorem

If $T_{E,\sigma}$ is a maximal torus in O_q then the complete reflex algebra (and hence for all reflex algebras) we have $E^\Phi \hookrightarrow Cliff^+(V,q)$ with the canonical involution restricting to σ on E^Φ.

Decompose $E \otimes_k \overline{k} = \underset{\rho}{\times} \overline{k} e_\rho$. Then define:

$$e_\phi = \prod_{\rho \in \phi} \frac{1}{\rho(\lambda)} e_\rho \otimes e_{\rho \circ \sigma} \in Cliff^+(V,q) \otimes \overline{k}.$$

The collection e_ϕ for $\phi \in \Phi$ are orthogonal idempotents in $Cliff^+(V,q) \otimes \overline{k}$ on which the Galois group acts as it does on Φ. Thus:

$$E^\Phi = (\underset{\phi}{\times} \overline{k} e_\phi)^\Gamma \subset Cliff^+(V,q).$$

The Relationship to Tori in the Spin Group

The previous theorem tells us several interesting things:

1. It tells us that maximal tori in the spin group are associated to reflex algebras.
2. This in turn gives (some) justification for the role of reflex algebras in the theory of fields of definition for points on Shimura varieties.
3. If we wish to view an orthogongal Shimura variety as a moduli space of abelian varieties, it tells us something about the endomorphism algebras of its special points.
4. Finally, for the problem at hand It tells us that E^{Φ} splits the algebra $Cliff^{+}(V, q)$. It is this observation which led us to try to prove the next theorem.

Local Characterization (Non-archimedian)

Suppose k is a finite extension of $\mathbb{Q}_p$.

Theorem

Let (E, σ) be an étale algebra with involution of dimension n and (V, q) be a quadratic space of dimension n then the torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if:

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- E^{ϕ} *splits* $Cliff^{+}(V, q)$ *for all* σ*-types* ϕ*, equivalently* E^{Φ} *splits* $Cliff^{+}(V, q)$.

The Proof (Non-archimedian)

The idea of the proof is the following:

- $Cliff^+(V,q)$ is a matrix algebra over a quaternion algebra.
- E^ϕ splits it if and only if E^ϕ contains a quadratic subextension (or if the algebra is already split.)
- Every reflex algebra contains a quadratic extension if and only if E has a non trivial factor where 'z' is not a square (this was that fact we mentioned earlier).
- The Hasse invariant can be modified via a choice of λ if and only if E has a non trivial factor where 'z' is not a square (this is apparent if you stare at the formula long enough).

Local Characterization (Archimedean)

Suppose k is $\mathbb{R}$.

Theorem

Let (E,σ) be an étale algebra with involution of dimension n and (V,Q) be a quadratic space of dimension n then the torus:

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if the signature of q is of the form:

$$(2r+2i+t, n-(2r+2i+t)) \quad 0 \leq i \leq s$$

- *s is the number of real places of E^σ that become complex in E,*
- *r is the number of complex places of E^σ , and*
- *t is the number of real places of E^σ that stay real in E.*

Moreover, if $T_{E,\sigma} \hookrightarrow O_q$ then:

- *$D(q) = (-1)^{n/2}\delta_{E/k}$.*
- *E^ϕ splits $Cliff^+(V,q)$ for all ϕ.*

Local Everywhere Embedding

Now let k be a finite extension of $\mathbb{Q}$. We get locally everywhere embedding from global conditions.

Theorem

Let (E, σ) be an étale algebra with involution of dimension n and (V, Q) be a quadratic space of dimension n then the torus:

$$T_{E,\sigma} \otimes k_{\mathfrak{p}} \hookrightarrow O_q \otimes k_{\mathfrak{p}}$$

for all places (including infinite) $\mathfrak{p}$ of k if and only if

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- *E^ϕ splits Cliff$^+(V, q)$ for all ϕ.*
- *The signature condition is plausible at all real places.*

Local-Global?

Unfortunately, an example of Prasad - Rapinchuk shows a local global principle for the embedding of tori fails in general.

Example

Let $p_1 \cong 1 \pmod 4$ and $p_2 \cong 1 \pmod 8$ with p_2 a square mod p_1.
Set:

$$(F_1, \sigma_1) = (\mathbb{Q}(\sqrt{p_1}), \sqrt{p_1} \mapsto -\sqrt{p_1}) \qquad \text{and}$$

$$(F_2, \sigma_2) = (\mathbb{Q}(\sqrt{p_1})(\sqrt{p_2}), \sqrt{p_2} \mapsto -\sqrt{p_2})$$

where the action σ_2 on $\sqrt{p_1}$ is trivial.
Then $(F, \sigma) = (F_1 \times F_2, \sigma_1 \times \sigma_2)$ does not satisfy the local-global principle.
Indeed, if q is any quadratic form for which $T_{F,\sigma} \hookrightarrow O_q$ then letting $\tilde{q}$ be the quadratic form identical to q except with Hasse invariants at p_1, p_2 inverted, then $T_{F,\sigma}$ does not inject into $O_{\tilde{q}}$ even though it does locally everywhere.

Local-Global

Theorem (Prasad-Rapinchuk)

If for any two factors E_1, E_2 of E there exists a non-archimedian places $\mathfrak{p}$ of k and $\mathfrak{p}_1, \mathfrak{p}_2 | \mathfrak{p}$ of E_1, E_2 respectively such that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both non split over E^σ. Then the local global principle is satisfied for E.

In particular the local-global principle holds if E is a field.

More generally one can show that the local-global principle holds if σ 'comes from the Galois group'. That is, if σ comes from the restriction of a single element of the Galois group of the normal closure of the composite of the fields from which E is made.

Fields with Involutions

Independently of their result, one can show that when E is a field, there exists a global choice of $\lambda \in E^\times$ if and only if there exists one locally everywhere. (Since I am motivated by understanding special fields, the case where E is a field really is the case I care most about.)

Theorem

Let (E, σ) be a number field with involution with $dim(E) = dim(V)$. The torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- E^ϕ *splits* $Cliff^+(V, q)$ *for all* ϕ.
- *The signature condition is plausible at all real places.*

“Non-Maximal” Tori ($dim(V) = dim(E) + 1$)

What can we say about the case of maximal tori coming from fields in orthogonal groups of odd dimension?

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) = dim(E) + 1$ then the torus

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if

- *E^{ϕ} splits $Cliff^{+}(V, q)$ for all ϕ.*
- *The ‘signature’ condition is plausible at all real places.*

Non-Maximal Tori ($dim(V) = dim(E) + 2$)

What about the case of almost maximal tori coming from fields in orthogonal groups of even dimension?

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) = dim(E) + 2 = n + 2$ then the torus

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if

- $(E^{\phi}(\sqrt{(-1)^n D(q)\delta_{E/k}}))$ *splits* $Cliff^{+}(V, q)$ *for all* ϕ.
- *The 'signature' condition is plausible at all real places.*
- *Local-global conditions 'work out'.*

Non-Maximal Tori ($dim(V) > dim(E) + 2$)

What can we say about all the other tori coming from fields?

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) > dim(E) + 2$ then

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if the signature condition is plausible at all real places.

What About Special Points?

The special field for a CM-algebra is the unique one which contributes the $(2,*)$ to the signature. That factor is the only part of the torus that is relevant to the special point.

Some remarks:

1. For CM-algebras and (V,q) of signature $(2,n)$ the signature conditions are always plausible.
2. For CM-algebras there is no local global obstruction. Indeed, since σ comes from the Galois group, there exists a p for which σ acts as Frobenius.
3. If $T_{E,\sigma} \hookrightarrow O_q$ there is no obstruction to making E the special field.
4. We are thus left with the discriminant condition when $dim(E) = dim(V)$ and the appropriate splitting conditions when $dim(V) - dim(E) \leq 2$.

Further Questions

Some questions that this work raises:

- What happens integrally? For example, what can we say about the structure, or integral invariants, of lattices with quadratic forms of the shape:
$$Tr_{E/k}(\lambda x \sigma(x)).$$
A natural question is given a lattice Λ and $\mathcal{O} \subset E$ when does $\mathcal{O}^{\times} \hookrightarrow O(\Lambda)$?
- An important source of interest in special points is the Galois action on them. Can we use the structure we described (for example the inclusion $E^{\Phi} \hookrightarrow Cliff^{+}(V, q)$) to understand the Galois action on special points. What about the Hecke action?
- The same problem we have tried to answer but for other algebraic groups? Tori in unitary groups and symplectic groups come with some similar structures.

The End

Thank you.