# Introduction to Propositional Logic

This is going to be a fair bit of terminology and definitions.

The concepts here aren't necissarily hard, but may take some time to absorb!

You are strongly encouraged to read carefully over the first chapter of the textbook more than once to help it sink in.

**There will be questions on the exams that are definitions that I quickly gloss over today but which are directly from chapter one of the textbook.**

## Assertions

An assertion is a **sentence** that where it makes sense to say that it is either true or false.

**Examples**

- There are more than 25 people in this room.
- Andrew Fiori is over 50 years olds.
- Bananas are a type of berry.
- Strawberries are a type of berry.
- Either bananas are a type of berry or strawberries are a type of berry, but definitely not both.

The primary thing that makes an assertion an assertion is that it is asserting a thing.

The thing may be true, may be false, we may need more information to decide, it may be impossible to know, but an assertion is a sentance that makes an assertion about about the truth of a thing that could at least plausibly be true, or false.

As a simple test, if a sentence could be proceeded by the phrase

*It is true that*

or

*It is not true that*

and still make sense, then it is likely an assertion.

## Some *assertions* about assertions

- The instructor will sometimes refer to assertions as statements because other people use that terminology.
- Determining if a sentence is an assertion may require knowing the definitions of all the words in the sentence, but often it doesn't.
- It may not be possible to know if an assertion is actually true or false without more information.

Can Questions be Assertions? **NO!**
Exclamations are also not assertions!

## Deductions

A **deduction** is a series of hypothesis, each of which is an assertion, followed by a conclusion, which is also an assertion.

We generally interpret a deduction as trying to make the claim that

*if ever you find yourself in a situation where you know the hypothesis are true, then you also know the conclusion is true.*

A deduction can be thought of as a *meta-assertion*, it somehow makes an assertion about assertions, but rather than being true or false, we say it is **valid** or **invalid**.

A deduction is **valid** if the conclusion is necissarily true whenever all the hypotheses are true.
A deduction is **invalid** if it is possible for the conclusion to be false even if all the hypotheses are true.

It is both helpful and important not to confuse deductions and assertions, the extra terminology is intended to help you think and communicate more clearly about them.

It **never** makes sense to say

*this deduction is true*

**Hypothesis**
Socrates is a man.
All men are mortal.
**Conclusion**
Socrates is mortal.

The above deduction is valid!

The above deduction tells us nothing about cats or dogs whose names happen to be Socrates!
**The truth/falsehood of hypothesis does not in and of itself change whether or not the deduction is valid**.

The above deduction is also not so useful if we lived in a world where not all men are mortal!
**The truth/falsehood of hypothesis does not in and of itself change whether or not the deduction is valid**.

But, if we live in a world where all men are mortal, and we come across a man named Socrates, then it is definitely true that he is mortal!
**The deduction is valid because it tells us something about the conclusion when the hypothesis are true**

Even though the deduction is valid, the deduction does not let me conclude that Socrates is a man! there are many cats names Socrates!!!

**Hypothesis**
Roses are red.
Violets are violet.
**Conclusion**
I am currently in a building at the university of Lethbridge.

This assertion is **not** valid! **even though the conclusion is true!!!**

**Hypothesis**
The professor prepared these slides.
The professor is very careful.
**Conclusion**
There are no typos in these slides.

Just like with assertions, it may be hard to tell if a deduction is valid, and it may be hard to tell if a deduction is applicable to a given situation.

- It is unclear if the deduction is valid, certainly many people would argue it isn't.
- Even if we assume it were valid, none of you know if the hypothesis are true!

Knowing a deduction is valid does not tell you the Hypotheses are true!!

## Converting assertions and deductions into symbols

In order to work with logical assertions and deductions, for example to check their validity or to simplify them, it will be useful to rewrite them using symbols and then develop a system of rules we know work with these symbols.

- This allows us to carefully cut through ambiguity that may exist in the English language.
- It gives us a systematic way to work with assertions, deductions and ultimately proofs.

All of the symbols we define will form a language, and like any language there is a grammar, and the way we translate between languages must take into account the grammar.

## Symbolization Key

Because not every single assertion has a predefined symbol attached to it, we often need to define the meaning of symbols we plan to use.

A **symbolization key** lets us assign different letters to refer to diffent specific logical assertions.

**Symbolization Key**
C: Cats are cute
D: Dogs are big
A: Assertions in symbolization keys usually do not contain logical connectives.
L: Connectives are words with logical meanings like: *and; or; not; etc...*

Once we have a symbolization key, we can make logical assertions like:

$$C \text{ and } D$$

to mean things like

$$\textit{Cats are cute} \text{ and } \textit{Dogs are big}$$

## Logical Connectives

Some words that we use come with natural logical meanings. We generally don't include these words in assertions in truth tables, because understanding the assertion is best done by thinking about the meaning of these words.

**Logical Connectives** are those words, phrases, and language constructs that let us combine simple assertions into complex assertions.

- Not ($\neg$)
- And ($\&$, note that some people use $\wedge$)
- Or ($\vee$, note that some people (many programming languages) use other things)
- If ... then ($\Rightarrow$)
- If and only if ( $\iff$ )

Each of these language constructs has an intuitive meaning.
But sometimes people disagree about the exact meaning, we will need to make the interpretation precise, to avoid any ambiguity.
We can make our interpretation of these logical connectives precise by describing their truth tables.

## Interpretting Connectives / Truth Tables

**Symboliztion Key**

**A: The bear is big   B: The dog is small**

The bear is not big: is written as $\neg A$ and is true precisely when $A$ is false.

The bear is big and the dog is small: is written as $A \& B$ and is only true if both are true.

The bear is big or the dog is small: is written as $A \vee B$ and is true so long as at least one is true.

If the bear is big then the dog is small: is written as $A \Rightarrow B$ and is only false if $A$ is true and $B$ is false.

The bear is big if and only if the dog is small: is written as $A \iff B$ and the claim is that $A$ and $B$ have *the same* truth value.

Truth tables allow us to understand the logical meaning of the different connectives.

| A | $\neg A$ |
|---|---|
| F | T |
| T | F |

| A | B | $A \& B$ | $A \vee B$ | $A \Rightarrow B$ | $A \iff B$ |
|---|---|---|---|---|---|
| F | F | F | F | T | T |
| F | T | F | T | T | F |
| T | F | F | T | F | F |
| T | T | T | T | T | T |

One can think of the truth table as **defining** the meaning of these symbols, which is needed, because not everyone agrees on the natural meaning. If you want to mean something not described by these symbols either combine them.

## Converting English To Symbols/ Parenthesis and Order of Operations

If the bear is big then both the dog is small and the cat is cute.

**Symboliztion Key**

A: The bear is big  B: The dog is small  C: The cat is cute

If A then both B and C.

$$A \Rightarrow (B \,\&\, C)$$

**Note:** We do not define order of operations between these terms (except perhaps that $\neg$ happens before all else). So if you omit parenthesis, then an expression is likely ambiguous:

$$A \Rightarrow B \,\&\, C \qquad ?=? \qquad \begin{cases} (A \Rightarrow B) \,\&\, C \\ A \Rightarrow (B \,\&\, C) \end{cases}$$

As such, it is **helpful and important** to use parenthesis whenever there might be ambiguity about what you are saying.

When translating between the langauge of Logic and English keeping track of the order of operations for complex expressions can be a real challenge, and you must be very careful around ambiguity that can be created here.

## Terminology ( Fast Overview)

A **Tautology** is an assertion in logic which is always true, regardless of the truth of any *variables* appearing in it.

The standard example of a tautology is the **law of the excluded middle**, which is the rule which tells us that

$$A \vee (\neg A)$$

is always true.

A **Contradiction** is an assertion in logic which is always false, regardless of the truth of any *variables* appearing in it.

An assertion $\mathcal{A}$ is a contraditiction if and only if $\neg(\mathcal{A})$ is a tautology.
The standard example of a contradiction is then $A \,\&\, (\neg A)$ which is always false.

Two assertions are **Logical Equivalence** if their truth/falsehood is identical for all possible truth values of its *variables*.

Two assertions, $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent if and only if the assertion $(\mathcal{A} \iff \mathcal{B})$ is a tautology.
The expression $\neg(\neg A) \equiv A$ indicates that these are logically equivalent, this is one of the **rules of negation**.

## Some Logical Equivalences ( Fast Overview)

There are a number of important logical equivalences discussed in the textbook.
**You should read about these in the textbook, you will strongly benefit from knowing these!!**

- Rules of Negation

$$\neg(\neg A) \equiv A$$

$$\neg(A \vee B) \equiv (\neg A)\,\&\,(\neg B) \qquad \neg(A\,\&\,B) \equiv (\neg A) \vee (\neg B)$$

$$\neg(A \Rightarrow B) \equiv A\,\&\,(\neg B) \qquad \neg(A \iff B) \equiv (\neg A)\,\&\,(B) \equiv (A)\,\&\,(\neg B)$$

- Rules of Commutativity

$$(A \vee B) \equiv (B \vee A) \qquad (A\,\&\,B) \equiv (B\,\&\,A) \qquad (A \iff B) \equiv (B \iff A)$$

- Rules of Associativity

$$(A \vee B) \vee C \equiv A \vee (B \vee C) \qquad (A\,\&\,B)\,\&\,C \equiv A\,\&\,(B\,\&\,C)$$

## Even More Terminology (Fast Overview)

**You should read about these in the textbook!!**

The **Converse** of $\mathcal{A} \Rightarrow \mathcal{B}$ is $\mathcal{B} \Rightarrow \mathcal{A}$.

The logical implications of the converse are often very different than the implications of the original statement.
The converse is often useful terminology to talk about assertions, but it is rarely useful as terminology in a proof.

The **Contrapositive** of $\mathcal{A} \Rightarrow \mathcal{B}$ is $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$

**A statement and its contrapositive are logically equivalent! This can often be useful.**

$$\mathcal{A} \Rightarrow \mathcal{B} \equiv (\neg\mathcal{B}) \Rightarrow (\neg\mathcal{A})$$

This equivalence is often useful in proofs, though we can work around not using it, it is good to be aware of it.

Both of these things only make sense in relation to $\mathcal{A} \Rightarrow \mathcal{B}$ type statements.
You can't take the contrapositive of $((A \,\&\, B) \vee C)$

One key thing that logic provides us with is a language in which we can be precise and make arguments.

This contrasts to most natural languages (eg English) in which there are many ambiguous sentences, and in fact, sentences which though gramatically correct have no meaning whatsoever.

To succeed in this course you will need to:

- Be able to distinguish between well formed and not well formed sentences in logic.

$$A \Rightarrow (B \,\&\, C) \qquad vs \qquad A \Rightarrow B \neg C$$

  Though I will never ask you 'is the following sentence an assertion?', you will need to write plenty of assertions so it is important that you can identify if what you are writing 'makes sense'.

- Be able to translate between natural language (English) and the language of logic.
- Be able to write precise and unambiguous sentences in natural language (English). (This is a good life skill outside this course aswell).

The first two tasks complement the third, because thinking about how one might do the translation can allow one to identify ambiguity in their own writing.