Theorem

A **Theorem** is a **valid** deduction.

One of the key activities in higher mathematics is identifying whether or not a deduction is actually a theorem and then trying to convince other people that you are right.

The way we convince people is generally to refer to sufficiently many things that they already know are correct.

So the first thing I want to do here, is give you a list of things that you basically already know are correct.
Each of the deductions on the next page explains the natural understanding we have of the meaning of the symbols involved.

Some Basic Theorems

1. Repeat: $A, \therefore A$
2. &-introduction $A,\ B,\ \therefore A \,\&\, B$
3. &-elimination $A \,\&\, B,\ \therefore A$ $\quad A \,\&\, B,\ \therefore B$
4. $\vee$-introduction $A,\ \therefore A \vee B$ $\quad B,\ \therefore A \vee B$
5. $\vee$-elimination $A \vee B,\ \neg A,\ \therefore B$ $\quad A \vee B,\ \neg B,\ \therefore A$
6. $\Rightarrow$-elimination $A \Rightarrow B,\ A,\ \therefore B$ **also called Modus Ponens**
7. $\Leftrightarrow$-introduction $A \Rightarrow B,\ B \Rightarrow A,\ \therefore A \iff B$
8. $\Leftrightarrow$-elimination $A \iff B,\ \therefore A \Rightarrow B$ $\quad A \iff B,\ \therefore B \Rightarrow A$
9. Proof by cases. $A \vee B,\ A \Rightarrow C,\ B \Rightarrow C\ \therefore C$

We have already seen a bunch of other theorems

1. Law of the Excluded Middle $\therefore A \vee \neg A$.
2. Rules of Negation **eg:** $\neg(A \,\&\, B),\ \therefore (\neg A) \vee (\neg B)$.
3. Rules of Commutativity **eg:** $(A \,\&\, B),\ \therefore B \,\&\, A$.
4. Logical Equivalence of Contrapositive. $A \Rightarrow B,\ \therefore (\neg B) \Rightarrow (\neg A)$.

What is a proof?

For the purpose of this class

A **proof** of a deduction consists of series of assertions beginning with the hypotheses and ending with the conclusion such that each assertion in the proof is an **immediate** consequence of the proceeding assertions.
Moreover, each assertion must come with an explanation of why/how it follows immediately from previous assertions.

The explanations usually involve a reference to a deduction that you already know is a theorem. (eg: those on the previous slide).

In the real world is very common for people writing proofs to skip steps (because they think something is immediate) or to omit explanations (because they think that something is obvious).
In future courses you will likely end up asking instructors/professors *do I need to include a proof of this step?* As a student, if you are asking the answer is usually **yes**. In this course even more so!

Why are you writing a proof?

In most contexts, before you write a proof, it is good to ask yourself this question. Some common answers:

- I don't actually know if a deduction is valid, and a proof is the only way to be sure.
- I think a deduction is valid, and a good proof might let me understand why.
- I know the deduction is valid, but lots of other people don't and they all need to be convinced.
- Some specific person doesn't believe me that a deduction is valid, and I need to convince them.
- Some jerk knows a deduction is valid, but is insisting I prove it for them anyways.

Knowing your audience lets you decide what details to include/exclude

- A proof for yourself only needs to include the steps that aren't obvious to you. (But be careful, because people often think incorrect things are obviously true)
- A proof for your friend only needs the details they don't think are obvious.
- A proof for experts only needs the details experts don't think are obvious.
- A proof for a jerk probably needs to include all the details.
  In this course include all details, One thing I am trying to evaluate is if you know how to write the details, so show me that you do!!!

Example (of a proof)

- We need to have a sequence of assertions, together with explanations of why we know these follow from previous assertions!

Prove the deduction:

$$(A \vee B) \Rightarrow C, \quad B, \quad \therefore C$$

Organizing Proofs

A **two-column proof** is a technique for organizing proofs so that we can easily check that each step is properly explained.

For the first part of this course we will require that you write proofs in this format!
It will help remind you of the need to explain each step and make precise assertions.

We will move away from this style later in the course.

Example (of two-column proofs)

- In a two column proof we write the assertion in the left column, and the justification in the right hand column!
- We number lines to make them easy to refer to.
- We seperate the hypotheses from the rest of the assertions for clarity.

Prove the deduction:

$$(A \vee B) \Rightarrow C, \quad B, \quad \therefore C$$

Fill in the right hand column of this two-column proof:

1	$(A \,\&\, \neg B) \Rightarrow (B \vee C)$	
2	$\neg(B \vee \neg A)$	
3	$\neg B \,\&\, A$	
4	$A \,\&\, \neg B$	
5	$B \vee C$	
6	$\neg B$	
7	C	

What does it prove?

$\Rightarrow$-Introduction

The idea of $\Rightarrow$-Introduction is that the best want to convince you that

$$\mathcal{A} \Rightarrow \mathcal{B}$$

is to prove that if I knew A, I would know B, but that is proving the deduction

$$\mathcal{A} \therefore \mathcal{B}$$

is **valid**.

In order to make use of $\Rightarrow$-Introduction, we must include a **subproof** that the deduction $\mathcal{A} \therefore \mathcal{B}$ (given everything we already know) is valid.
To convince you $A \Rightarrow B$, I just write a proof that A implies B.

- The subproof can reference earlier lines from the outer proof.
- The outer proof can't reference lines inside a sub-proof (except using the $\Rightarrow$-introduction rule [or Contradition])

Examples (of proofs using $\Rightarrow$-Introduction)

Recall:

- The subproof can reference earlier lines from the outer proof.
- The outer proof can't reference lines inside a sub-proof (it can only ever point at the whole the proof, and mention what the subproof proved.)

Prove the deduction:

$$(A \vee B) \Rightarrow C, \quad \therefore A \Rightarrow C$$

Prove the deduction:

$$A \Rightarrow B, \quad B \Rightarrow C, \quad \therefore A \Rightarrow C$$

Example of a Proof By Cases

Recall:

Proof by cases. $A \lor B$, $A \Rightarrow C$, $B \Rightarrow C$ $\therefore C$

- The key to the rule is that I know $A \lor B$ and I know both $A \Rightarrow C$ and $B \Rightarrow C$.
- So the proof will have in three parts, before I even get to use the rule.

Prove the deduction:

$$(A \lor B) \Rightarrow (C \lor \neg D), \quad C \lor D, \quad \therefore A \Rightarrow C$$

Contradiction

The role of a contradiction is to slightly shorten proofs that tend to follow the same pattern.
If $\mathcal{B}$ is a contradiction, and we can prove $\mathcal{A} \Rightarrow \mathcal{B}$ is a tautology using $\Rightarrow$-Introduction, then we can conclude $\neg\mathcal{A}$.
ie. The deduction:

$$\neg\mathcal{B}, \mathcal{A} \Rightarrow \mathcal{B} . \therefore \neg\mathcal{A}$$

is valid.

We will leave it as an exercise to prove this deduction.
In order to make use of contradiction we include a subproof that shows $\mathcal{A} \Rightarrow \mathcal{B}$.
Important notes:

- In the subproof we generally indicate that the hypothesis $\mathcal{A}$ is expected to generate a contradiction.
- It must be clear that $\mathcal{B}$ is a contradiction.

Examples (of proofs by contradiction)

Recall:

- The subproof can reference earlier lines from the outer proof.
- The outer proof can't reference lines inside a sub-proof (except using a special rule)

Prove the deduction:

$$(A \vee B) \Rightarrow C, \quad A \Rightarrow \neg C, \quad \therefore \neg A$$

Example

Prove the deduction:

$$A \to B, \quad C \to \neg B, \quad \therefore \neg A \vee \neg C$$

Hint: The first step is to decide on a strategy!.

Prove the same deduction using a different strategy!
Hint: For a proof by cases use logical equivalence of contrapositive.

Any proof that can be done with a proof by cases, or proof by contradiction could be done with the other, or with neither, but they are often good strategies to employ when we get stuck.
A common question is

when do I use either one

- If there are two cases, and you can't see why either of them wouldn't be true (so that you can do an $\vee$-elim) than a proof by cases makes sense.
- If there is a specific thing you think is true (or not true), and you can't think of something else to do, a contradiction makes sense.
- If all else fails, use a contradiction to prove precisely the conclusion or use the law of the excluded middle to make up two cases to consider.

What is hard about proofs?

- Deciding on your next step.
  Two-column proofs often make this a bit easier, because you can think about the small number of rules you are allowed to use.
- Correctly writing down the next step.
  This is something you need to practice, and something you want to get feedback on from pretty much anyone
- Correctly explaining the next stop.
  Two-column proofs make this easier; you don't need to worry about the english language, and there are not many possible explanations
- Deciding on a strategy.
  This takes practice, but whenever you get stuck, you want to consider all the strategies you know, and think about what you would need to prove to use them.
- Presenting the proof in an organized way.
  Two-column proofs make this easier, because they dictate an organization scheme.

Counterexamples

Counterexamples are often the best strategy to show that a deduction is not valid.
To show that:

$$\mathcal{A} \Rightarrow \mathcal{B}$$

is **not** a tautology, we need to show it is possible for this expression to be false.

- You need to find an assignment of the "variables" so that the hypotheses are true, and the conclusion is false.
- You need only find one counterexample.
- There may be many examples that are **not** counterexamples.
- Counterexamples do not necissarily let you conclude anything about any other deductions which involve $\mathcal{A}$ and/or $\mathcal{B}$.

Examples (of counterexamples)

- You need to find an assignment of the “variables” so that the hypotheses are true, and the conclusion is false.
- We only need to find one counterexample.
- The process by which we find the counterexample is not really part of the justification.

Show the deduction is **not** valid

$$(A \,\&\, B) \Rightarrow C, \quad \therefore A \Rightarrow C$$