Sets

The first sort of mathematical object we will talk about in this course are sets. A **Set** encapsulates the logical notion of a having a well defined collection of well defined objects. Sets are one of the foundational objects in modern mathematics.

A **Set** is any collection of objects.

The **elements** of a set are those objects that are actually in the set.
The **cardinality** of a set is the **size** of the set (so the number of elements in the set).

This is not a course in axiomatic set theory. Our definition glosses over some technical foundational issues which may be addressed properly in other courses. These technical issues should not interfere with your understanding.

Examples of Sets

- $A = \{1, 2, 3, 4, 5\}$.
  The set A has exactly 5 elements.

- $B = \{\text{red}, \text{green}, \text{blue}\}$.
  The set B has exactly 3 elements.

- $C = \{\} = \emptyset$.
  The set C has exactly 0 elements.

- $D = \{\emptyset\}$.
  The set D has exactly 1 element.
  I cannot emphasise enough that C and D are different sets!!
  C contains 0 elements, and D contains 1 element!!
  Historically at least 20% of the class will at least once write $\{\emptyset\}$ when they mean $\emptyset$ on an exam.
- $E = \{\{1, 2\}, \{3, 4\}, 1, 5, \text{red}, \{\{1\}\}\}$.
  The set E has exactly 6 elements, (3 of these happen to be sets).
  Examples like this won't generally appear in this course.

Note: Sets do not have an ordering or repetitions, there is no first element,

$$\{1, 2\} = \{2, 1\} = \{2, 1, 2, 2\}.$$

Main Assertions about sets

If one is talking about collections, there are several types of assertions one is prone to making:

- The element 'x' is in my set 'A' ($x \in A$, x is an element of A)
  eg: It is true that $\text{'cat'} \in \{\text{'cat'}, \text{'dog'}\}$
- The element 'x' is **not** in my set 'A' ($x \notin A$, x is not an element of A)
  eg: It is true that $\text{'duck'} \notin \{\text{'cat'}, \text{'dog'}\}$
- Every element in my set A is also in by other set B ($A \subset B$, A is a subset of B)
  eg: It is true that $\{\text{'cat'}, \text{'bear'}\} \subset \{\text{'cat'}, \text{'dog'}, \text{'bear'}\}$
- Some element in my set A is **not** in my other set B ($A \not\subset B$, A is not a subset of B)
  eg: It is true that $\{\text{'cat'}, \text{'bear'}\} \not\subset \{\text{'cat'}, \text{'dog'}\}$
- My collections have **no** elements in common, we say that they are **disjoint** (notation we will eventually see is $A \cap B = \emptyset$)

Understanding the meaning of these symbols is equivalent to understanding

what sets are.

Understanding the truth values of these assertions for a particular set is equivalent to understanding exactly

what the specific set is.

If I know which elements are in a set, I know what set I am talking about, if I don't, ... then I don't.

Examples with Sets

- $A = \{1, 2, 3, 4, 5\}$.
  Notice that $1 \in A$ but $17 \notin A$!

- $B = \{\text{red}, \text{green}, \text{blue}\}$.
  Notice that $\{\text{red}, \text{green}\} \subset B$ but $B \not\subset \{\text{red}, \text{green}\}$

- $C = \{\} = \emptyset$.
  Notice that $C \subset A$ and $C \subset B$.
  in general the empty set is a subset of every set!

  Notice that $C \notin C$, but $C \subset C$!.
- $D = \{\emptyset\}$.
  Notice that $D \subset D$, and $C \subset C$, and $C \in D$, but that $D \notin D$.
  I will not make trick questions using things like this on exams... but odds are someone in the class will make a mistake about this anyways.
  The empty set, $\emptyset$, is a subset of every set, but the emptyset is **not usually** an element of most sets we think about.
- $E = \{\{1, 2\}, \{3, 4\}, 1, 5, \text{red}, \{\{1\}\}\}$.
  Notice that $\{1, 2\} \in E$ but $\{1, 2\} \not\subset E$.
  This may be counterintuitive, I won't setup things like this on exams.

Assertions about the objects that make up sets

The reason we make collections is we care about the objects in them and so want to be able to make assertions about them.
Sometimes we want to think about the *same assertion* about many different objects.

A **predicate** is a sentence that becomes an assertion once we specify the value of some variables within it.
The distinction between a predicte and an assertion is that a predicate (in principal) depends on the variable.

- Predicates can have as many variables as one wants, but we will deal with predicates that are
  - unary (one variable)

    eg: $P(x) =$ 'x is even'
  - and binary (two variable)

    eg: $x \ T \ y =$ 'x is taller than y'

  The description of the meaning of predicates is often included in a symbolization key.
- Depending on the value of the the variable the truth of the resulting assertion might change.

Examples

We have a predicte:

$$\textbf{eg:}\ P(x) = \text{`}x\text{ is even'}$$

What are the truth values of

- $P(1)$?
- $P(2)$?
- $P(\text{`}cat')$? A predicate typically only makes sense for certain types of objects, and it is normally a good idea to specify what type of objects you will be talking about when you specify your predicate!
  We will come back to the problem this presents shortly.

Predicates give us a useful way to define subsets.
Given a set A, and a predicate $P(x)$, we can define a new set with the notation:

$$B = \{x \in A \mid P(x)\}$$

If $A = \{1, 2, 3, 4, 5\}$ then what is the set:

$$\{x \in A \mid P(x)\}?$$

Universe of Discourse

Set theorey creates many foundational issues that can lead to paradoxes if one is not careful. Additionally, in order to make sure everyone is on the same page, and understands the sorts of things our predicates are talking about we often consider the following.

By the **Universe of discourse** $\mathcal{U}$, we mean the collection of all thing which are allowed to be in sets.
It is often implicit, and so we often abbreviate

$$\{x \in \mathcal{U} \mid P(x)\} \quad \text{by} \quad \{x \mid P(x)\}$$

Whenever we are doing propositional (or first order) logic, we typically have a universe of discourse in mind and at times it may be useful to make this explicit.
The universe of discourse $\mathcal{U}$ could be

- the set of natural numbers.
- the set of colours.
- the collection of all animals.
- the collection of all natural numbers, and all subsets of the natural numbers, and all colours.

Operations on sets

There are several very common constructions people perform with sets. In order to make talking about these easier they get there own notation, we will define these in a moment:

- Union
- Intersection
- Set difference
- Set complement

Each of these can be understood by a Venn diagram, but it will be useful to also have a definition.

Union

The union of two sets. $A \cup B = \{x \in \mathcal{U} \mid (x \in A) \vee (x \in B)\}$ so:

$$(x \in (A \cup B)) \iff ((x \in A) \vee (x \in B))$$

Given that the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$A = \{1, 3, 5, 7\} \qquad B = \{3, 4, 5\} \qquad C = \{2, 4, 6\}$$

- $A \cup B = \{1, 3, 4, 5, 7\}$.
- $A \cup C = \{1, 2, 3, 4, 5, 6, 7\}$.
- $B \cup C =?$.

Intersection

The intersection of two sets. $A \cap B = \{x \in \mathcal{U} \mid (x \in A) \,\&\, (x \in B)\}$ so:

$$(x \in (A \cap B)) \iff ((x \in A) \,\&\, (x \in B))$$

Given that the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$A = \{1, 3, 5, 7\} \qquad B = \{3, 4, 5\} \qquad C = \{2, 4, 6\}$$

- $A \cap B = \{3, 5\}$.
- $A \cap C = \{\} = \emptyset$.
- $B \cap C = ?$.

We can also now give notation for the assertion that sets A and B are **disjoint**.

$$A \cap B = \emptyset$$

Set Difference

The set difference of two sets. $A \setminus B = \{x \in \mathcal{U} \mid (x \in A) \,\&\, (x \notin B)\}$ so:

$$(x \in (A \setminus B)) \iff ((x \in A) \,\&\, (x \notin B))$$

Given that the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$A = \{1, 3, 5, 7\} \qquad B = \{3, 4, 5\} \qquad C = \{2, 4, 6\}$$

- $A \setminus B = \{1, 7\}$.
- $A \setminus C = \{1, 3, 5, 7\}$.
- $B \setminus C = ?$.

Set Complement

The complement of a set. $\overline{A} = \{x \in \mathcal{U} \mid x \notin A\}$ so:

$$(x \in \overline{A}) \iff (x \notin A)$$

Given that the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$A = \{1, 3, 5, 7\} \qquad B = \{3, 4, 5\} \qquad C = \{2, 4, 6\}$$

- $\overline{A} = \{2, 4, 6\}$.
- $\overline{B} = \{1, 2, 6, 7\}$.
- $\overline{C} = ?$.

Example Operations on Sets

Given that the universe of discourse is $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$A = \{1, 3, 5, 7\} \qquad B = \{3, 4, 5\} \qquad C = \{2, 4, 6\}$$

what is:

$$(A \cap B) \cup C.$$

Examples of Venn Diagrams

We can draw Venn diagrams to try to help us understand the configurations of sets:

- $A \subset B$.
- $A \cap B$.
- $X \subset (A \cap B)$.
- $A \cup (B \cap C)$
- $X \subset (A \cup (B \cap C))$

A Venn Diagram is not itself a proof, but they can be helpful for understanding what sorts of things are true, and what sorts of things are not true, both of which can be helpful while writing a proof.

Example proofs about sets

- It is often useful to have your symbolization key describe what sets you are considering.
- We now have a few extra rules for dealing with assertions about sets:
  - Definition of .... rules allow you to use the definition of a set operation.

Prove the following deduction:

$$(x \in A) \,\&\, (x \in B), \quad \therefore (x \in A) \vee (x \in B)$$

Prove the following deduction:

$$x \in A \cap B, \quad (x \in A) \Rightarrow (x \in C), \quad \therefore x \in B \cap C$$

Explain how you know the following deduction is **not** valid:

$$x \in A \cup B, \quad \therefore x \in A \cap B$$

The power set of a set.

This is probably the most complex construction we will give to get a new set from an old one.

$$\mathcal{P}(A) = \text{'the set which contains every subset of } A\text{'} \\ = \{X \in \mathcal{U} \mid X \subset A\}$$

so this means:

$$(X \in \mathcal{P}(A)) \iff (X \subset A)$$

implicitly this means in particular that 'X is a set'.

Example: What is the power set of $\{5, 8, 9\}$?

Using the power set construction may require you to enlarge your universe of discourse!
In particular, you need that all of the subsets you are considering are part of the universe of discourse...
This creates problems if you also want to talk about set complements...
There are lots of ways to work around this... this is unlikely to be all that important in anything you have to deal with in this course.