You can't prove a predicate *is true* because a predicate is not an assertion, you can't prove *it is valid* as it is not a deduction!
If someone asks you to prove $P(x)$, it is not totally clear what they mean.

Here are examples of assertions we can make about a predicate.

- No matter which object $x \in \mathcal{U}$ we consider, $P(x)$ will always be true.

$$\forall x, P(x) \qquad \forall x \in \mathcal{U}, P(x) \qquad \forall x \in A, P(x)$$

for all x, the assertion $P(x)$ is true.

- No matter which object $x \in \mathcal{U}$ we consider, $P(x)$ will always be false.

$$\forall x, \neg P(x) \qquad \forall x \in \mathcal{U}, \neg P(x) \qquad \forall x \in A, \neg P(x)$$

for all x, the assertion $P(x)$ is false.

- There is an example of an object $x \in \mathcal{U}$ for which, $P(x)$ will be true.

$$\exists x, P(x) \qquad \exists x \in \mathcal{U}, P(x) \qquad \exists x \in A, P(x)$$

there exists x, for which the assertion $P(x)$ is true.

- There is no example of an object $x \in \mathcal{U}$ for which, $P(x)$ will be true.

$$\neg\exists x, P(x) \qquad \neg\exists x \in \mathcal{U}, P(x) \qquad \neg\exists x \in A, P(x)$$

there does not exists x, for which the assertion $P(x)$ is true.

Are any of these logically equivalent?
The variable name, x, is a placeholder, it could be anything, a, b, y, z, γ, φ, ξ.

Examples - Converting natural language sentances

Symbolization Key:

$\mathcal{U}$: The set of all animals.

B: The set of all bears.

F: The set of all fish.

j: Jasper.

d: Dory.

$x\ E\ y$: 'x likes to eat y.

- Jasper is a bear.
- Dory is a fish.
- Jasper likes to eat Dory.
- Some bears like to eat Dory.
- All bears like to eat Dory.
- Jasper likes to eat certain fish.
- Jasper likes to eat all fish.
- Some bears like to eat some fish.
- All bears like to eat all fish.
- For all bears there is some fish they like to eat.
- There are some fish that all bears like to eat.
- If every bear likes to eat every fish then Dory likes to eat Jasper.

One key point is that $\forall\ \exists$ and $\exists\ \forall$ are different, so you need to be careful.

"Larger" Example

Consider the following symbolization key:

$\mathcal{U}$: The set of all living things	C: The set of all cats
A: The set of all animals	D: The set of all dogs
B: The set of all bears	xLy: "x is larger than y"

translate the following assertions **into the notation of First-Order Logic**:

Either every bear is larger than every dog and cat, or there exists a cat which is larger than every dog.

Step 1: Notice the or:

Either every bear is larger than every dog and cat, or there exists a cat which is larger than every dog.

$$(\quad \cdots \quad) \vee (\quad \cdots \quad)$$

Step 2: Now just the blue part: every bear is larger than every dog and cat,

$$\forall b \in B, (\quad \cdots \quad)$$

Step 3: What does the orange stuff mean? b is larger than every dog and cat, means

b is larger than every dog and b is larger than every cat,

$$(\forall d \in D, bLd) \ \& \ (\forall c \in C, bLc)$$

Summarizing where we are (Step 3 goes into step 2 which goes into step 1!):

$$(\forall b \in B, ((\forall d \in D, bLd) \& (\forall c \in C, bLc))) \vee (\quad \cdots \quad)$$

Step 4: Now the green part there exists a cat which is larger than every dog.

$\exists x \in C$, x is larger than every dog

$$\exists x \in C, \forall y \in D, xLy$$

Step 5: Put it all together

$$(\forall b \in B, ((\forall d \in D, bLd) \& (\forall c \in C, bLc))) \vee (\exists x \in C, \forall y \in D, xLy)$$

Logical Equivalences of Quantifier Expressions

There are many quantifier expressions that are logically equivalent!

One consequence is that there may not be a unique answer to a question.

We do not expect you to be able to identify on sight that two logically equivalent expressions are such.

On the next slide we will discuss some that you need to know, but here are some that might influence how you think about translating expressions:

Consider the following symbolization key:

$\mathcal{U}$: The set of all living things C: The set of all cats F: The set of all cute things

- There exists a cute cat

  $\exists x \in \mathcal{U}, (x \in C \,\&\, x \in F) \quad \exists x \in C, x \in F \quad \exists x \in F, x \in C \quad \exists x \in \mathcal{U}, x \in C \cap F$

- All cats are cute cats

$\forall x \in C, x \in F \quad \forall x \in \mathcal{U}, (x \in C \Rightarrow x \in F) \quad \forall x \in C, (x \in F \,\&\, x \in C) \quad \forall x \in C, x \in F \cap C$

How you read the sentance, often changes how you translate it, and how you think about it, even among logically equivalent statements, recognizing this can in the long term be very powerful.

Common Errors With Quantifiers/Sets/Translation

All of the 'math symbols' we have defined all have very specific 'grammar' rules.

The operations of propositional logic, $\&$, $\vee$, $\implies$.

require the left and right hand side to be assertions

The 'algebra' operations on sets $\cup$, $\cap$, $\setminus$.

require the left and right hand side to both be sets

The predicates of sets, $\in$, $\notin$, $\subset$, $\not\subset$.

require the left hand side to be a thing in the universe and the right hand side to be a set.

The quantifiers of first order logic, $\exists$, $\forall$.

always give exactly one comma, the thing which you are talking about

So it never makes sense to write

$$x \in A \neg B$$

or

$$\forall x \in A, x \in B, x \in C$$

and

$$x \in A \,\&\, \neg B$$

only makes sense it B is an assertion, and not if B is an assertion and **not** a set!!

The expressions you write should be things which you could hypothetically evaluate if you knew exactly what the objects were!

Examples - Negating Quantifiers

Just like negating statements in propositional logic, being able to negate quantifier expressions is very helpful in proofs!
It is important to remember the logical equivalences these imply.

- The key thing is to remember that if something isn't true for all examples... then there exists a counterexample.
- Conversely, if there doesn't exist an example, then everything is a counterexample.

Consider the two sentences:

- There are no bears that like to eat Dory.
- Every bear does not like to eat Dory.

These are logically equivalent, and so $\neg \exists x \in A, P(x) \equiv \forall x \in A, \neg P(x)$.

Consider the two sentences:

- not every bear wants to eat Dory.
- There is a bear that does not want to eat Dory.

These are logically equivalent, and so $\neg \forall x \in A, P(x) \equiv \exists x \in A, \neg P(x)$.

Simplify the following assertion. (so that $\neg$ does not appear).

$$\neg \forall d \in D, \left[\Big((d \in S) \,\&\, (d \notin T)\Big) \vee \exists e \in E, \Big((d \in T) \Rightarrow (e \in S)\Big)\right]$$

We have

$$\begin{aligned}
&\neg \forall d \in D, \left[\Big((d \in S) \,\&\, (d \notin T)\Big) \vee \exists e \in E, \Big((d \in T) \Rightarrow (e \in S)\Big)\right] \\
&\equiv \exists d \in D, \neg \left[\Big((d \in S) \,\&\, (d \notin T)\Big) \vee \exists e \in E, \Big((d \in T) \Rightarrow (e \in S)\Big)\right] \\
&\equiv \exists d \in D, \left[\neg \Big((d \in S) \,\&\, (d \notin T)\Big) \,\&\, \neg \exists e \in E, \Big((d \in T) \Rightarrow (e \in S)\Big)\right] \\
&\equiv \exists d \in D, \left[\Big(\neg(d \in S) \vee \neg(d \notin T)\Big) \,\&\, \forall e \in E, \neg \Big((d \in T) \Rightarrow (e \in S)\Big)\right] \\
&\equiv \exists d \in D, \left[\Big((d \notin S) \vee (d \in T)\Big) \,\&\, \forall e \in E, \Big((d \in T) \,\&\, (e \notin S)\Big)\right]
\end{aligned}$$

Notice that at each step, we move the negation sign deeper into the expression, and are just using one of the simple negation rules.

Examples Vacuous truth - Some Tautologies with Quantifiers

- If there are no counterexamples... then everything is an example... even if there are no examples.

Consider the two sentences:

- Every fish on the moon is 100m long.
- There are no fish on the moon that are not 100m long.

The statements are logically equivalent, and are both true, **vacuously**

- $\forall x \in \emptyset, P(x)$ is vacuously true.
- $\exists x \in \emptyset, \neg P(x)$ clearly false.

Quantifiers and Assertions about Sets

Assertions about sets are often the same as an assertion involving a quantifier.

- $A \subset B$ is the same as $\forall a \in A, a \in B$.
- $A \neq \emptyset$ is the same as $\exists a, a \in A$.

A consequence is that to prove:

Theorem

Given any two sets A and B we must have:

$$A \cap B \subset A \cup B$$

We must prove

$$\forall x \in A \cap B, x \in A \cup B$$

How to deal with proofs involving quantifiers will come soon .