Rules for Quantifiers

In order to prove assertions that involve quantifiers we will need some rules that let us manipulate them. Just like with all the other logical connectives we want rules:

- $\exists$-Introduction
- $\exists$-Elimination
- $\forall$-Introduction
- $\forall$-Elimination

Introductions generally allow you to prove deductions where the conclusion includes the symbol whereas eliminations let you make use of hypothesis that include the symbol

Our proofs about quantifiers will not be written in two column format so we often do not use these rules by name.
When you write proofs you must still ensure that each assertion follows from the previous assertions using deduction that you know to be true and include enough of an explanation so that anyone reading your work would also understand why/how an assertion follows from previous assertions.

∃-Introduction - Rabit in the hat proofs

In order to carry out ∃-Introduction one typically first does some scratch work off to the side to come up with an **example**
The proof that one must give is to demonstrate that the purported example is an example.

The best way to demonstrate that there is a rabit in a hat, is to pull the rabit out of the hat and show it to people, you do not need to explain why there is a rabit in the hat, how you knew it was there, or the mystery that enables you to pull it from the hat.

The only real alternative to prove an ∃ is to prove "the double negative", that is

not all rabits are not in my hat

maybe I could do this by contradiction by assuming all rabits are not in my hat, and seeing what this would imply...

∃-Introduction Example

Prove that:

$$\exists x \in \mathbb{N}, 20 < x^2 < 100$$

$$\exists x \in \mathbb{R}, x^2 + 9 = 10$$

Sentence choices

The Start

- Consider the example $x = 5$, we know that $x \in \mathbb{N}$ and moreover we see that....
- Let $x = 5$, we see that x satisfies...
- Consider $x = 5$, then
- Let $a = 5$, we see that a satisfies...
- Consider the example $a = 5$, we know that $a \in \mathbb{N}$ and moreover we see that....

You now of course need to prove something about x (or a).

The End

- We have thus shown that $x = 5$ is an example of an element of $\mathbb{N}$ such that $20 < x^2 < 100$, hence we conclude
$$\exists x \in \mathbb{N}, 20 < x^2 < 100$$
- Thus x is an example of an element which proves
$$\exists x \in \mathbb{N}, 20 < x^2 < 100$$
- So $a = 5$ is an example of an element from $\mathbb{N}$ which proves
$$\exists x \in \mathbb{N}, 20 < x^2 < 100$$

The key element of the opening sentence is making it clear what example you are talking about, the key element of the closing sentence is to summarize that you showed me an example of something and what it proves.

$\exists$-Elimination

If I am certain that an object with some properties exists, then I can give it a name (so as to be able to refer to it) and start using it and/or talking about it.

The most common misuse of this rule, is to try to assume extra things about the object. Just because I know there is a rabit in my hat, does not mean I know for free that the rabit is white, or has floppy ears. If I want to claim that it does... I must prove it somehow.

If I know an object exists, the most natural way to use that in my proof is to start talking about the thing which I know exists.
It seems unlikely that my knowing something exists, but never talking about it, will ever allow me to convince you of much of anything.

$\exists$-Elimination Example

Prove that in the universe of discourse of real numbers, and supposing A is a set that the following deduction is valid

$$\exists x \in A, ((x^2 > 4) \,\&\, (x > 0)), \qquad \therefore \exists x \in A, x > 2$$

Whenever you have a hypothesis that is an exist statement, you should anticipate using $\exists$-elimination, and it can probably be one of the first things you do.

This is especially important if your conclusion is also an $\exists$ statement.
if you are trying to convince me a thing exists, knowing other things exist might help!

Sentence choices

Using The Hypothesis

- By hypothesis we know $\exists x \in A, (x^2 > 4) \,\&\, (x > 0)$ which means there is an element of A, call it a, such that $(a^2 > 4)$ and $(a > 0)$.
- By hypothesis $\exists x \in A, (x^2 > 4) \,\&\, (x > 0)$, select such an element and call it a, so $a \in A$ and $(a^2 = 1)$ and $(a > 0)$.
- By hypothesis there exists some $a \in A$ such that $(a^2 > 4)$ and $(a > 0)$.
- By hypothesis there exists some $x \in A$ such that $(x^2 > 4)$ and $(x > 0)$.

Now you need to use this object (named a, or x, or something else) to actually prove whatever you wanted to prove.

As you can see there are many choices for how you phrase the use of this rule, and these are only some of them. You are free to pick your own, the best ones make it clear:

- which assertion that you know you are using.
- that you are giving a name to some object that is now known to exists.
- which set your new object lives in.
- what properties your new object has.

$\forall$-Elimination

If I know something is true about all objects in a set. And you ask me if it is true for a particular object in the set. Then the answer is yes.

A common missuse of this rule involves assuming that an example actually exists. Just because all of rabits in my hat have floppy ears, does not mean there is a rabit in my hat.

However, If I know there is a rabit, named Hopper, in my hat, and I know that all of the rabits in my hat have floppy ears. Then I can conclude that Hopper has floppy ears, because Hopper is a rabit in my hat, and all rabits in my hat have floppy ears.

In proofs the more we know about the objects we are talking about, so if ever you have a rabit in a proof, and there is a forall statement that would let you tell me more things about the rabit, it will normally be useful to do so.

$\forall$-Elimination Example

Suppose A, B, and C are sets. Prove the deduction:

$$A \subset B, \qquad A \neq \emptyset, \qquad \therefore B \neq \emptyset$$

Whenever you have a hypothesis that is a $\forall$ statement, it is likely that you will eventually be finding an object which is in the set of things we are talking about.
why did a hypothesis tell me all rabits are cute, if my proof does not involve any rabits?
As soon as that happens, you probably will want to use $\forall$-Elimination.

Sentence choices

Explain that you have a particular element

- By hypothesis the set A is not empty, and so we know there exists an element, call it x, which is in A.
- Because by hypothesis $A \neq \emptyset$ this means $\exists x, x \in A$, pick such an element and call it a.
- By hypothesis $A \neq \emptyset$, so there exists an element $x \in A$.
- The hypothesis that $A \neq \emptyset$ means that $\exists x \in A$, let a be any such element.

Here we are using $\exists$-elimination.

Explain that you know a forall statement

- Since $A \subset B$ we know $\forall a \in A, a \in B$.
- Since $A \subset B$ we know $\forall x \in A, x \in B$.
- Since $A \subset B$ we know every element of A is an element of B.

Here we are explaining the meaning of a symbol.

Explain what these two things mean together

- In particular, because $a \in A$ we know $a \in B$.
- In particular, because a is an element of A it is also an element of B.
- Because we know $a \in A$ and $\forall a \in A, a \in B$ we know $a \in B$.
- Because we know $a \in A$ and $\forall x \in A, x \in B$ by setting $x = a$ we know $a \in B$.
- Because we know a is an element of A, and every element of A is an element of B, we know a is an element of B.

Here we are using $\forall$-elimination, the important thing is to make it clear that we know something about every element of a set, that we have a particular element of that set, and then to make it clear what this tells us about this particular element.

Another Example

Symbolization Key:
$\mathcal{U}$: The set of all animals.
B: The set of all bears.
xNy: 'x is near y'
$R(x)$: 'x is running away'
m: 'some partiuclar animal'

Prove the deduction

$$\forall x \in \mathcal{U}, (x \in B) \Rightarrow (mSx), \quad \forall x \in \mathcal{U}, ((mSx)\&(xNm)) \Rightarrow R(m), \quad \exists x \in B, xNm, \quad \therefore R(m)$$

$\forall$-Introduction

To prove that something is true about all $x \in X$, then either I must go through each element $x \in X$, and write a seperate proof that this assertion is true about x or...

I can write a single proof that would work for each $x \in X$ and make it clear that the proof works for an arbitrary $x \in X$. That is, even if my worst enemy was given the chance to pick any $x \in X$ he would like, my proof would still work.

The most common missuse of this rule is to only prove that there exists an $x \in X$ for which $P(x)$ is true, rather than showing it for all.
You need to be very careful not to assume extra things about the arbitrary x that you chose.
Proving that white rabits have fluffy tails does not imply that all rabits have fluffy tails!

The only real alternative to this rule if you want to prove a $\forall$ statement, is a proof by contradiction, show that if a counterexample existed, this would lead to a contradiction.

$\forall$-Introduction Example

Suppose A, B, and C are sets. Prove the deduction:

$$A \subset B, \qquad B \subset C, \qquad \therefore A \subset C.$$

Whenever you are trying to prove a forall statement, it is likely that the first thing you want to do is to setup for a $\forall$-Introduction.
Recognizing the proof pattern of $\forall$ statements and $\Rightarrow$ statements gives you the first line of at least 50% of proofs in this course and elsewhere!

Sentence choices

Line 1 - **the setup for $\forall$-introduction**

- Let $a \in A$ be arbitrary.
- Let a be an arbitrary element of A.
- Let a be an arbitrary element of A, so $a \in A$.
- Suppose a is chosen to be an arbitrary element of A.
- Select an arbitrary element a from A.

Line 2 - **explaining the meaning of a symbol**

- By hypothesis we know that $A \subset B$ and so we know $\forall a \in A, a \in B$.
- Since $A \subset B$ we know $\forall x \in A, x \in B$.
- Since $A \subset B$ we know every element of A is an element of B.

Line 3 - **a first use of $\forall$-elimination**

- In particular, because $a \in A$ we know $a \in B$.
- In particular, because a is an element of A it is also an element of B.
- Because we know $a \in A$ and $\forall a \in A, a \in B$ we know $a \in B$.
- Because we know $a \in A$ and $\forall x \in A, x \in B$ by setting $x = a$ we know $a \in B$.
- Because we know a is an element of A, and every element of A is an element of B, we know a is an element of B.

Line 4 - **explaining the meaning of a symbol**

- By hypothesis we know that $B \subset C$ and so we know $\forall a \in B, a \in C$.
- Since $B \subset C$ we know $\forall b \in B, b \in C$.
- Since $B \subset C$ we know $\forall x \in B, x \in C$.
- Since $B \subset C$ we know every element of B is an element of C.

Line 5 - **a second use of $\forall$-elimination**

- In particular, because $a \in B$ we know $C \in B$.
- In particular, because a is an element of B it is also an element of C.
- Because we know $a \in B$ and $\forall a \in B, a \in C$ we know $a \in C$.
- Because we know $a \in A$ and $\forall x \in B, x \in C$ by setting $x = a$ we know $a \in C$.
- Because we know a is an element of B, and every element of B is an element of C, we know a is an element of C.

Line 6 - **the conclusion of our $\forall$-introduction**

- Because $a \in A$ was arbitrary, we have shown $\forall a \in A, a \in C$.
- Because $a \in A$ was arbitrary, we have shown $\forall x \in A, x \in C$.
- Because $a \in A$ was arbitrary, we have shown every element of A is an element of C.
- Because a was an arbitrary element of A, we have shown every element of A is an element of C.

Line 7 - **explaining the meaning of notation**

- Because every element of A is an element of C this means $A \subset C$.
- We have shown $\forall x \in A, x \in C$ which by definition means $A \subset C$.
- Therefore by definition $A \subset C$.

General Proof Writing Comments

The first thing you should think about when you start writing a proof is:

what am I trying to prove?

Then think about

Does the meaning of that imply I should be doing a $\forall$-introduction?

or possibly

Does the meaning of that imply I should be doing a $\exists$-introduction?

or maybe

Does the meaning of that imply I should be doing a $\Rightarrow$-introduction?

Then look at the hypothesis and ask:
Do they tell me things exist?

Use a $\exists$-elimination.

Do they tell me something about all objects of some type?

If I have one of those, then use $\forall$-elimination.