Limits

The concept of a limit is basically the thing which makes Calculus work. What we will be talking about here is a version of limits that often appears in Real Analysis.

We call an infinite list $a_1, a_2, a_3, \ldots$ of real numbers a **sequence** of real numbers.

Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers and L is any other real number. We say that the sequence **converges** to L (and write $a_n \to L$) if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$$

Recall that the absolute value of a number is just the 'size' of the number (gets rid of negative sign).

Examples

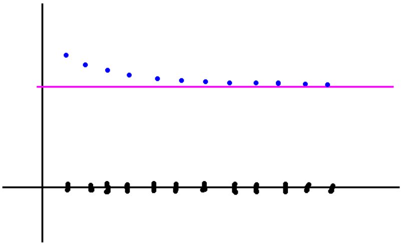
- $1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, \qquad n, \qquad \ldots$
- $1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots, \qquad 1, \qquad \ldots$
- $1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots, \qquad (-1)^{n+1}, \qquad \ldots$
- $1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8 \ldots, \qquad 1/n, \qquad \ldots$
- $3/5, 5/8, 7/11, 9/14, 11/17, 13/20, 15/23, 17/26, \ldots \qquad (2n+1)/(3n+2), \qquad \ldots$

Sequences don't have to have formuals, or we don't need to know them to work with them, but it is often reassuring to imagine they do.

Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers and L is any other real number. We say that the sequence **converges** to L (and write $a_n \to L$) if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$$

The intuitive way to think about limits is that the list of numbers, a_n is getting closer and closer to the limit L as we go further and further into the list.



Facts about absolute values

Working with limits means using absolute values, here is a list of rules, you have probably seen them before, you may have forgetten some of them.

For $x, y, z \in \mathbb{R}$ we know:

1. $|x| \geq 0$
2. $|-x| = |x|$
3. $|x+y| \leq |x| + |y|$ this is called the triangle inequality
4. $|xy| = |x||y|$
5. $-|x| \leq x \leq |x|$
6. $\exists N \in \mathbb{N}, N > |x|$ one can end up using this alot with limits
7. If $|x| < |y|$ and $z \neq 0$ then $|xz| < |yz|$.
8. If $|x| > |y| > 0$ then $\frac{1}{|x|} < \frac{1}{|y|}$.

Intuition of limits

Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers and L is any other real number. We say that the sequence **converges** to L (and write $a_n \to L$) if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$$

- The assertion $|a_n - L| < \epsilon$ means a some particular n, that a_n is closer than ϵ to L.
  With $\epsilon = 1/2$, $L = 0$, $n = 5$ and $a_n = 1/n$ we have
  $$|1/5 - 0| < 1/2$$
- $\forall n > N, |a_n - L| < \epsilon$ then says that the entire sequence after a particular N (*the end of the sequence*) is closer than ϵ to L.
  With $\epsilon = 1/10$, $L = 0$, $N = 10$ and $a_n = 1/n$ if $n > 10$ then $|1/n - 0| < 1/10$ so after the 10th element in the sequence, everything is closer than $1/10$.
- $\exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$ says that there is actually some notion of the *end of the sequence* which is closer than ϵ to L.
  For example with $\epsilon = 1/100$, $L = 0$, $a_n = 1/n$ we can find some value N which shows
  $$\exists N \in \mathbb{N} \forall n > 10, |1/n - 0| < 1/100$$
  eg N=100th spot, everything in the sequence is pretty close to zero.
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$ says that no matter how close I insist the word close actually means, then the sequence eventually gets that close.

Example Theorems About Limits

We will need the definition while we do some examples.

If a_1, a_2, a_3, . . . is an infinite list of real numbers. and L is another real number. We say that the sequence (the infinite list) **converges** to L (and write $a_n \to L$) if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$$

Notice that there are three quantifiers, this should tell us how most of our proofs about limits will start!

- $a_n = \frac{1}{n}$, $\therefore a_n \to 0$.
- $a_n = \frac{2n+1}{3n+2}$, $\therefore a_n \to \frac{2}{3}$.

Example Theorems About Limits (Using Hypothesis)

We will need the definition while we do some examples.

If a_1, a_2, a_3, ... is an infinite list of real numbers. and L is another real number. We say that the sequence (the infinite list) **converges** to L (and write $a_n \to L$) if and only if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \epsilon$$

Notice that there are three quantifiers, this should tell us how we can use it as a hypothesis.

- $a_n \to 10$, $\therefore \exists N, \forall n > N, a_n < 20$.
- $a_n = 10b_n$, $b_n \to 6$, $\therefore a_n \to 60$.
- $a_n = b_n + 3c_n$, $b_n \to 2$, $c_n \to 3$, $\therefore a_n \to 11$.
- $a_n = b_n^2$, $b_n \to 10$, $\therefore a_n \to 100$.

Some facts that are useful that you can't use unless you prove them yourself!

these are good exercises, the last one is kinda tricky.

Suppose $a_n \to L_1$ and $b_n \to L_2$ then:

- if $c_n = (a_n + C)$ then $c_n \to (L_1 + C)$.
- if $c_n = (Ca_n)$ then $c_n \to CL_1$.
- if $c_n = (a_n + b_n)$ then $c_n \to (L_1 + L_2)$
- if $c_n = (a_n b_n)$ then $c_n \to (L_1 L_2)$ (this is a bit tricky to prove (that is, there is a trick to it), you will not need to use/prove it in this course but it is important for actually figuring out when limits will exist and what they will be)
- if $a_i, L_1 \neq 0$ and $c_n = \frac{1}{a_n}$ then $c_n \to \frac{1}{L_1}$ (this is even trickier to prove, you will definitely not need to use/prove it in this course but it is important for actually figuring out when limits will exist and what they will be)