Cartesian Products

The **Cartesian Product** of two sets A and B, is the set $A \times B$ whose elements are all of the **ordered pairs** (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

- An ordered pair (a, b) is **NOT** a set.
- The order in an ordered pair matters $(1, 2) \neq (2, 1)$.
- It is almost always the case that $A \times B \neq B \times A$.

People often use Cartesian Products to define things like vector spaces (eg $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the real plane.

Even if you just have sets $A = \{1, 2, 3, 4\}$ and $B = \{red, blue, green\}$, then you can still picture it as a grid.

What is the cardinality of $A \times B$?

The number of elements in $A \times B$ is the size of A times the size of B

Examples Proofs Cartesian Products

Show that if $A = \emptyset$, then

$$A \times B = \emptyset$$

Show that if $A \neq \emptyset$ and $A \times B \subset A \times C$ then

$$B \subset C$$

Similarly we can show that

- If $A \times B \subset C \times D$ then

  $A = \emptyset$ or $B = \emptyset$ or ($A \subset C$ and $B \subset D$)

- If $A \times B = B \times A$?

  then $A = \emptyset$ or $B = \emptyset$ or $A = B$

Functions - informal

The idea of a function is that it is a thing which, upon being given an input (which should be some value in a set we call the domain) it somehow (through some well defined process) gives us a single value (which should be some value in a set we call the codomain).

There are lots of ways of trying to describe functions:

- Using an abstract concept.
- They can be described by a table of values.
- They can be described by a graph.
- They can be described by a formula.
- They can be described by a picture.
- A formal definition which leaves them impossible to actually evaluate for any value of input.

Just because you think you are describing a function does not mean that you are actually describing a function (we will see what the conditions are when we give the actual definition on the next slide).

Functions - formal

A **function**, f, from a set A to a set B is a **subset** $f \subset A \times B$ such that for each $a \in A$ there exists a **unique** $b \in B$ such that $(a, b) \in f$.

We will write $f : A \to B$ to indicate that f is a function from A to B.

We will write

$$f(a) = b \qquad \Leftrightarrow \qquad (a, b) \in f$$

to describe the value of the function at a.

The intuition to this definition is that we are describing a function by its *graph*.

The conditions which determine if something is a function can be expressed precisely as:

- is $f(a)$ defined for each $a \in A$:

  $$\forall a \in A, \exists b \in B, (a, b) \in f$$

  Verticle line test: each verticle line intersects the graph somewhere.

- is $f(a)$ well defined for each $a \in A$:

  $$\forall a \in A, \forall b_1 \in B, \forall b_2 \in B, (a, b_1) \in f \,\&\, (a, b_2) \in f \Rightarrow (b_1 = b_2)$$

  Verticle line test: each verticle line intersects the graph **at most once**.

Functions - formal

We call A the **domain** of f.

We call B the **codomain** of f. (We will rarely use this terminology)

We call the set $R = \{r \in B \mid \exists a \in A, (a, r) \in f\}$ the **range** of f. (It is often very hard to determine the exact range of a function, which is why the terminology codomain exists)

It is often useful to visuallize a function by its graph (as we will do shortly), even if it isn't always practical for the sets A and B we are using the intuition can be helpful.

If is possible to describe functions where the domain/codomain are cartesian products

$$f : A \times B \to C \qquad f(a, b) = c$$

or

$$gA \to B \times C \qquad g(a) = (b, c)$$

it is often harder to visuallize such things, but even so, they have graphs which are subsets of

$$f \subset (A \times B) \times C \qquad g \subset A \times (B \times C)$$

we won't typically deal with such functions in this course.

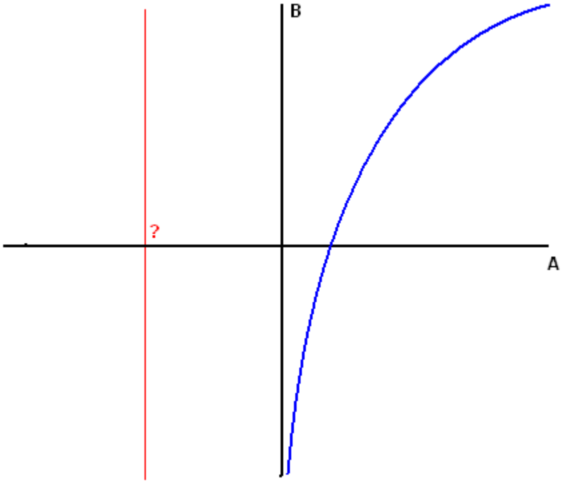
Examples/non-examples of functions

The usual way you think of evaluating a function works.
Importantly we notice this curve passes the vertical line test, because any vertical line intersects exactly once

Examples/non-examples of functions

This Curve fails the vertical line test for some lines, even though it passes for others.

Examples/non-examples of functions



This Curve fails the vertical line test, because some vertical lines don't hit anything.

Examples/non-examples of functions

- There is a function M from the set P of all people to itself which assigns to every person, the person who gave birth to them (so, 'their mother'). Why is this (or isn't it) a function?
- If we replace M by S, ie. sister, then this is definitely **not** a function. Why?

Examples/non-examples of functions

- There is a function
$$\exp : \mathbb{R} \to \mathbb{R}$$
from the set $\mathbb{R}$ of all real numbers to itself which assigns to every real number, the value e^x. We might write
$$\exp(x) = e^x$$
- It is not a function if we try to define
$$\exp : \mathbb{Q} \to \mathbb{Q},$$
even though I can still write down the symbols
$$\exp(x) = e^x$$
Why?
- There is a function
$$i : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$
which assigns to every non-zero real number the value $\frac{1}{x}$. So
$$i(x) = \frac{1}{x}$$
- But $i : \mathbb{R} \to \mathbb{R}$ given by $i(x) = \frac{1}{x}$ is not a function.

Examples of functions we can't evaluate everywhere

A function we can explicitly describe, but can't actually evaluate.

The busy beaver function - Informal
The busy beaver function is a function:

$$\Sigma : \mathbb{N} \to \mathbb{N}$$

which for has the property that:
For each natural number n, the function $\Sigma(n)$ is equal to:
The longest possible runtime (in clock cycles) for a halting program of length n (in bytes)

The busy beaver function - Formal
Maximum number of 1's on the output tape of a 2-symbol, n-state halting Turing machine. (subject to a fairly specific definition of what a 2-symbol, n-state Turing machine is.)

This is an explicitly defined function... but we **provably** cannot compute $\Sigma(n)$ for most n.
We actually only currently know it for $0, 1, 2, 3, 4$.

Examples of functions we can't evaluate everywhere

A function which we formally assert exists, without even describing how to evaluate it.

Axiom of Choice (non standard formulation that glosses over some technicalities.) Given any set X, there is a function C_X from the set $\mathcal{P}(X) \setminus \{\emptyset\}$ to X which to every non-empty subset of $A \subset X$ assigns some value $a \in A$.

Consider the case $X = \mathbb{R}$, let $C_\mathbb{R}$ be a function from $\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ to $\mathbb{R}$ guarenteed by the axiom of choice.
Then $C_\mathbb{R} : \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$ is a function... but for the most part I don't know how to evaluate it.
So we know

- $C_\mathbb{R}(\{1, 2, 3\}) \in \{1, 2, 3\}$, but we don't know which.
- $C_\mathbb{R}(\{2, 3, 4, 5, 6, 7, 8\}) \in \{2, 3, 4, 5, 6, 7, 8\}$, but we don't know which.

The function makes choices for us, which seems straight forward enough.

Assuming there are functions that can make infinitely many choices will lead to

- The Banach-Tarski Paradox
- The existance of non-measurable sets.

What are we going to do with functions?

We are going to define terminology that makes some assertions about functions:

- One-to-one functions
- Onto functions
- Bijective functions

We are going to define some common constructions involving functions:

- Compositions of functions
- Inverses of functions.
- Direct Image and Pre-image functions.

We are going to prove some theorems about these assertions, these constructions, and the relationships between them.

One-to-one functions

Definition

We say a function $f : A \to B$ is **One-to-one** or **injective** if and only if:

$$\forall a_1 \in A, \forall a_2 \in A, (f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2)$$

(These are sometimes also said to be **monomorphisms**, or **monic**, the definitions of those words is technically different but equivalent in this context.)

Intuitively this means that:

no two values of A get sent to the same value in B

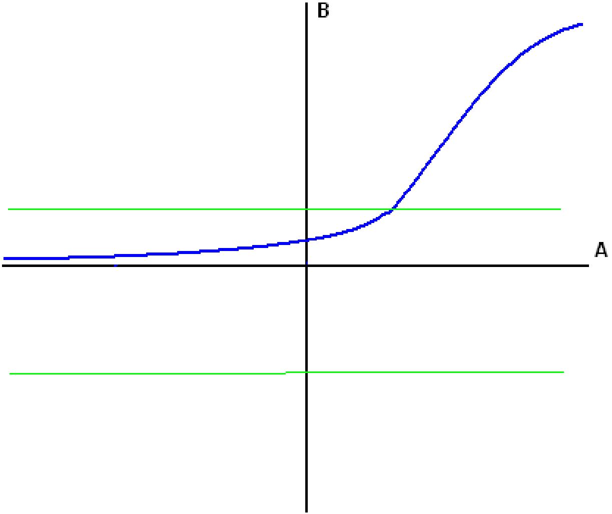
About the graph it says:

If you draw a horizontal line $y = b$

then there is at most one point $(x, f(x))$ which is on the graph and the line.

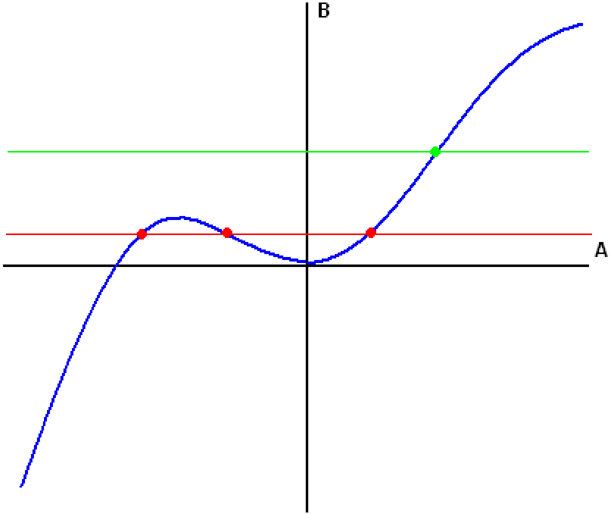
if there were two points $(a_1, f(a_1))$, $(a_2, f(a_2))$ with $y = b$, then $f(a_1) = b = f(a_2)$ so $a_1 = a_2$.

Example Pictures



This is injective, because no horizontal line intersects more than once.

Example Pictures



This is not injective, because there is a horizontal line that intersects twice.

Showing a function is one-to-one

Imagine some person gives you a function f, and asks you to prove:

$$\forall x \in A, \forall y \in A, (f(x) = f(y)) \Rightarrow (x = y)$$

even without knowing f, what can we say about the proof?

Generic Structure

Let $x \in A$ be arbitrary. because of the outer forall

Let $y \in A$ be arbitrary. because of the inner forall

Assume $f(x) = f(y)$. because we are doing a subproof for $\Rightarrow$-intro

This means ... this will depend on the definition of f

We want to show that $x = y$

filling in the proof will require manipulating the definition of f and maybe the knowledge that $x, y \in A$.

Examples about one-to-one functions

We say a function $f : A \to B$ is **One-to-one** or **injective** if and only if:

$$\forall x \in A, \forall y \in A, (f(x) = f(y)) \Rightarrow (x = y)$$

- Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is one-to-one.
- Show that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2 + 2x + 1$ is **not** one-to-one.
- Show that $h : \{x \in \mathbb{R} \mid x > 0\} \to \mathbb{R}$ given by $h(x) = x^2 + 2x + 1$ is one-to-one.
- Show that if $f : \mathbb{R} \to \mathbb{R}$ is a one-to one function and $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = 3f(x)$, then $g(x)$ is a one-to-one function.
- If you have an injective fuction $f : A \to B$, then **intuitively** what can you say about the relative sizes?

Onto functions

Definition

We say a function $f : A \to B$ is **Onto** or **surjective** if and only if:

$$\forall b \in B, \exists a \in A, f(a) = b$$

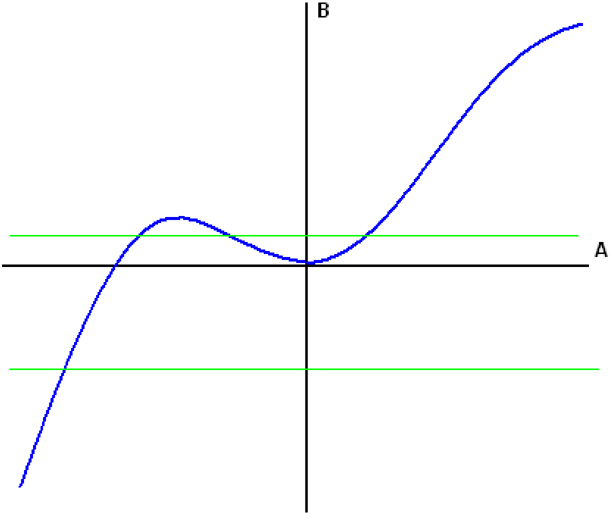
(These are sometimes also said to be **epimorphisms**, or **epic**, though the definitions of those words is technically different but equivalent in this context.)

Intuitively this means that:
Every element of B is the range of f.

About the graph it says:

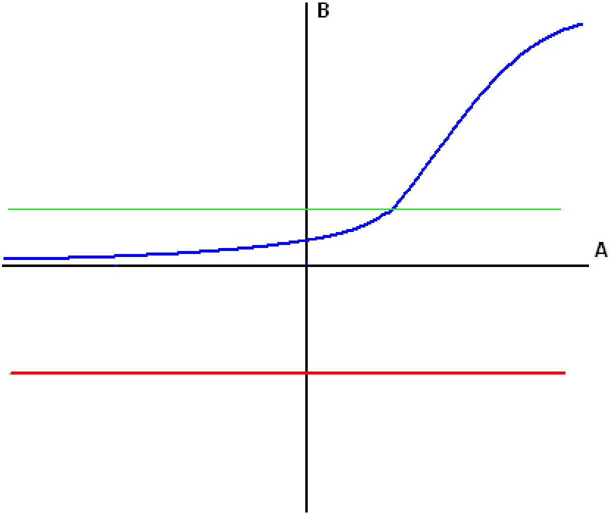
If you draw a horizontal line $y = b$
then there is at least one point $(x, f(x))$ which is on the graph and the line.
if there were no value a, with (a, b) on the graph, then there is no $a \in A$ with $f(a) = b$.

Example Pictures



This is surjective, because every horizontal line intersects at least once.

Example Pictures



This is not surjective, because there is a horizontal line that does not intersect.

Showing a function is onto

We are proving $\forall b \in B, \exists a \in A, (f(a) = b)$.

Generic Structure
Let $b \in B$ be arbitrary. because of the outer forall
Let $a =$??? is an element of A because of the inner exists
We want to show that $f(a) = b$ To give the proof of the there exists intro.
This usually just requires evaluating f, if f has a nice definition this will be easy

The hard part is usually filling in the ??? above

Examples with Onto functions

We say a function $f : A \to B$ is **Onto** or **surjective** if and only if:

$$\forall b \in B, \exists a \in A, f(a) = b$$

- Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is onto.
- Show that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2 + 2x + 1$ is **not** onto.
- Show that $h : \mathbb{R} \to \{x \in \mathbb{R} \mid x \geq 0\}$ given by $h(x) = x^2 + 2x + 1$ is onto.
- Show that if $f : \mathbb{R} \to \mathbb{R}$ is a onto function and $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = 3f(x)$, then $g(x)$ is a onto function.
- If you have a surjective fuction $f : A \to B$, then **intuitively** what can you say about the relative sizes?

Bijective functions

Definition

We say a function $f : A \to B$ is **Bijective** if and only if it is both one-to-one and onto.

(These are sometimes also said to be **isomorphisms**, or **iso**, though the definitions of those words is technically different but equivalent in this context, we will prove this, though we will not define the word.)

What does this say about the set $f \subset A \times B$?

Combining what one-to-one says, and what onto says we get...

For every horizontal line, there exists a unique point where it intersects the graph.

What does this remind you of?

Alternative Sympol Definition

We say a function $f : A \to B$ is **Bijective** if and only if it is both one-to-one and onto.

You can interpret this to say:

$$\forall y \in B, \text{there exists a } \textbf{unique } x \in A, f(x) = y$$

many people write this as:

$$\forall y \in B, \exists! x \in A, f(x) = y$$

this is often not helpful in writing proofs.

Showing a function is bijective

We are proving two things, that it is injective, and that it is surjective.
So we divide our proofs into two parts, where first we do the one, and second we do the other.

Examples with Bijective functions

We say a function $f : A \to B$ is **Bijective** if and only if it is both one-to-one and onto.

- Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is bijective.
- Show that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2 + 2x + 1$ is **not** bijective.
- Show that if $f : \mathbb{R} \to \mathbb{R}$ is a bijective function and $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = 3f(x)$, then $g(x)$ is a bijective function.
- If you have a bijective fuction $f : A \to B$, then **intuitively** what can you say about the relative sizes?

Composition of functions

Definition

Given a pair of functions $f : A \to B$ and $g : B \to C$ we can define the **Composition**, $g \circ f : A \to C$ of the functions g and f to be:

$$g \circ f(x) = g(f(x))$$

If we define it in terms of sets we can define $g \circ f \subset (A \times C)$ by:

$$g \circ f = \{(a, c) \in A \times C \mid \exists b \in B, ((a, b) \in f) \,\&\, ((b, c) \in g)\}$$

We would then need to prove, that this set satisfies all the axioms to give a function. There isn't really a great graph interpretation of this... unless you can picture things in 4 dimensions.

Some facts about composition

Theorem
Given three functions $f : A \to B$, $g : B \to C$, and $h : C \to D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

That is, the order of composition doesn't matter, and so we write:

$$h \circ g \circ f$$

It is rarely true that

$$f \circ g = g \circ f$$

in fact, it is rarely true that both of those two things even make sense!!!

Computing Compositions

Suppose f and g are both functions from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = x^2 + 2$ and $g(y) = 3y + 7$.
What is $f \circ g$ and what is $g \circ f$?

When dealing with compositions of functions it is often useful to arange for every function involved to have a different *variable name*.
The name of the variable in the description of the function doesn't change its meaning, but having the same variable appear more than once can lead to errors.

Some Proofs About Compositions

We say a function $f : A \to B$ is **Onto** or **surjective** if and only if:

$$\forall b \in B, \exists a \in A, f(a) = b$$

We say a function $f : A \to B$ is **One-to-one** or **injective** if and only if:

$$\forall a_1 \in A, \forall a_2 \in A, (f(a_1) = f(a_2)) \Rightarrow (a_1 = a_2)$$

Suppose $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is injective.
Prove that f is injective.

Suppose $f : A \to B$ and $g : B \to C$ are both surjective functions.
Prove that $g \circ f$ is surjective.

The following are also true (excercises)

- If both f and g are injective the composition is injective.
- If both f and g are bijective the composition is bijective.
- If $g \circ f$ is surjective then g is surjective.

Hints about proofs with injective/surjective/bijective

- If ever you have a function $f : A \to B$, and an element $a \in A$, you probably want to explain everything you know about $f(a)$.
  If ever you have a function f, but no elememt a, you are probably eventually going to want to find one!
  Functions are mostly useful when you are evaluating them, if you have to prove something about a function, you probably want to explain something about evaluating it somewhere at some point.
- If ever you have a function $g : B \to C$, and an element $b \in B$, you probably want to explain everything you know about $g(b)$.
  If ever you have a function g, but no elememt b, you are probably eventually going to want to find one!
- If ever you have a **surjective** function $h : A \to B$, and an element $b \in B$, you probably want to use this to tell me about the element $x \in A$ with $h(x) = b$.
  If you have a surjective function but no element b... you probably want to find b
- If ever you have an injective function $k : B \to C$, and two elements $b_1, b_2 \in B$, you probably want to think about whether or not $k(b_1) = k(b_2)$, and if so explain why this means $b_1 = b_2$.
  If you have an injective function but no elements b_1, b_2... you probably want to find some

The inverse function

This is another way in which we get new functions from old functions

Definition

Given a pair of functions $f : A \to B$ and $g : B \to A$ we say that g is the inverse of f if and only if:

$$(\forall a \in A, g \circ f(a) = a) \,\&\, (\forall b \in B, f \circ g(b) = b).$$

If g is the inverse of f we will often write $g = f^{-1}$.

Intuitively we think of the inverse of the function f as the function which 'undoes' f, or the function that lets me solve solve for x in:

$$y = f(x) \qquad x = f^{-1}(y)$$

We will see in a sec that this is how we find inverses of functions.

Alternative Interpretation

Recall that $g \circ f : A \to A$ and $f \circ g : B \to B$.

The condition $g \circ f(a) = a$ says $g \circ f = \mathrm{Id}_A$ where $\mathrm{Id}_A : A \to A$ defined by $\mathrm{Id}_A(a) = a$.

The condition $f \circ g(b) = b$ says $f \circ g = \mathrm{Id}_B$ where $\mathrm{Id}_B : B \to B$ defined by $\mathrm{Id}_B(b) = b$.

Computing Inverses

Definition

Given a pair of functions $f : A \to B$ and $g : B \to A$ we say that g is the inverse of f if and only if:

$$(\forall a \in A, g \circ f(a) = a) \,\&\, (\forall b \in B, f \circ g(b) = b).$$

If g is the inverse of f we will often write $g = f^{-1}$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 7$, find the inverse, and confirm that it is an inverse.

Consider the function $g : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{1\}$ given by $g(x) = \frac{x+2}{x+1}$, confirm that:

$$h(x) = \frac{2-x}{x-1}$$

is the inverse.

I would not ask you to do this with insanely complex functions on an exam, because it is just asking you to make calculation errors, but it is a very useful algebra skill to have.

The existance of an inverse

Question

Given $f : A \to B$, under what conditions can their possibly exist a function $g : B \to A$ so that:

- $(\forall a \in A, g \circ f(a) = a)$? What would happen if $f(a) = f(b)$?
  So we have the deduction:
  $$(\forall a \in A, g \circ f(a) = a) \quad \therefore f \text{ is injective}$$
- $(\forall b \in B, f \circ g(b) = b)$? What is $g(b)$ an example of?
  So we have the deduction:
  $$(\forall b \in B, f \circ g(b) = b) \quad \therefore f \text{ is surjective}$$
- $(\forall a \in A, g \circ f(a) = a)$ & $(\forall b \in B, f \circ g(b) = b)$
  So we have the deduction:
  $$g \text{ is the inverse of } f \quad \therefore f \text{ is bijective}$$

Can we get a converse to the answers to the above?

- Yes (mostly), and you can prove it. (exercise)
- Maybe... only if you assume the **axiom of choice** (you can think about why it should be true, but proving it may require infinitely many choices)
- Yes, and you can prove it.

Proof of existence of inverse

Theorem

A function has an inverse if and only if it is bijective.

This is a also a bonus question on assignment.

- We already sketched above that f having an inverse implies f is bijective.
- For the converse (bijective implies has an inverse) one step is to show that if f is bijective then the graph

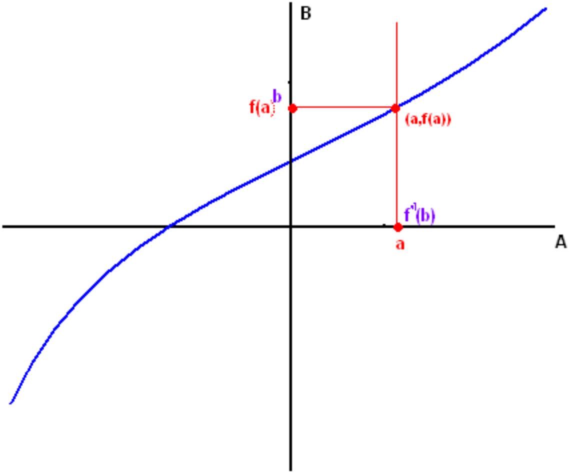
$$g = \{(y, x) \in B \times A \mid (x, y) \in f\}$$

  describes a function, that is show that it passes the vertical line test.
- The other step is then to prove that this function is the inverse of f, that is, the function g whose graph is

$$\{(y, x) \in B \times A \mid (x, y) \in f\}$$

  is actually the inverse of f.
  This part is awkward to explain, but lets look at a picture about why it is true..

This function is
bijective,

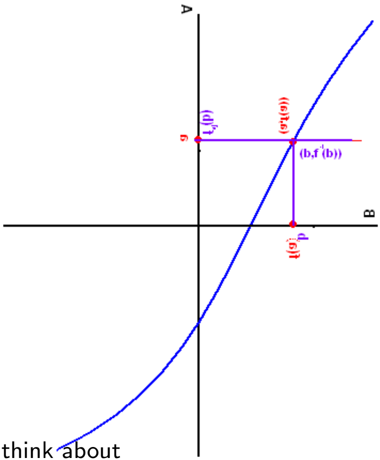


To understand the inverse think about

$$f^{-1}(b) = a \Leftrightarrow f(a) = b$$

To draw a graph of the function f^{-1}, just need to flip it in the line $y = x$.

This function is bijective,

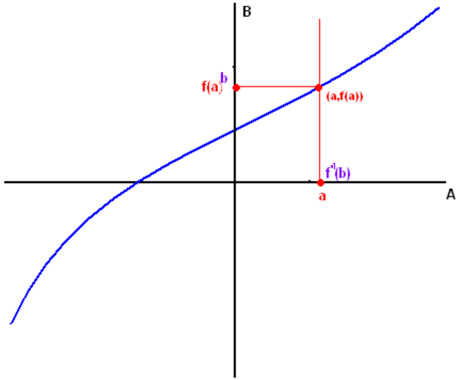


To understand the inverse think about

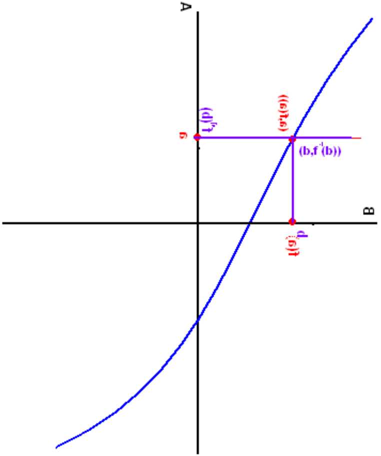
$$f^{-1}(b) = a \Leftrightarrow f(a) = b$$

To draw a graph of the function f^{-1}, just need to flip it in the line $y = x$.

The graph for an inverse just comes from swapping axis... and so flipping the graph.



$\Rightarrow$



This is why injective/surjective are important, the horizontal line tests become vertical line tests!

Proofs about Inverses

If $f : A \to B$ and $g : B \to A$ then g is the inverse of f if and only if:

$$(\forall a \in A, g \circ f(a) = a) \,\&\, (\forall b \in B, f \circ g(b) = b).$$

Suppose $f : A \to B$ and $g : B \to C$ are functions with inverses respectively $f^{-1} : B \to A$ and $g^{-1} : C \to B$.

Prove that $f^{-1} \circ g^{-1}$ is the inverse of $f \circ g$.

A few other theorems about inverses

Many things are true about inverses! Below is a sample of some often useful facts.

Theorem

Suppose $f : A \to B$ is a function. If g and h are both inverses of f, then $g = h$.

Idea: To prove $g = h$ we must prove

$$\forall b \in B, g(b) = h(b)$$

functions are equal if they always take on same value

Let $x \in B$ be arbitrary, then because they are both inverses we have

$$f(g(x)) = x = f(h(x))$$

so by the injectivity of f, we know $g(x) = h(x)$.

Theorem

If g is the inverse of f then f is the inverse of g. That is $(f^{-1})^{-1} = f$.

Idea: check directly that f satisfies the definition of being the inverse of f^{-1}.

Filling in the details we leave as an exercise.

The (direct)-image of a function

If $f : A \to B$, and $X \subset A$, the direct image of X under f, is

the stuff that X gets sent to by f.

Formal definition Construction/Definition

Given a function $f : A \to B$ we can construct a function (which, by an abuse of notation, we will also call f because we are terrible people, some people use the notation f_*) $f : \mathcal{P}(A) \to \mathcal{P}(B)$ using the rule

$$f(X) = \{b \in B \mid \exists x \in X, f(x) = b\}$$

This gives us a way to make sense of writing $f(X)$ whenever $X \subset A$.
Given a subset of the domain $X \subset A$ we call the set $f(X) \subset B$ the (direct-)**image** of X under f.

Practical Definition: The fact that the direct image is a function will never really matter in this course. If you want you can just think of it as notation, given a function f and a subset X of the domain we define

$$b \in f(X) \Leftrightarrow \exists a \in X, f(a) = b$$

Example Problem

$$b \in f(X) \Leftrightarrow \exists a \in X, f(a) = b$$

Prove that if $f : A \to B$ is a function, and $X, Y \subset A$ then

$$f(X \cap Y) \subset f(X) \cap f(Y)$$

The main thing that happens in proofs about direct image is you use the definition of the direct image.

whenever you know something is in a direct image, explain what this means.

whenever you want to prove something is in a direct image, prove the thing that this means.

The pre-image of a function

If $f : A \to B$, and $Y \subset B$, the pre image of Y is

the stuff that gets sent to Y by f.

Construction/Definition

Given a function $f : A \to B$ we can construct a function (which, by an abuse of notation, we will also call f^{-1} because we are terrible people, some people use the notation f^*) $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ using the rule

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$$

This gives us a way to make sense of writing $f^{-1}(Y)$ whenever $Y \subset B$.
Given a set $Y \subset B$ we call the set $f^{-1}(Y) \subset B$ the **pre-image** (or inverse-image) of Y under f.

The notation is only vaguely connected to the idea of the inverse:

- $f : A \to B$ might not have an inverse.
- $f : \mathcal{P}(A) \to \mathcal{P}(B)$ might not have an inverse.

Practical Definition: The fact that the pre(inverse)-image is a function will never really matter in this course. If you want you can just think of it as notation, given a function f and a subset Y of the codomain we define

$$a \in f^{-1}(Y) \Leftrightarrow f(a) \in Y$$

Example Problem

$$a \in f^{-1}(Y) \Leftrightarrow f(a) \in Y$$

Prove that if $f : A \to B$ is a function, and $X, Y \subset B$ then

$$f^{-1}(X) \cap f^{-1}(Y) \subset f^{-1}(X \cap Y)$$

Prove that if $f : A \to B$ is a function, and $X \subset B$ then

$$f(f^{-1}(X)) \subset X$$

The main thing that happens in proofs about pre-image is you use the definition of the pre-image.

whenever you know something is in a pre-image, explain what this means.
whenever you want to prove something is in a pre-image, prove the thing that this means.