Example of Induction

Find out if it is true that for all $n \in \mathbb{N}$ we have:

$$\sum_{i=0}^{n} i^4 = \frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5}$$

n	0	1	2	3	4	5	6
n^4	0	1	16	81	256	625	1296
$\sum$	0	1	17	98	354	979	2275

So lets check if this makes sense:

- $(\frac{1}{5})0(1)(\frac{1}{2})(\frac{-1}{3}) = 0$
- $(\frac{1}{5})1(2)(\frac{3}{2})(\frac{5}{3}) = 1$
- $(\frac{1}{5})2(3)(\frac{5}{2})(\frac{17}{3}) = 17$
- $(\frac{1}{5})3(4)(\frac{7}{2})(\frac{35}{3}) = (2)(7)(7) = 98$
- $(\frac{1}{5})4(5)(\frac{9}{2})(\frac{59}{3}) = (2)(3)(59) = 354$
- $(\frac{1}{5})5(6)(\frac{11}{2})(\frac{89}{3}) = (11)(89) = 979$

But there are infinitely many n to check... this could take a while...

Lets check something else, notice that:

$$\sum_{i=0}^{n} i^4 = \left(\sum_{i=0}^{n-1} i^4\right) + n^4$$

So if the theorem was true, at least for n and $n-1$ then we would have:

$$\frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5} = \frac{(n-1)(n)(n-\frac{1}{2})(n^2-n-\frac{1}{3})}{5} + n^4$$

But this is something we can actually check! And indeed:

$$\frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5} = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$\frac{(n-1)(n)(n-\frac{1}{2})(n^2-n-\frac{1}{3})}{5} = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

But I can run these calculations backwards, so we can turn this around.

Theorem

Suppose $n \in \mathbb{N}$ is greater than zero. If

$$\left(\sum_{i=0}^{n-1} i^4\right) = \frac{(n-1)(n)(n-\frac{1}{2})(n^2-n-\frac{1}{3})}{5}$$

Then

$$\left(\sum_{i=0}^{n} i^4\right) = \frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5}$$

Sketch of Proof

$$\begin{aligned}
\left(\sum_{i=0}^{n} i^4\right) &= \left(\sum_{i=0}^{n-1} i^4\right) + n^4 && \text{definition of sums}\\
&= \frac{(n-1)(n)(n-\frac{1}{2})(n^2-n-\frac{1}{3})}{5} + n^4 && \text{hypothesis}\\
&= (\frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n) + n^4 && \text{algebra}\\
&= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n && \text{algebra}\\
&= \frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5} && \text{algebra}
\end{aligned}$$

Let $P(n)$ the assertion $\sum_{i=0}^{n} i^4 = \frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5}$.
The above argument proves the implication:

$$\forall n > 0, P(n-1) \Rightarrow P(n)$$

We have also verified $P(0), P(1), P(2), P(3), P(4)$.
Lets verify some more:

- We know $P(4)$ and we know $P(4) \Rightarrow P(5)$ (since $5 > 0$) therefore we know $P(5)$.
- We know $P(5)$ and we know $P(5) \Rightarrow P(6)$ (since $6 > 0$) therefore we know $P(6)$.
- We know $P(6)$ and we know $P(6) \Rightarrow P(7)$ (since $7 > 0$) therefore we know $P(7)$.
- We know $P(7)$ and we know $P(7) \Rightarrow P(8)$ (since $8 > 0$) therefore we know $P(8)$.
- We know $P(8)$ and we know $P(8) \Rightarrow P(9)$ (since $9 > 0$) therefore we know $P(9)$.

This could still take a while... but at least the calculations are easier!
Who here thinks $P(17)$ is true? $P(521342)$? $P(12351235235235)$?

So the original theorem should be true for all n. But how do we explain this?

Induction - Formalism

Let $P(n)$ be any predicate on the set $\mathbb{N}$.
We have the following deduction:

$$P(0), (\forall n > 0, (P(n-1) \Rightarrow P(n))), \therefore \forall n \in \mathbb{N}, P(n)$$

The intuition is just what you saw above.
What it lets you do is bypass needing to write out infinitely many steps!
(We will sketch a proof of this later).

You may also have seen:

$$P(0), (\forall n \geq 1, (P(n-1) \Rightarrow P(n))), \therefore \forall n \in \mathbb{N}, P(n)$$

or

$$P(0), (\forall n \geq 0, (P(n) \Rightarrow P(n+1))), \therefore \forall n \in \mathbb{N}, P(n)$$

or

$$P(1), (\forall n \geq 1, (P(n) \Rightarrow P(n+1))), \therefore \forall n \geq 1, P(n)$$

or

$$P(1), (\forall n \geq 2, (P(n-1) \Rightarrow P(n))), \therefore \forall n \geq 1, P(n)$$

these are all equivalent! (except the exact conclusion is a bit different)

Examples

Strategy:

1. We need to define a predicate P on $\mathbb{N}$.
   In examples you have to do, this will typically be exactly the thing you are trying to prove.
2. Need to check a base case (typically $P(0)$ or $P(1)$).
   In examples you have to do, this will typically be exactly the first case you are trying to prove.
   But note: in principal this isn't always the case, see below!
3. Need to check that $P(n-1) \Rightarrow P(n)$.
   This is typically the hard part.

Prove that if $a \neq 1$

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}$$

Prove that

$$\forall n \geq 0, 2^n \geq 2n$$

hint: $n = 0, 1$ should be a base cases!

Recurance Sequences

A recurance sequence is just a sequence of numbers which satisfies a recurance relation.
A recurance relation on a sequence is a *rule* which says that:

$$\forall n > N, a_n \text{ can be determined using a formula based on } a_m \text{ for } m < n$$

For example:

$$a_n = a_{n-1} + a_{n-2} \quad n > 1$$

This gives a recurance relation.
A sequence which satisfies a recurence relation is then determined by initial values:

$$a_0 = 1 \quad a_1 = 3 \quad a_n = a_{n-1} + a_{n-2} \quad n > 1$$

determines the whole sequence!

Consider the recurrence sequence:

$$a_0 = 0 \qquad a_n = a_{n-1} + 6n^2 \quad n > 0$$

What are the first few terms?
Prove that for all $n \in \mathbb{N}$ we have

$$a_n = n(n+1)(2n+1)$$

Other forms of Induction

- Slightly Stronger Induction:

$$P(0), P(1), (\forall n > 1, (P(n-2) \,\&\, P(n-1) \Rightarrow P(n))), \therefore \forall n \in \mathbb{N}, P(n)$$

- Strong Induction:

$$\forall n \in \mathbb{N}, ((\forall k < n, P(k)) \Rightarrow P(n)), \therefore \forall n \in \mathbb{N}, P(n)$$

- Induction on sets X that are not $\mathbb{N}$.
  1. The key feature of induction is an ordering (or at least a partial ordering) which is bounded below.

Stronger Induction Example

Define a sequence by the following rule:

$$a_0 = 3 \quad a_1 = 9 \qquad\qquad a_n = 2a_{n-1} - a_{n-2} \text{ for } n > 1$$

Prove by induction that for all n we have $a_n = 6n + 3$.

Proof

Let $P(n)$ be the assertion that $a_n = 6n + 3$.

Base Cases

The case $n = 0$ is obvious, because $a_0 = 3 = 6(0) + 3 = 6n + 3$.
The case $n = 1$ is also obvious, because $a_1 = 9 = 6(1) + 3 = 6n + 3$.

Inductive Case

We assume that $n > 1$ and that both $P(n-1)$ and $P(n-2)$ are true.
We calculate

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} && \text{because } n > 1 \\ &= 2(6(n-1)+3) - 6(n-2)+3 && \text{inductive hypothesis for } a_{n-1}, a_{n-2} \\ &= 6n + 3 \end{aligned}$$

And so this proves that $P(n)$ is true.
Because we proved the base cases, $P(0), P(1)$ and we proved that $(P(n-2) \,\&\, P(n-1)) \Rightarrow P(n)$, we conclude that $P(n)$ is true for all n, and thus that $a_n = 6n + 3$.
We can see in the formula why we need both $P(n-1)$ and $P(n-2)$.

Well ordering (Why induction works)

Given a set $S \subset \mathbb{N}$, we say $m \in S$ is the smallest element if and only if:

$$\forall n \in S, m \leq n$$

Theorem

Every non-empty subset $S \subset \mathbb{N}$ of the natural numbers has a smallest element.

Theorem

The following deduction is valid.

$$P(0), \forall n > 0, (P(n-1) \Rightarrow P(n)), \therefore \forall n \in \mathbb{N}, P(n)$$

Proof Sketch

Let $S = \{n \in \mathbb{N} \mid \neg P(n)\}$ we need to prove $S = \emptyset$.
We will prove this by contradiction, so assume that S is not empty.
Let $m \in S$ be the smallest element. So $\neg P(m)$.
We claim that $P(m)$ is true, we shall prove this by considering cases $m = 0$ or $m \neq 0$ If $m = 0$ then we know $P(m) = P(0)$ is true so we are done.

If $m \neq 0$ then we know $m - 1 \notin S$, because m was the smallest element and $0 \leq m - 1 < m$. Since $m - 1 \notin S$ we know $\neg\neg P(m-1)$ so we know $P(m-1)$.

Other Proofs Using Well Ordering

Theorem (division algorithm)
Given any two integers $a > 0$ and $b \geq 0$ there exists integers q and r with $0 \leq r < a$ such that:

$$b = aq + r$$

(the above is true without the assumption on positivity)
Proof Sketch Let R be the set of integers:

$$R = \{r \in \mathbb{N} \mid \ \& \ a|(b-r)\}$$

We must show the set is not empty
The set R is not empty because $b \in R$, as $b - b = 0$ and so $a|(b-b)$.
Let r be the smallest element of R (which exists by the well ordering of $\mathbb{N}$).
We wish to prove that $r < a$, we shall do this by contradition.
Suppose for the purpose of contradition that $r \geq a$.Then $r - a \geq 0$ and because $a|(-a)$ and $a|r$ we can show (you can show!) that $a|(r-a)$.Therefore $r - a \in R$, but $r - a < r$, this contradicts the fact that r was the smallest element of R.
We conclude that $r < a$.

you can now show that there exists q so that $b = aq + r$!

Expectations

Induction (and well ordering) are very useful strategies for proving theorems and are applicable in many contexts.

In general, identifying when an induction proof is helpful can be tricky, even identifying 'what n should describe', or the predicate $P(n)$ can be a puzzle.

In this course we will generally tell you to use induction, and it should be clear what the predicate and base cases will be.

You need to know how to setup an induction proof

- Introduce the statement you will be proving by induction and the variable (n) you will be inducing on.
- Identify and prove the base case(s).
- Prove the inductive case.
- Give a summary that explains that you have just given a proof by induction.