What is a vector?

A vector in $\mathbb{R}^n$ is an ordered list of real numbers.

$$(x_1, \ldots, x_n)$$

where $x_i \in \mathbb{R}$.
So $(1, 2, 3)$ is a vector in $\mathbb{R}^3$, and $(\pi, \sqrt{3})$ is a vector in $\mathbb{R}^2$.
It is important that we realize that $(1, 2, 3)$ is not the same as $(3, 2, 1)$.
A vector in $\mathbb{C}^n$ is then just an ordered list of complex numbers.

$$(x_1, \ldots, x_n)$$

where now $x_i \in \mathbb{C}$.

What are vectors good for?

A vector (in $\mathbb{R}^n$) is just a list of real numbers... so what is a list of real numbers good for?

- Maybe they describe locations in space.
- Maybe your list of numbers describe probabilities.
- Maybe your list of numbers describes the number of goals each player on your favorite sports team has.
- Maybe they describe the states in a finite state machine.
- Maybe they describe the populations of wolves and hares.
- Maybe they are taken from the table of values of some function.
- Maybe they describe how many of each colour of smartie I have.

A list of numbers could be any list of numbers.

In many cases doesn't only having the list of numbers lose information?

Yes and no, if you remember the translation scheme:

$$(2, 2, 4, 1/2, 1)$$
$$\updownarrow$$
$$2\text{'cups of sugar'} + 2\text{'cups of butter'} + 4\text{'eggs'} + (1/2)\text{'cups of flour'} + 1\text{'cup of cocoa'}$$

then we don't lose information. But if we forget the translation scheme, then the list loses a lot of value.

Remembering the translation scheme is very important in most applications, so it can be useful to actually keep the translation as part of the notation. Oddly there is also some implicit extra information in presenting things as an ordered list.

As an example of not doing this, in physics it is very common to write:

$$(a, b, c) \leftrightarrow a\vec{i} + b\vec{j} + c\vec{k} = b\vec{j} + a\vec{i} + c\vec{k}$$

because somehow, what you have is more important than the order.

What can I do with a list of numbers?

Depending on what your list of numbers represents, most random things you can do with them are pretty meaningless.
For example, with:

$$2\text{cups of sugar} + 2\text{cups of butter} + 4\text{eggs} + (1/2)\text{cups of flour} + 1\text{cup of cocoa}$$

What is the meaning of $(2)(2)(4)(1/2)(1)$?I am not convinced that number has anything to do with the quality of my brownies.
However,

$$2(2, 2, 4, 1/2, 1) = (4, 4, 8, 1, 2)$$

looks like I just doubled the amount of brownies I am making.

So what are we going to do with lists of numbers?

We will focus on certain types of operations which are often meaningful.
However, properly interpretting what they mean will always depend on context!

- If you have two lists of numbers of the same length, you can add them up, this generically corresponds to combining two collections of things.
- If you have a list of numbers, you can multiply it by a constant, this corresponds to uniformly increasing how much of everything you have.
- If you have two lists of numbers, you might be interested in 'how far apart' they are?
  - In terms of strict distance
  - In terms of the 'angle between them' (or the correlation)

  Is $(2, 2, 4, 1/2, 1)$ more like $(2, 0, 4, 1/2, 1)$ or like $(4, 4, 8, 1/2, 2)$?
  In any case, you need to be careful when talking about distance, because butter is way more important than flour in brownies.
- Processes which translate one list into another.

Multiplying Lists by Constants

Given any vectors (list of numbers) $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and some constant c in $\mathbb{R}^n$ we can always write down the vector (list of numbers)

$$c\vec{x} = (cx_1, cx_2, \ldots, cx_n)$$

which is also in $\mathbb{R}^n$.

Examples:

$$\begin{aligned}
5(1,3,5) &= (5,15,25) \\
3(7,5) &= (21,15) \\
\pi(\sqrt{2},21) &= (\sqrt{2}\pi, 21\pi) \\
2(6) &= (12) \\
17() &= () \\
0(2,4,6,8) &= (0,0,0,0)
\end{aligned}$$

Adding Lists Together

Given any two list of numbers $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we can always write down the vector (list of numbers)

$$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n)$$

which is also in $\mathbb{R}^n$.

Examples:

$$\begin{gathered}
(1, 3, 5) + (2, 3, 4) = (3, 6, 9) \\
(7, 5) + (0, 0) = (7, 5) \\
(\sqrt{2}, 21) + (\pi, -3) = (\sqrt{2} + \pi, 18) \\
(6) + (-7) = (-1) \\
() + () = () \\
(0, 0, 0, 0) + (0, 0, 0, 0) = (0, 0, 0, 0)
\end{gathered}$$

Linear Combinations of Lists

I can do both of these at the same time, and with any number of lists

$$\vec{x}^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)}),\ \vec{x}^{(2)} = (x_1^{(2)}, \ldots, x_n^{(2)}),\ \ldots,\ \vec{x}^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$$

and corresponding real numbers $c_1, \ldots, c_k$ then I can work with

$$\begin{aligned}
&c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \cdots + c_k\vec{x}^{(k)} \\
=&c_1(x_1^{(1)}, \ldots, x_n^{(1)}) + c_2(x_1^{(2)}, \ldots, x_n^{(2)}) + \cdots + c_k(x_1^{(k)}, \ldots, x_n^{(k)}) \\
=&(c_1x_1^{(1)} + c_2x_1^{(2)} + \cdots + c_kx_1^{(k)}, c_1x_2^{(1)} + c_2x_2^{(2)} + \cdots + c_kx_2^{(k)}, \ldots, c_1x_n^{(1)} + c_2x_n^{(2)} + \cdots + c_kx_n^{(k)})
\end{aligned}$$

For example

$$2(1,2,3) - 6(1,1,1) + 2(3,2,1) = (2,2,2)$$

It is worth noticing that when you have a list of lists of numbers, you may need to have a lot of super/subscripts :-(

Evaluating Functions on Lists

We often define functions: $f : \mathbb{R}^n \to ??$ by explaining a formula based on the list, eg:

$$f : \mathbb{R}^n \to \mathbb{R} \qquad f(x_1, \ldots, x_n) = x_1 + 2x_2 + \cdots + nx_n$$

So if given $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n)$, and real numbers a, b and if we define

$$\vec{z} = a\vec{x} + b\vec{y} \in \mathbb{R}^n.$$

What is

$$f(\vec{z}) = f(a\vec{x} + b\vec{y})?$$

If we write $\vec{z} = (z_1, z_2, \ldots, z_n)$ then we will know that

$$f(\vec{z}) = z_1 + 2z_2 + \cdots + nz_n.$$

Luckily we know how to do that as $z_i = ax_i + by_i$, we have:

$$\begin{aligned} f(a\vec{x} + b\vec{y}) &= f(ax_1 + by_1, \ldots, ax_n + by_n) \\ &= (ax_1 + by_1) + 2(ax_2 + by_2) + \cdots + n(ax_n + by_n) \end{aligned}$$

In order to apply the *formula*, we write the list in a format compatible with the *formula*. This goes both ways, if someone is nice and tells you that *the rule* $g(a\vec{x} + b\vec{y}) = a$ actually is a function (do we remember what that means?), this is useful if you need to evaluate $g(7\vec{x} + 3\vec{y})$... but not useful to directly evaluate $g(1, 2, 3, 4, 5, 6)$.

Note *the rule*, $g(a\vec{x} + b\vec{y}) = a$, may or may not describe a function in different contexts

Concatenating Lists Warning: this is a fairly advanced thing to do

Given any two list of numbers $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, we can always write down the list of numbers

$$(\vec{x}, \vec{y}) = (x_1, \dots, x_n, y_1, \dots y_m)$$

which represents the image of $(\vec{x}, \vec{y}) \in \mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{n+m}$.

One has to be very careful here...
the choice of order $(\vec{x}, \vec{y})$ vs $(\vec{y}, \vec{x})$ matters.
Also notice that if your first list is

$$\text{2cups of sugar} + \text{2cups of butter} + \text{4eggs} + (1/2)\text{cups of flour} + \text{1cup of cocoa}$$

and your second list is:

$$\text{1cups of sugar} + (1/2)\text{cups of milk} + \text{1cup of cocoa}$$

you might want to combine the two entries for sugar and cocoa... or you might want to keep them separate.
This type of problem, in a more abstract sense, is a common source of errors in linear algebra (and baking)
We will talk about this sort of thing implicitly more later when we discuss *direct sums*, in that context we will see why it is useful to have no overlap between the lists being concatenated.

Distances between lists

Given any two list of numbers $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \dots, y_m) \in \mathbb{R}^n$, we can write down a function that describes the distance between them.

$$||\vec{x} - \vec{y}|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

This may not be the function you are looking for!
If the units on the things in your list don't agree, (number of eggs vs cups), then this quantity may not represent exactly what you want it to.
There are lots of other reasons a different function might be more natural.

- Butter is way more important than flour.
- You have different entries for brown/white sugar.
- You really need to consider what the different things in your list are, and how they all relate to each other.

The generalized idea of distances and angles is something we will discuss near the end of the course when we talk about *inner product spaces*.

Angles between lists (correlation)

Given any two list of numbers $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \ldots, y_m) \in \mathbb{R}^n$, we can write down a function that describes the 'angle' between them.

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n$$

This is the dot product.

To get the angle out of this you need to recall that

$$\vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos(\theta)$$

in any case, this quantity on its own still captures some information.

This formula **defines** the angle provided $||\vec{x}||$ and $||\vec{y}||$ are not 0.
An important case is that $\vec{x} \cdot \vec{y} = 0$ if and only if the angle is $\pi/2$, so the vectors are perpendicular, or one of the vectors is all 0's

As above though, someone might want to renormalize differently depending on what the stuff in your list means

The generalized idea of distances and angles is something we will discuss near the end of the course when we talk about *inner product spaces*.

Projecting a vector onto another

Given any two list of numbers $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \ldots, y_m) \in \mathbb{R}^n$, we can write down a function that describes the projection of the first onto the second.

$$\mathrm{Proj}_{\vec{x}}(\vec{y}) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||}\vec{x}.$$

This is basically the same as the above...
You are looking for the extent to which the first vector points in the same direction as a second vector.
as above though, someone might want to renormalize differently depending on what the stuff in your list means

The generalized idea of projections is that comes up very briefly when we discuss *direct sum decompositions*.

Taking a Cross Product

Cross products are a fascinating and deep construction arising in $\mathbb{R}^0$, $\mathbb{R}^1$, $\mathbb{R}^3$ and $\mathbb{R}^7$ based on a triality that exists on a related vector space (of one higher dimension) and is in some sense responsible for the fact that only certain spheres are parallelizable (a deep fact in differential geometry)

The cross product in $\mathbb{R}^0$ is given by

$$() \times () = ()$$

The cross product in $\mathbb{R}^1$ is given by

$$(x_1) \times (y_1) = (0)$$

The cross product in $\mathbb{R}^3$ is given by

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

The cross product in $\mathbb{R}^7$ is given by

$$(x_1, \ldots, x_7) \times (y_1, \ldots, y_3) = (x_2 y_3 - x_3 y_2 + x_4 y_5 - x_5 y_4 + x_7 y_6 - x_6 y_7, \ldots)$$

let us not mention this again!

Questions people ask about vectors.

- Is it possible to combine several lists in some way so as to obtain a specific other list? and if so how?
  Solve:
  $$a_1\vec{x}_1 + \cdots + a_r\vec{x}_r = \vec{y}$$
- If I can't do that, how close can I get?
  (This is often least squares approximation, which is a form of regression analysis)
  doing this requires a good notion of distance!
  The essential idea is always:
  The path that goes from the closest point to the actual point will be perpendicular to all of the directions in which I could adjust things

In both of these cases, the underlying calculation that needs to be performed is to solve a system of linear equations. (Which is something you should have done a lot in previous linear algebra courses, and that comes up again in one the next few things we will want to review)

Another questions people ask about vectors.

What is the best way to represent the information I have by a list of numbers?

Sometimes information in a list can be represented in two different ways:

$$1\text{cups of white sugar} + 1\text{cups of brown sugar} + 2\text{cups of butter}$$

vs

$$2\text{cups of sugar} + 0\text{white/brown sugar bias} + 2\text{cups of butter}$$

or

$$1.5\text{cups of white sugar} + 0.5\text{cups of brown sugar} + 2\text{cups of butter}$$

vs

$$2\text{cups of sugar} + 0.5\text{white/brown sugar bias} + 2\text{cups of butter}$$

You can translate by noticing that:

$$1\text{cups of sugar} = 0.5\text{cups of white sugar} + 0.5\text{cups of brown sugar}$$

and

$$1\text{white/brown sugar bias} = 1\text{cups of white sugar} - 1\text{cups of brown sugar}$$

Understanding how to leverage this idea is something we will come back to later in the course.