Complex Numbers

Consider two real numbers x, y.

- What is $2 + x$?
- What is $x + y$?
- What is $(2 + x)(3 + y)$?
- What is $(2x + 3y)(3x + 5y)$?
- What is the inverse of $3 + x$?
- What one fact do I know for sure about x^2?

Now imagine now that $x, y \in \mathbb{C}$, what changes about the above?
It no longer even makes sense to ask if $x^2 > 0$, because $x > y$ doesn't make sense for complex numbers.

Complex Numbers

What was the point of all that?

If you don't have specific complex numbers in mind, if you are just working with arbitrary constants that are place holders, then complex numbers are not really different than real numbers.

Complex Numbers

Consider the two specific real numbers $x = 2 + \sqrt{7}$ and $y = 3 + 4\pi$.

- What is $2 + x$?
- What is $x + y$?
- What is $(2 + x)(3 + y)$?
- What is $(2x + 3y)(3x + 5y)$?
- What is the inverse of $3 + x$?

Now imagine now that $\sqrt{7}$ is replaced by $\sqrt{-1}$, what changes about the above?
I have to replace all the $\sqrt{7}$'s by $\sqrt{-1}$'s and all the places where I wrote $(\sqrt{7})^2 = 7$ by $(\sqrt{-1})^2 = -1$
You shouldn't be scared of i, it is in fact it is a much easier number to understand than say $\pi + e$. What is the inverse of

$$1 + \pi + e?$$

vs

$$1 + i$$

Complex Numbers

What was the point of all that?

Working with explicit real numbers that are not small rational numbers is kindof a pain, and needing to do explicit things like compute inverses even more so.
This isn't something which is annoying about complex numbers vs real numbers, it is something which is annoying about numbers that are not in the set

$$\{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\}$$

If it were up to me, we would never need to work explicitly with such annoyingly complicated numbers as 11.

But the universe disagrees.

Actually doing computations with complex numbers

We can explicitly represent any complex number in the form:

$$z = a + bi$$

where a and b are real numbers, and i is a complex numbers such that $i^2 = -1$.
We often call this the cartesion form of a complex number.
We call a the **real part**, sometimes denote by $a = \mathrm{Re}(z)$
of z, and b the **imaginary part**, sometimes denoted by $b = \mathrm{Im}(z)$.
Now, imagine that $a + bi$ and $c + di$ are complex numbers. We have the following useful rules:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- $(a + bi)^{-1} = \frac{1}{a^2+b^2}(a - bi)$.

Polar form

We can also explicitly represent any complex number in the form:

$$z = a + bi = r\cos(\theta) + r\sin(\theta)i = re^{i\theta}$$

Where $r \geq 0$ and $\theta \in [0, 2\pi)$. This is called the **polar** form of a complex numbers. We call θ the **argument** of the complex number.
(You can write the same things with r negative and θ outside that range, but it can be useful to pick the reduced form)
The conversion in the one direction is easy (if you can evaluate cosine and sine), in the other direction use that:

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}(b/a)$$

which is also easy if you can evaluation arctan....
Now, imagine that $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$ are complex numbers. We have the following useful rules:

- $r_1 e^{i\theta_1} + r_2 e^{i\theta_2} =$ There is a formula, but no one uses it..
- $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.
- $(r_1 e^{i\theta_1})^{-1} = \frac{1}{r_1} e^{-i\theta_1}$.

Complex Conjugates

If $z = a + bi$ is a complex number, then the complex conjugate is

$$\overline{z} = a - bi$$

One of the key properties of $\overline{z}$ is the fact that:

$$z\overline{z} = a^2 + b^2 = r^2$$

is always positive

This lets us define the **absolute value** of a complex number as

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} = r$$

This then lets us define the length of a complex vector:

$$||(z_1, z_2, z_3)|| = \sqrt{z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3}}$$

and the **Hermitian** inner product

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1\overline{y_1} + x_2\overline{y_3} + x_3\overline{y_3}$$

Note that this agrees with the dot product for $\mathbb{R}^n$ when all complex numbers are actually real numbers.

Fundamental Theorem of Algebra

Let $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be a polynomial with $a_i \in \mathbb{C}$.
Then there exists $\lambda_i \in \mathbb{C}$ so that

$$P(z) = \prod_{i=1}^{n} (z - \lambda_i)$$

The proofs of this theorem are topological.

Suppose that $a_i \in \mathbb{R}$.
Then if λ is a root, so too is $\overline{\lambda}$.