Vector Spaces - informal

A vector space V over $\mathbb{R}$ is first and foremost a set of things which we call vectors.

If $\vec{v}_1$ and $\vec{v}_2$ are vectors in V, that is $\vec{v}_1, \vec{v}_2 \in V$, and a, b are real numbers, $a, b \in \mathbb{R}$ then I should be able to make sense of

$$a\vec{v}_1 + b\vec{v}_2$$

Namely, this should be a vector in V, that is $(a\vec{v}_1 + b\vec{v}_2) \in V$.

Also, all those rules we take for granted about adding and multiplying should work out, because if it wasn't true that:

$$(a\vec{v}_1 + b\vec{v}_2) + (c\vec{v}_1 + d\vec{v}_2) = (a + c)\vec{v}_1 + (b + d)\vec{v}_2$$

then working with these things would be a serious pain.

Vector Spaces - Formal

A vector space V over $\mathbb{R}$ is a set V, together with two maps:

$$\cdot : \mathbb{R} \times V \to V \qquad \text{and} \qquad + : V \times V \to V$$

(the first map describes how to make sense of the expressions like $a\vec{v}_1$, the second how to make sense of expressions like $\vec{v}_1 + \vec{v}_2$.)
These two maps satisfy the following rules.

1. $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V$ we have $\vec{0} + \vec{v} = \vec{v}$.
2. $\forall \vec{v} \in V$ we have $\exists \vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$.
3. $\forall \vec{v}_1, \vec{v}_2 \in V$ we have $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$.
4. $\forall \vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$ we have $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$.
5. $\forall \vec{v} \in V$ we have $1\vec{v} = \vec{v}$.
6. $\forall \vec{v} \in V, \forall a, b \in \mathbb{R}$ we have $a(b\vec{v}) = (ab)\vec{v}$.
7. $\forall \vec{v} \in V, \forall a, b \in \mathbb{R}$ we have $(a + b)\vec{v} = (a\vec{v}) + (b\vec{v})$.
8. $\forall \vec{v}_1, \vec{v}_2 \in V, \forall a \in \mathbb{R}$ we have $a(\vec{v}_1 + \vec{v}_2) = (a\vec{v}_2) + (a\vec{v}_1)$.

Interpretation

All of these rules basically say everything works the way you think it should.

Examples

Give me some examples of sets $\mathcal{X}$, for which you have in the past taken two elements $\mathcal{A}, \mathcal{B} \in \mathcal{X}$ and then written:

$$\mathcal{A} + \mathcal{B}$$

Give me some examples of sets $\mathcal{X}$, for which you have in the past taken an element $\mathcal{A} \in \mathcal{X}$ and a real number C then written:

$$C\mathcal{A}$$

Can you think of some sets where you have done both of these things?
Have those rules from the last slide ever not been true?

$\mathbb{R}^n$ is a Vector Space

1. The vector $(0, 0, \ldots, 0)$, is a vector such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ we have
$$(0, 0, \ldots, 0) + (x_1, \ldots, x_n) = (x_1, \ldots, x_n)$$
so we have shown there is a vector that does what $\vec{0}$ must do, so set
$$\vec{0} = (0, 0, \ldots, 0)$$
and we see rule 1 works out.
2. For any vector $\vec{v} = (x_1, \ldots, x_n)$ the vector $\vec{w} = (-x_1, \ldots, -x_n)$ satisfies
$$\vec{v} + \vec{w} = (x_1, \ldots, x_n) + (-x_1, \ldots, -x_n) = (x_1 - x_1, \ldots, x_n - x_n) = (0, 0, \ldots, 0)$$
so rule 2 works out.
3. For any vectors $\vec{v}_1 = (x_1, \ldots, x_n)$ and $\vec{v}_2 = (y_1, \ldots, y_n)$ we have
$$\begin{aligned}\vec{v}_1 + \vec{v}_2 &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n)\\ &= (x_1 + y_1, \ldots, x_n + y_n)\\ &= (y_1 + x_1, \ldots, y_n + x_n)\\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n)\\ &= \vec{v}_2 + \vec{v}_1\end{aligned}$$
so rule 3 works out.

4. For any three vectors $\vec{v}_1 = (x_1, \ldots, x_n)$, $\vec{v}_2 = (y_1, \ldots, y_n)$ and $\vec{v}_3 = (z_1, \ldots, z_n)$

$$\begin{aligned}
(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 &= ((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) + (z_1, \ldots, z_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, \ldots, z_n) \\
&= (x_1 + y_1 + z_1, \ldots, x_n + y_n + z_n) \\
&= ((x_1, \ldots, x_n) + (y_1 + z_1, \ldots, y_n + z_n) \\
&= ((x_1, \ldots, x_n) + ((y_1, \ldots, y_n) + (z_1, \ldots, z_n)) \\
&= \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)
\end{aligned}$$

so rule 4 works out.

5. For any vector $\vec{v} = (x_1, \ldots, x_n)$ we have

$$1 \cdot \vec{v}v = 1 \cdot (x_1, \ldots, x_n) = (1 \cdot x_1, \ldots, 1 \cdot x_n) = (x_1, \ldots, x_n) = \vec{v}$$

so rule 5 works out.

6. For any vector $\vec{v} = (x_1, \ldots, x_n)$, and any real numbers a, b we have

$$\begin{aligned}
a \cdot (b \cdot \vec{v})) &= a \cdot (b \cdot (x_1, \ldots, x_n)) \\
&= a \cdot (bx_1, \ldots, bx_n) \\
&= ((ab)x_1, \ldots, (ab)x_n) \\
&= (ab) \cdot (x_1, \ldots, x_n) \\
&= (ab) \cdot \vec{v}
\end{aligned}$$

so rule 6 works out.

7. For any vector $\vec{v} = (x_1, \ldots, x_n)$ and any two real numbers a, b we have

$$\begin{aligned}(a+b)\vec{v} = &= (a+b)\cdot(x_1, \ldots, x_n) \\ &= ((a+b)x_1, \ldots, (a+b)x_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n) \\ &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= a\cdot(x_1, \ldots, x_n) + b\cdot(x_1, \ldots, x_n) \\ &= a\cdot\vec{v} + b\cdot\vec{v}\end{aligned}$$

   so rule 7 works out.

8. For any vector $\vec{v}_1 = (x_1, \ldots, x_n)$ and $\vec{v}_2 = (y_1, \ldots, y_n)$ and any real number a we have

$$\begin{aligned}a\cdot(\vec{v}_1 + \vec{v}_2) = a((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) &= a\cdot(x_1 + y_1, \ldots, x_n + y_n) \\ &= (a(x_1 + y_1), \ldots, a(x_n + y_n)) \\ &= (ax_1 + ay_1, \ldots, ax_n + ay_n) \\ &= (ax_1, \ldots, ax_n) + (ay_1, \ldots, ay_n) \\ &= a\cdot(x_1, \ldots, x_n) + a\cdot(y_1, \ldots, y_n) \\ &= a\cdot\vec{v}_1 + a\cdot\vec{v}_2\end{aligned}$$

   so rule 8 works out.

This proves that $\mathbb{R}^n$ is a vector space.

Examples

In principal to check that each of the following is a vector space, we must *figure out what* $\vec{0}$ *is*, and then check all 8 rules.

- $\mathbb{R}^n$.
- The space of n by m matricies.
- "polynomials in as many variables as you would like"
- Rational "functions" from $\mathbb{R}$ to $\mathbb{R}$.
- Functions from any set X into $\mathbb{R}$.
- Power series.
- The set of Taylor series of infinitely differentiable functions.
- Fourier Series.
- Sequences of real numbers.

Most of the vector spaces you deal with in this class can be thought of as subsets of $\mathbb{R}^n$. There are a very small number of examples that are mentioned on the assignment, and I will occasionally mention some other examples in class as motivation.

Alot of things follow from these 8 rules: Here are some facts to know

- If $\vec{v} \in V$ then $0\vec{v} = \vec{0} \in V$.

  Proof: Let $\vec{v} \in V$ be arbitrary and denote by $\vec{w}$ the vector from rule 2 for which $0\vec{v} + \vec{w} = \vec{0}$ then

$$\begin{aligned}
\vec{0} &= 0\vec{v} + \vec{w} && \text{rule 2}\\
&= (0+0)\vec{v} + \vec{w} && \text{because } 0 = (0+0) \text{ for real numbers}\\
&= (0\vec{v} + 0\vec{v}) + \vec{w} && \text{rule 7}\\
&= 0\vec{v} + (0\vec{v} + \vec{w}) && \text{rule 4}\\
&= 0\vec{v} + \vec{0} && \text{rule 2}\\
&= \vec{0} + 0\vec{v} && \text{rule 3}\\
&= 0\vec{v} && \text{rule 1}
\end{aligned}$$

- The element $\vec{w} = \vec{0}$ is the only element such that $\forall \vec{v} \in V$ we have $\vec{w} + \vec{v} = \vec{v}$.
  Proof: Suppose $\vec{w}_1$ and $\vec{w}_2$ both satisfy $\forall \vec{v} \in V$ we have $\vec{w}_1 + \vec{v} = \vec{v} = \vec{w}_2 + \vec{v}$.
  Then what is $\vec{w}_1 + \vec{w}_2$?

$$\begin{aligned}
\vec{w}_1 &= \vec{w}_1 + \vec{w}_2 && \text{assumption } \forall \vec{v}, \vec{v} + \vec{w}_2 = \vec{v}\\
&= \vec{w}_2 + \vec{w}_1 && \text{rule 3}\\
&= \vec{w}_2 && \text{assumption } \forall \vec{v}, \vec{v} + \vec{w}_1 = \vec{v}
\end{aligned}$$

- For all $a \in \mathbb{R}$, we have $a\vec{0} = \vec{0}$.
  Proof:

$$\begin{aligned}
a\vec{0} &= a(0 \cdot \vec{0}) && \text{see above}\\
&= (a0)\vec{0} && \text{rule 6}\\
&= 0 \cdot \vec{0} && \text{algebra}\\
&= \vec{0} && \text{see above}
\end{aligned}$$

- If $a \neq 0$ and $a\vec{v} = \vec{0}$ then $\vec{v} = \vec{0}$.

Proof:
$$\begin{aligned}
\vec{v} &= (\tfrac{1}{a}a)\vec{v} && \text{rule 5}\\
&= \tfrac{1}{a}(a\vec{v}) && \text{rule 6}\\
&= \tfrac{1}{a}\vec{0} && \text{assumption } a\vec{v} = \vec{0}\\
&= \vec{0} && \text{see before}
\end{aligned}$$

- If $\vec{v} \in V$ then $(-1)\vec{v} + \vec{v} = \vec{0} \in V$

Proof:
$$\begin{aligned}
(-1)\vec{v} + \vec{v} &= (-1)\vec{v} + (1)\vec{v} && \text{rule 5}\\
&= (-1+1)\vec{v} && \text{rule 7}\\
&= (0)\vec{v} && \text{algebra}\\
&= \vec{0} && \text{see before}
\end{aligned}$$

- If $\vec{v} \in V$ then the element $\vec{w} = (-1)\vec{v}$ (usually just written $-\vec{v}$) is the only element such that $\vec{w} + \vec{v} = \vec{0}$.

Proof: Suppose $\vec{w}_1$ and $\vec{w}_2$ both satisfy $\vec{w}_1 + \vec{v} = \vec{0} = \vec{w}_2 + \vec{v}$.
$$\begin{aligned}
\vec{w}_1 &= \vec{0} + \vec{w}_1 && \text{rule 1}\\
&= (\vec{w}_2 + \vec{v}) + \vec{w}_1 && \text{assumption } \vec{w}_2 + \vec{v} = \vec{0}\\
&= \vec{w}_2 + (\vec{v} + \vec{w}_1) && \text{rule 4}\\
&= \vec{w}_2 + (\vec{w}_1 + \vec{v}) && \text{rule 3}\\
&= \vec{w}_2 + \vec{0} && \text{assumption } \vec{w}_1 + \vec{v} = \vec{0}\\
&= \vec{0} + \vec{w}_2 && \text{rule 4}\\
&= \vec{w}_2 && \text{rule 2}
\end{aligned}$$

There are many facts whose proofs work like this... Let us **NEVER AGAIN** be so pedantic. Addition and multiplication of vectors works the way you think it does.

Non-Examples

Just because I define a $+$ sign and a $\cdot$ sign doesn't mean the result is a vector space. Those 8 rules exist for a reason.

- $\{\vec{0}, \text{cat}\}$ with the rules for addition
  - $\text{cat} + \text{cat} = \vec{0}$,
  - $\text{cat} + 0 = \vec{cat}$,
  - $0 + \text{cat} = \vec{cat}$,
  - $0 + 0 = \vec{0}$

  and the rules for multiplication
  - $a\text{cat} = \text{cat}$ for $a \neq 0$
  - and $0\text{cat} = \vec{0}$.

  Which rule is broken?

well if the rules did work then

$$\text{cat} = 2\text{cat} = (1+1)\text{cat} = (1\text{cat}) + (1\text{cat}) = \text{cat} + \text{cat} = \vec{0}$$

But $\text{cat} \neq \vec{0}$ so at least one of those steps does not work in this example.

Weird actual examples

Just because I define a $+$ sign and a $\cdot$ sign and the definition is wierd doesn't mean the result won't be a vector space.

One can construct wierd things which end up satisfying these rules

You won't need to do things with these types of examples in this class.

Consider the set $V = \{x \in \mathbb{R} | x > 0\}$.
Define $'+' : V \times V \to V$ by

$$\vec{v_1}\,'+'\,\vec{v_2} = (v_1 v_2).$$

We define $+$ on this vector space to be the product in $\mathbb{R}^+$.... which is wierd.
Define $\cdot : \mathbb{R} \times V \to V$ by

$$a\vec{v_1} = v_1^a$$

and we define scalar multiplication to be exponentiation.

These operations satisfy all of the rules which define a vector space!

Why Abstract Vector Spaces?

There are many examples of vector spaces that are not just $\mathbb{R}^n$. The observation that in many ways they work just like $\mathbb{R}^n$ is useful because:

- Once you know theorems for vector spaces they also apply to these other settings.
- A lot of the techniques for vector spaces can be applied in these other settings.
    - Putting 'dot' products on spaces of functions is a critical part of Fourier Analysis.
    - Studying differentiation and differential equations is often the equivalent of studying linear transformations.

Concrete Example - Digital Audio Recording

Sounds can be represented by graphs, acurately reproducing the sound requires storing enough data to accurately reproduce the graph.

For complicated sounds, this would require plotting a lot of data points.

Most complicated sounds are approximately just superpositions of simple sounds:

$$[complexsound] \approx a[simplesound1] + b[simplesound2] + c[simplesound3]$$

so I only need to know a, b, c and the graphs of the simple sounds.

Simple sounds have very simple graphs $\sin(n\pi x)$, so require very little data to describe (just remember the single number n).

Problem
How can we find the coefficients (that is a, b, and c)?
Solutions
Construct a "dot product" so that the coefficients come from projections.
This is basically Fourier analysis, and is one of the core ideas of signal processing.

Why Abstract Vector Spaces?

Because even $\mathbb{R}^n$ can have different descriptions.

- Two lists of numbers, might have the same length, say 5, but thinking of them as both being in the same $\mathbb{R}^5$ can be misleading.
  If one of the vectors is in V, and one is in W, you can think of these as 'different sets' even if they are both essentially $\mathbb{R}^5$.
  It is a good idea not to confuse a list that tracks the outcomes of sporting events with one that tracks the number of cats we all own.
- We have also aready seen, that there can be more than one way to assign a list of numbers to something as simple as baking ingredients (the brown/white sugar bias). Understanding the fact that there are different ways to identify V with $\mathbb{R}^n$ is easier if you don't already come with a preconcieved notion of how they should match up.
- It will make talking about subspaces easier.

Vector Space Definition - Summary

A vector space is a set, where addition and scalar multiplication make sense, and work the way you think they should.

In this class most, but not all, examples will end up being subsets of $\mathbb{R}^n$.

The next thing we want to talk about, is the special case of subsets of vector spaces as vector spaces.

Natural Questions to ask about Abstract Vector Spaces

- Is the set with the operations described on it actually a vector space?
  this can be technical but you need to understand what this means.
- What meaning can I associate to objects in some abstractly described vector space?
  This question is open ended.