Vector Subspaces - Informal

A subspace is a space, that is inside another space.

A vector subspace (which we will also just call a subspace) of a vector space V is just a subset $W \subset V$, which is still a vector space without needing to muck around with the definitions of $+$ and $\cdot$.

If V with $+$ and $\cdot$ is a vector space, and W is a subset of V, then by restricting the domain of $+$ and $\cdot$ we definitely have maps:

$$\cdot : \mathbb{R} \times W \to V \qquad + : W \times W \to V$$

and although we can easily just restrict the domain, the first major condition to being a subspace is that the **range** of the above maps lands in W so that I can also restrict the codomain so that I actually have maps:

$$\cdot : \mathbb{R} \times W \to W \qquad + : W \times W \to W$$

It is important to realize this is a serious condition, you can't normally just randomly change the codomain of a function!

Vector Subspaces - Formal

Given a vector space V (with $+$ and $\cdot$) we say that W is a **subspace** of V if $W \subset V$ and

1. $\vec{0} \in W$.
2. $\forall \vec{v}_1, \vec{v}_2 \in W$ we have $\vec{v}_1 + \vec{v}_2 \in W$.
3. $\forall \vec{v} \in W$ and $a \in \mathbb{R}$ we have $a\vec{v} \in W$.

The last two rules say that we can restrict the codomain as on the previous slide

Theorem

If V is a vector space and W is a subspace of V, then W is a vector space using the maps

$$\cdot : \mathbb{R} \times W \to W \qquad + : W \times W \to W$$

Lemma 2.I.2.9 p. 92 of Hefferon's textbook says something very similar to this

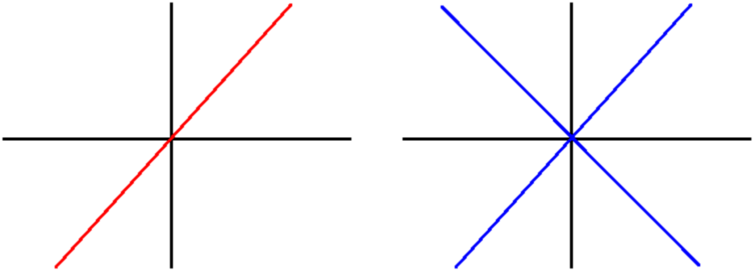
Proof Idea: Rule 2 and 3 for subspaces say that we can restrict the domain to get functions. In order to check rules 1-8 of being a vector space for these functions notice that

- Rules 3-8 for vector spaces are just forall statements, which follow immediately.
- Rule 2 for vector spaces follows from $\vec{v} + (-1)\vec{v} = \vec{0}$ and that $(-1)\vec{v} \in W$ if $\vec{w} \in W$ (rule 3 for subspaces).
- Rule 1 for vector spaces is rule 1 for subspaces.

Example (Picture)

Consider the two following subsets of $V = \mathbb{R}^2$.

$$W_1 = \{(x,y) \in V \mid x = y\} \qquad W_2 = \{(x,y) \in V \mid x^2 = y^2\}$$



One of these is a subspace, one is not!

If you pick any two vectors on W_1, and add them, the result is still on the line.

If you pick one vector from each of the two lines in W_2, and add them, the result is probably not on either line!

The pictures aren't proofs though, we will give an example of a proof shortly!

Examples (Proofs)

- The emptyset $W = \emptyset \subset V$ is **never** a subspace of V.
  proof: It fails rule 1 of being a subspace, because $\vec{0} \notin \emptyset$.
- The set $W = \{\vec{0}\} \subset V$ is always a subspace of V.
  proof: We must check the three conditions to be a subspace of W:
  1. By definition $\vec{0} \in W$.
  2. Given any two vectors in W, they are both $\vec{0}$ and we know
  $$\vec{0} + \vec{0} = \vec{0} \in W$$
  3. Given any vector in W it is $\vec{0}$, and given any $a \in \mathbb{R}$ we know
  $$a\vec{0} = \vec{0} \in W$$
- The set $W = \{(a, 2a, 3a) \in \mathbb{R}^3 \mid a \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
  proof: We must check the three conditions to be a subspace of $\mathbb{R}^3$:
  1. Taking $a = 0$ we see that $\vec{0} = (0, 0, 0) = (0, 2(0), 3(0)) \in W$, and so $\vec{0} \in W$.
  2. Let $(a, 2a, 3a), (b, 2b, 3b)$ be two arbitrary elements in W then by taking $c = a + b$ we see that
  $$(a, 2a, 3a) + (b, 2b, 3b) = (c, 2c, 3c)$$
  is also an element of W.
  3. Let $(a, 2a, 3a)$ be two arbitrary element in W and x an arbitrary element in $\mathbb{R}$ then by taking $c = ax$ we see that
  $$x(a, 2a, 3a) = (c, 2c, 3c)$$
  is also an element of W.

Examples (continued)

- The set $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$ is a subspace of $\mathbb{R}^3$.
  We must check the three conditions to be a subspace of $\mathbb{R}^3$:
  1. We see that $\vec{0} = (0, 0, 0)$, satisfies $(0) + (0) - (0) = 0$ and so $\vec{0} \in W$.
  2. Let $(x_1, y_1, z_1), (x_2, y_2, z_2)$ be two arbitrary elements in W so that we know $x_1 + y_1 - z_1 = 0$ and $x_2 + y_2 - z_2 = 0$ then
     $$(x_3, y_3, z_3) = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
     will have
     $$x_3 + y_3 - z_3 = (x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) = (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) = 0$$
     so is also an element of W.
  3. Let (x_1, y_1, z_1) be two arbitrary element in W, so we know $x_1 + y_1 - z_1 = 0$ and let c an arbitrary element in $\mathbb{R}$ then
     $$(x_2, y_2, z_2) = c(x_1, y_1, z_1) = (cx_1, cy_1, cz_1)$$
     will have
     $$x_2 + y_2 - z_2 = cx_1 + cy_1 - cz_1 = c(x_1 + y_1 - z_1) = c(0) = 0$$
     so is also an element of W.

Subspaces in $\mathbb{R}^3$

In $\mathbb{R}^3$ (and more generally $\mathbb{R}^n$) subspaces are not so mysterious.
We will see that in $\mathbb{R}^3$ all of the subspaces must be of the form:

- $\{0\}$
- A line through the origin
- A plane containing the origin
- All of $\mathbb{R}^3$.

Why Subspaces?

It turns out that the answers to a lot of questions that people end up asking about vectors and matricies

- Can be reinterpretted as a question about subspaces.
- Have answers which will actually just be subspaces.

This is true of both the problems of determining if a system of equations has a solution, and the problem of describing all the solutions to a system of equations.

Theorem
If W_1 and W_2 are subspaces of a vector space V then $W_1 \cap W_2$ is also a subspace.

This is a direct check, which we leave as an exercise.

Theorem
If we have $W \subset U \subset V$ with V a vector space, and U a subspace of V then:

$$W \text{ is a subspace of } U \text{ if and only if } W \text{ is a subspace of } V$$

This is a straight forward check, just notice that the conditions to be a subspace don't depend on the weather you check relative to U of V.

Natural Questions to ask about Subspaces

- Is the subset $X \subset V$ actually a subspace?
  Basically always solved by checking the three conditions.
- Is the specific vector $\vec{x} \in V$ actually in the subspace W?
  How to check this clearly depends on the definition of the subspace!! We will see a couple ways to introduce subspaces, each will ultimately come with a way to check.
- What are the *simplest* descriptions of the subspace W?
  This question is open ended, but it does make sense to ask for alternate descriptions.