Linear Combinations

Let V be a vector space, and let $\vec{x}_1, \ldots, \vec{x}_\ell \in V$ be a collection of vectors. Given $a_1, \ldots, a_\ell \in \mathbb{R}$ we call the vector

$$\vec{v} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell$$

a **Linear Combination** of $\vec{x}_1, \ldots, \vec{x}_\ell$.

Examples

1. The vector $(1,2,3,4)$ is a linear combination of $(4,3,2,1)$ and $(1,1,1,1)$, because

   $$(1,2,3,4) = 5(1,1,1,1) + (-1)(4,3,2,1)$$

2. Determine if $(6,5,4,3)$ can be expressed as a linear combination of $(1,2,3,4)$ and $(4,3,2,1)$.
   We need to try to solve

   $$(6,5,4,3) = a(1,2,3,4) + b(4,3,2,1)$$

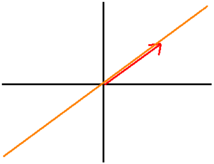
3. Determine if $(1,2,1,2)$ can be expressed as a linear combination of $(1,2,3,4)$ and $(4,3,2,1)$.
   We need to try to solve

   $$(1,2,1,2) = a(1,2,3,4) + b(4,3,2,1)$$

These problems always turn into solving a system of equations.

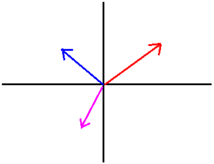
Examples (Picture)

Consider the vector $(3, 2)$, which vectors can be written as linear compinations of it?



it can only be the vectors on the line $3y = 2x$.

What if we have the vectors $(3, 2)$, $(-2, 2)$, and $(-1, -2)$?



Then it can be any vector in $\mathbb{R}^2$.

Spans

Let V be a vector space, let X be any subset of V. Define the **Span** of the set X to be:

$$\mathrm{Span}(X) = \{\vec{v} \in V \mid \exists \vec{x}_1, \ldots, \vec{x}_\ell \in X, \vec{v} \text{ is a linear combination of } \vec{x}_1, \ldots, \vec{x}_\ell\}$$

(**technical note:** even if the set X is infinite, the linear combinations only include finitely many of the X.)

Likewise, given a list of vectors, $\vec{x}_1, \ldots, \vec{x}_n$ we define

$$\mathrm{Span}(\vec{x}_1, \ldots, \vec{x}_n) = \{\vec{v} \in V \mid \exists a_1, \ldots, a_n \in \mathbb{R}, \vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n\}$$

These definitions agree when the set X is finite, as given $X = \{\vec{x}_1, \ldots, \vec{x}_\ell\}$ then

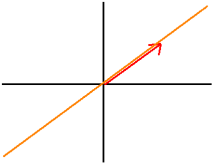
$$\mathrm{Span}(X) = \mathrm{Span}(\vec{x}_1, \ldots, \vec{x}_\ell) = \{a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell \mid a_1, \ldots, a_\ell \in \mathbb{R}\}$$

Put another way:

$$\vec{v} \in \mathrm{Span}(\vec{x}_1, \ldots, \vec{x}_\ell) \Leftrightarrow \exists a_1, \ldots, a_\ell,\ \vec{v} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell$$

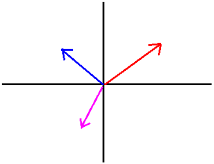
Examples (Picture)

Consider again the vector $(3, 2)$,



the span of $(3, 2)$ is the line $3y = 2x$.

If we have the vectors $(3, 2)$, $(-2, 2)$, and $(-1, -2)$,



then we have $\mathbb{R}^2 = \mathrm{Span}((3, 2), (-2, 2), (-1, -2))$.

Examples

By definition

$$\mathrm{Span}((1,2,3,4),(4,3,2,1)) = \{r(1,2,3,4) + s(4,3,2,1) \mid r,s \in \mathbb{R}\}$$

Determine if $(6,5,4,3) \in \mathrm{Span}((1,2,3,4),(4,3,2,1))$?
We need to try to solve

$$(6,5,4,3) = a(1,2,3,4) + b(4,3,2,1)$$

I feel I may have done that already!

Determine if $(1,2,1,2) \in \mathrm{Span}((1,2,3,4),(4,3,2,1))$?
We need to try to solve

$$(1,2,1,2) = a(1,2,3,4) + b(4,3,2,1)$$

I feel I may have done that already!

When the set 'X' is finite, these questions are identical to the previous questions about checking if something is a linear combination!

Theorem about Spans

Theorem

Given $X \subset V$ the set $\mathrm{Span}(X)$ is a subspace. It follows that $\mathrm{Span}(X)$ is the smallest subspace of V containing X.

Proof:

We must check the three subspace conditions

1. $\vec{0}$ is by convention equal to the empty sum, and so is a linear combination.
2. If $\vec{y} = a_1\vec{x}_1 + \ldots + a_\ell\vec{x}_\ell$ and $\vec{z} = a_{\ell+1}\vec{x}_{\ell+1} + \ldots + a_{\ell+r}\vec{x}_{\ell+r}$ so that $\vec{y}, \vec{z} \in \mathrm{Span}(X)$ are two arbitrary elements then
$$\vec{y} + \vec{z} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell + a_{\ell+1}\vec{x}_{\ell+1} + \cdots + a_{\ell+r}\vec{x}_{\ell+r}$$
so their sum is also in the span.
3. If $\vec{y} = a_1\vec{x}_1 + \ldots + a_\ell\vec{x}_\ell$ is an arbitrary element then
$$x\vec{y} = (xa_1)\vec{x}_1 + \ldots + (xa_\ell)\vec{x}_\ell$$
is also in the span

Note It is a useful/important technical convention that an *empty* sum is $\vec{0}$ and so

$$\mathrm{Span}(\emptyset) = \{\vec{0}\}$$

Theorems about Spans

Theorem

Suppose that $W \subset V$ is any subspace. If $\vec{v}_1, \dots, \vec{v}_n \in W$ then $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) \subset W$.

Proof:Let $\vec{v} \in \mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n)$ be arbitrary, then there are $a_1, \dots, a_n \in \mathbb{R}$ such that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

We know $\vec{v}_i \in W$, so each $a_i\vec{v}_i \in W$ because W is a subspace.But then their sum, $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$ is also in W, because again W is a subspace.This shows that $\vec{v} \in W$, and so $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) \subset W$.

Corollary

Suppose $\vec{v}_1, \dots, \vec{v}_n$ and $\vec{x}_1, \dots, \vec{x}_m$ are two collections of vectors in V. Then

$$\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) \subset \mathrm{Span}(\vec{x}_1, \dots, \vec{x}_m)$$

if and only if

$$\vec{v}_1, \dots, \vec{v}_n \in \mathrm{Span}(\vec{x}_1, \dots, \vec{x}_m)$$

Corollary

Suppose $\vec{v}_1, \dots, \vec{v}_n$ and $\vec{x}_1, \dots, \vec{x}_m$ are two collections of vectors in V. Then

$$\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n) = \mathrm{Span}(\vec{x}_1, \dots, \vec{x}_m)$$

if and only if

$$\vec{v}_1, \dots, \vec{v}_n \in \mathrm{Span}(\vec{x}_1, \dots, \vec{x}_m) \qquad \text{and} \qquad \vec{x}_1, \dots, \vec{x}_m \in \mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n)$$

This is often the best way to check if subspaces are subsets, this can take a lot of checks

Examples

Consider the vector subspace $W \subset \mathbb{R}^4$ given by:

$$W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w - x - y + z = 0 \text{ and } w - 2x + y = 0\}$$

Determine if

$$\mathrm{Span}((1, 2, 3, 4), (4, 3, 2, 1)) \subset W$$

Need to check if $(1, 2, 3, 4), (4, 3, 2, 1) \in W$.

Determine if

$$\mathrm{Span}((1, 2, 1, 2), (4, 3, 2, 1)) \subset W$$

Need to check if $(1, 2, 1, 2), (4, 3, 2, 1) \in W$.

Generators

Given a subset of vectors $X \subset V$ we say X **generates** V if and only if every vector in V can be written as a linear combination of vectors from X.
In symbols this means

$$\forall \vec{v} \in V, \exists \vec{x}_1, \ldots, \vec{x}_n \in X, \exists a_1, \ldots, a_n \in \mathbb{R}, \vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$$

Given a collection of vectors $\vec{x}_1, \ldots, \vec{x}_n \in V$, we say they **generate** V if and only if

$$\forall \vec{v} \in V,\ \exists a_1, \ldots, a_n \in \mathbb{R}, \vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$$

This is the same definition as for the set $X = \{\vec{x}_1, \ldots, \vec{x}_n\}$

This is very useful when X is finite, as it gives an alternative way to describe vectors in V, for example, say $X = \{(1,2,3),(3,2,1)\}$ is a generating set for V, then this would say

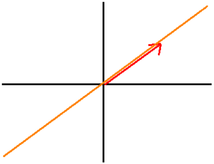
$$V = \{r(1,2,3) + s(3,2,1) \mid r, s \in \mathbb{R}\}$$

A good way to describe a vector (sub)space is to try to find a small set which generates it.
Most examples of vector spaces we deal with will be *finitely generated*

Theorem
Every vector space (and every subspace) has a generating set.
Proof Idea: The set V is a generating set.

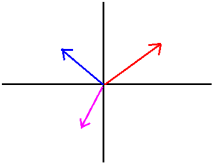
Examples (Picture)

So the vector $(3, 2)$



is a generator for the subspace $W = \{(x, y) \in \mathbb{R}^2 \mid 3y = 2x\}$.
But it is **not** on its own a generator for $\mathbb{R}^2$.

The vectors $(3, 2)$, $(-2, 2)$, and $(-1, -2)$,



are generators for $\mathbb{R}^2 = \mathrm{Span}((3, 2), (-2, 2), (-1, -2))$.
But they are **not** generators for $W = \{(x, y) \in \mathbb{R}^2 \mid 3y = 2x\}$.

Notes on terminology

The following sentences mean the same thing, (because people use them interchangeably):

- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are generators for V.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are a generating set for V.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ generate V.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are a spanning set for V.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ span V.

The last two are justified by the theorem:
Theorem The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are generators for V if and only if

$$V = \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$$

The proof is just to compare the definitions

So the span of the collection $\vec{v}_1, \ldots, \vec{v}_n$ is precisely the vector space they generate, that is

$$\vec{v}_1, \ldots, \vec{v}_n \textit{ generates } \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$$

Examples

Consider the vector subspace W in $\mathbb{R}^4$ defined by:

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + 3x_3 = 0 \text{ and } 4x_2 + x_4 = 0\}$$

Find a finite set X which generates it.
By solving the system of equations $2x_1 + 3x_3 = 0$ and $4x_2 + x_4 = 0$ we notice that evey solution can be written as:

$$(-\frac{3}{2}r, -\frac{1}{4}s, r, s) = r(-\frac{3}{2}, 0, 1, 0) + s(0, -\frac{1}{4}, 0, 1)$$

and so $(-\frac{3}{2}, 0, 1, 0)$, $(0, -\frac{1}{4}, 0, 1)$ is by definition a generating set.

Theorem
Gaussian elimination (when done correctly) will always find a generating set for a vector subspace defined by a system of linear equations.
This process always works because Gausian elimination finds all the solutions

From the previous slide, we know $(-\frac{3}{2}, 0, 1, 0)$, $(0, -\frac{1}{4}, 0, 1)$ generate

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + 3x_3 = 0 \text{ and } 4x_2 + x_4 = 0\}$$

Show that the set $\{(3, 1, -2, -4), (3, -1, -2, 4)\}$ also generates W.
First we *notice* that we can write:

$$(-\frac{3}{2}, 0, 1, 0) = \frac{1}{4}(3, 1, -2, -4) + \frac{1}{4}(3, -1, -2, 4)$$
$$(0, -\frac{1}{4}, 0, 1) = \frac{-1}{8}(3, 1, -2, -4) + \frac{1}{8}(3, -1, -2, 4)$$

How did I notice I could do that?
So let $\vec{v} \in W$ be arbitrary, then we know that there exists $r, s \in \mathbb{R}$ so that

$$\vec{v} = r(-\frac{3}{2}, 0, 1, 0) + s(0, -\frac{1}{4}, 0, 1).$$

But then we can write

$$\vec{v} = r\left(\frac{1}{4}(3, 1, -2, -4) + \frac{1}{4}(3, -1, -2, 4)\right) + s\left(\frac{-1}{8}(3, 1, -2, -4) + \frac{1}{8}(3, -1, -2, 4)\right)$$

So is also a linear combination of $(3, 1, -2, -4), (3, -1, -2, 4)$, hence these are also a generating set.
This process always works as explained by the next two theorems.

Suppose that $\vec{e}_1, \vec{e}_2$ are generators for a vector space V.
Prove that with

$$\vec{f}_1 = \vec{e}_1 + \vec{e}_2, \quad \vec{f}_2 = \vec{e}_1 - \vec{e}_2$$

we have $\vec{f}_1, \vec{f}_2$ are generators for V.

To prove that we have generators, the first line of the proof is often something like:

Let $\vec{v} \in V$ be arbitrary

we then just need to figure out how to write $\vec{v}$ in terms of $\vec{f}_i$

If you ever know you have generators, like $\vec{e}_1, \vec{e}_2$ above, and you have a vector $\vec{v}$, you probably want to use this fact:

As $\vec{v} \in V$ and $\vec{e}_1, \vec{e}_2$ are generators for a vector space V
we know that there are real numbers a_1, a_2 so that

$$\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2$$

Theorems about generating sets

Theorem

If $G = \{\vec{g}_1, \ldots, \vec{g}_n\}$ is a generating set for V, and $\vec{g}_n \in \mathrm{Span}(\vec{g}_1, \ldots, \vec{g}_{n-1})$ then $\vec{g}_1, \ldots, \vec{g}_{n-1}$ is a generating set for V.

Proof: Let $\vec{v} \in V$ be arbitrary.

Because $\vec{g}_1, \ldots, \vec{g}_n$ generate V we know there are $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$\vec{v} = a_1\vec{g}_1 + \cdots + a_n\vec{g}_n$$

Because $\vec{g}_n \in \mathrm{Span}(\vec{g}_1, \ldots, \vec{g}_{n-1})$ we know there are $b_1, \ldots, b_{n-1} \in \mathbb{R}$ so that

$$\vec{g}_n = b_1\vec{g}_1 + \cdots + b_{n-1}\vec{g}_{n-1}$$

Combining these we can write:

$$\begin{aligned}
\vec{v} &= a_1\vec{g}_1 + \cdots + a_n\vec{g}_n \\
&= a_1\vec{g}_1 + \cdots + a_{n-1}\vec{g}_{n-1} + a_n\left(b_1\vec{g}_1 + \cdots + b_{n-1}\vec{g}_{n-1}\right) \\
&= (a_1 + a_n b_1)\vec{g}_1 + \cdots + (a_1 + a_{n-1} b_1)\vec{g}_{n-1}
\end{aligned}$$

from which we see $\vec{v}$ is a linear combination of $\vec{g}_1, \ldots, \vec{g}_{n-1}$ and hence these are a generating set for V.

Theorem

If G is a generating set for V, and $G \subset \mathrm{Span}(S)$, then S is a generating set for V.

This is a homework exercise!.

Common Patterns in Patterns in Proofs About Generators and Spans

If you are trying to prove that something is a generating set for V, your first line should probably be something like:

Let $\vec{v} \in V$ be arbitrary

If you know that $\vec{g}_1, \ldots, \vec{g}_n$ is a generating set for V, and you ever have a vector $\vec{v} \in V$, it is likely that at some point you will want to explain:

As $\vec{v} \in V$, and $\vec{g}_1, \ldots, \vec{g}_n$ is a generating set for V, we know there exist $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$\vec{v} = a_1\vec{g}_1 + \cdots + a_n\vec{g}_n$$

Natural Questions to ask about generating sets

- Is the vector $\vec{v}$ contained in $\mathrm{Span}(X)$?
  if X is finite, this usually is solved using a system of equations as we have done.
- Does the set X generate the subspace W?
  This is often easier to solve using dimension (which we will see), but you can also solve it directly if you can solve the next question as we have done.
- Find a (small/finite) set X that generates the subspace W.
  Depending on how W is described, this can be solved using a system of equations as we have done. .
- What is the *smallest* set X that generates W?
  This is an important concept, which we will come back to soon.