Linear Combinations

Let $V$ be a vector space, and let $\vec{x}_1, \ldots, \vec{x}_\ell \in V$ be a collection of vectors. Given $a_1, \ldots, a_\ell \in \mathbb{R}$ we call the vector

$$\vec{v} = a_1 \vec{x}_1 + \cdots + a_\ell \vec{x}_\ell$$

a **Linear Combination** of $\vec{x}_1, \ldots, \vec{x}_\ell$.

**Examples**

1. The vector $(1, 2, 3, 4)$ is a linear combination of $(4, 3, 2, 1)$ and $(1, 1, 1, 1)$, because

$$(1, 2, 3, 4) = 5(1, 1, 1, 1) + (-1)(4, 3, 2, 1)$$

2. Determine if $(6, 5, 4, 3)$ can be expressed as a linear combination of $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$. We need to try to solve

$$ (6, 5, 4, 3) = a(1, 2, 3, 4) + b(4, 3, 2, 1) $$

3. Determine if $(1, 2, 1, 2)$ can be expressed as a linear combination of $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$. We need to try to solve

$$ (1, 2, 1, 2) = a(1, 2, 3, 4) + b(4, 3, 2, 1) $$

These problems always turn into solving a system of equations.
Consider the vector \((3, 2)\), which vectors can be written as linear combinations of it?

It can only be the vectors on the line \(3y = 2x\).

What if we have the vectors \((3, 2), (-2, 2),\) and \((-1, -2)\)?

Then it can be any vector in \(\mathbb{R}^2\).
Let $V$ be a vector space, let $X$ be any subset of $V$. Define the span of the set $X$ to be:

$$\text{Span}(X) = \{ \vec{v} \in V \mid \exists \vec{x}_1, \ldots, \vec{x}_\ell \in X, \vec{v} \text{ is a linear combination of } \vec{x}_1, \ldots, \vec{x}_\ell \}$$

(technical note: even if the set $X$ is infinite, the linear combinations only include finitely many of the $X$.)

Likewise, given a list of vectors, $\vec{x}_1, \ldots, \vec{x}_n$ we define

$$\text{Span}(\vec{x}_1, \ldots, \vec{x}_n) = \{ \vec{v} \in V \mid \exists a_1, \ldots, a_n \in \mathbb{R}, \vec{v} = a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n \}$$

These definitions agree when the set $X$ is finite, as given $X = \{ \vec{x}_1, \ldots, \vec{x}_\ell \}$ then

$$\text{Span}(X) = \text{Span}(\vec{x}_1, \ldots, \vec{x}_\ell) = \{ a_1 \vec{x}_1 + \cdots + a_\ell \vec{x}_\ell \mid a_1, \ldots, a_\ell \in \mathbb{R} \}$$

Put another way:

$$\vec{v} \in \text{Span}(\vec{x}_1, \ldots, \vec{x}_\ell) \iff \exists a_1, \ldots, a_\ell, \vec{v} = a_1 \vec{x}_1 + \cdots + a_\ell \vec{x}_\ell$$
Examples (Picture)

Consider again the vector \((3, 2)\),

the span of \((3, 2)\) is the line \(3y = 2x\).

If we have the vectors \((3, 2)\), \((-2, 2)\), and \((-1, -2)\),

then we have \(\mathbb{R}^2 = \text{Span}((3, 2), (-2, 2), (-1, -2))\).
Examples

By definition

\[ \text{Span}((1, 2, 3, 4), (4, 3, 2, 1)) = \{ r(1, 2, 3, 4) + s(4, 3, 2, 1) \mid r, s \in \mathbb{R} \} \]

Determine if \((6, 5, 4, 3) \in \text{Span}((1, 2, 3, 4), (4, 3, 2, 1))\)?
We need to try to solve

\[ (6, 5, 4, 3) = a(1, 2, 3, 4) + b(4, 3, 2, 1) \]

I feel I may have done that already!

Determine if \((1, 2, 1, 2) \in \text{Span}((1, 2, 3, 4), (4, 3, 2, 1))\)?
We need to try to solve

\[ (1, 2, 1, 2) = a(1, 2, 3, 4) + b(4, 3, 2, 1) \]

I feel I may have done that already!

When the set \(X\) is finite, these questions are identical to the previous questions about checking if something is a linear combination!
Theorem about Spans

**Theorem**
Given $X \subset V$ the set $\text{Span}(X)$ is a subspace. It follows that $\text{Span}(X)$ is the smallest subspace of $V$ containing $X$.

**Proof:**
We must check the three subspace conditions

1. $\vec{0}$ is by convention equal to the empty sum, and so is a linear combination.
2. If $\vec{y} = a_1 \vec{x}_1 + \ldots + a_\ell \vec{x}_\ell$ and $\vec{z} = a_{\ell+1} \vec{x}_{\ell+1} + \ldots + a_{\ell+r} \vec{x}_{\ell+r}$ so that $\vec{y}, \vec{z} \in \text{Span}(X)$ are two arbitrary elements then

   $$\vec{y} + \vec{z} = a_1 \vec{x}_1 + \cdots + a_\ell \vec{x}_\ell + a_{\ell+1} \vec{x}_{\ell+1} + \cdots + a_{\ell+r} \vec{x}_{\ell+r}$$

   so their sum is also in the span.
3. If $\vec{y} = a_1 \vec{x}_1 + \ldots + a_\ell \vec{x}_\ell$ is an arbitrary element then

   $$x\vec{y} = (xa_1)\vec{x}_1 + \ldots + (xa_\ell)\vec{x}_\ell$$

   is also in the span

**Note** It is a useful/important technical convention that an *empty* sum is $\vec{0}$ and so

$$\text{Span}(\emptyset) = \{\vec{0}\}$$
Theorems about Spans

**Theorem**
Suppose that $W \subset V$ is any subspace. If $\vec{v}_1, \ldots, \vec{v}_n \in W$ then $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n) \subset W$.

**Proof:** Let $\vec{v} \in \text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ be arbitrary, then there are $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n.$$ 

We know $\vec{v}_i \in W$, so each $a_i \vec{v}_i \in W$ because $W$ is a subspace. But then their sum, $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n$, is also in $W$, because again $W$ is a subspace. This shows that $\vec{v} \in W$, and so $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n) \subset W$.

**Corollary**
Suppose $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{x}_1, \ldots, \vec{x}_m$ are two collections of vectors in $V$. Then

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_n) \subset \text{Span}(\vec{x}_1, \ldots, \vec{x}_m)$$

if and only if

$$\vec{v}_1, \ldots, \vec{v}_n \in \text{Span}(\vec{x}_1, \ldots, \vec{x}_m)$$

**Corollary**
Suppose $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{x}_1, \ldots, \vec{x}_m$ are two collections of vectors in $V$. Then

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_n) = \text{Span}(\vec{x}_1, \ldots, \vec{x}_m)$$

if and only if

$$\vec{v}_1, \ldots, \vec{v}_n \in \text{Span}(\vec{x}_1, \ldots, \vec{x}_m) \text{ and } \vec{x}_1, \ldots, \vec{x}_m \in \text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$$

This is often the best way to check if subspaces are subsets, this can take a lot of checks.
Examples

Consider the vector subspace $W \subset \mathbb{R}^4$ given by:

$$W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w - x - y + z = 0 \text{ and } w - 2x + y = 0\}$$

Determine if

$$\text{Span}((1, 2, 3, 4), (4, 3, 2, 1)) \subset W$$

Need to check if $(1, 2, 3, 4), (4, 3, 2, 1) \in W$.

Determine if

$$\text{Span}((1, 2, 1, 2), (4, 3, 2, 1)) \subset W$$

Need to check if $(1, 2, 1, 2), (4, 3, 2, 1) \in W$. 
Generators

Given a subset of vectors $X \subset V$ we say $X$ generates $V$ if and only if every vector in $V$ can be written as a linear combination of vectors from $X$. In symbols this means

$$\forall \mathbf{v} \in V, \exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in X, \exists a_1, \ldots, a_n \in \mathbb{R}, \mathbf{v} = a_1 \mathbf{x}_1 + \cdots + a_n \mathbf{x}_n$$

Given a collection of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$, we say they generate $V$ if and only if

$$\forall \mathbf{v} \in V, \exists a_1, \ldots, a_n \in \mathbb{R}, \mathbf{v} = a_1 \mathbf{x}_1 + \cdots + a_n \mathbf{x}_n$$

This is the same definition as for the set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$

This is very useful when $X$ is finite, as it gives an alternative way to describe vectors in $V$, for example, say $X = \{(1, 2, 3), (3, 2, 1)\}$ is a generating set for $V$, then this would say

$$V = \{r(1, 2, 3) + s(3, 2, 1) | r, s \in \mathbb{R}\}$$

A good way to describe a vector (sub)space is to try to find a small set which generates it. Most examples of vector spaces we deal with will be finitely generated

**Theorem**

Every vector space (and every subspace) has a generating set.

**Proof Idea:** The set $V$ is a generating set.
Examples (Picture)

So the vector \((3, 2)\)

is a generator for the subspace \(W = \{(x, y) \in \mathbb{R}^2 \mid 3y = 2x\}\).

But it is **not** on its own a generator for \(\mathbb{R}^2\).

The vectors \((3, 2), (-2, 2),\) and \((-1, -2),\)

are generators for \(\mathbb{R}^2 = \text{Span}((3, 2), (-2, 2), (-1, -2))\).

But they are **not** generators for \(W = \{(x, y) \in \mathbb{R}^2 \mid 3y = 2x\}\).
Notes on terminology

The following sentences mean the same thing, (because people use them interchangeably):

- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are generators for $V$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are a generating set for $V$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ generate $V$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are a spanning set for $V$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ span $V$.

The last two are justified by the theorem:

**Theorem** The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are generators for $V$ if and only if

$$V = \text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$$

The proof is just to compare the definitions

So the span of the collection $\vec{v}_1, \ldots, \vec{v}_n$ is precisely the vector space they generate, that is $\vec{v}_1, \ldots, \vec{v}_n$ *generates* $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$
Examples

Consider the vector subspace $W$ in $\mathbb{R}^4$ defined by:

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + 3x_3 = 0 \text{ and } 4x_2 + x_4 = 0\}$$

Find a finite set $X$ which generates it.

By solving the system of equations $2x_1 + 3x_3 = 0$ and $4x_2 + x_4 = 0$ we notice that every solution can be written as:

$$(-\frac{3}{2}r, -\frac{1}{4}s, r, s) = r(-\frac{3}{2}, 0, 1, 0) + s(0, -\frac{1}{4}, 0, 1)$$

and so $(-\frac{3}{2}, 0, 1, 0), (0, -\frac{1}{4}, 0, 1)$ is by definition a generating set.

**Theorem**

Gaussian elimination (when done correctly) will always find a generating set for a vector subspace defined by a system of linear equations.

This process always works because Gaussian elimination finds all the solutions...
From the previous slide, we know \((-\frac{3}{2}, 0, 1, 0), (0, -\frac{1}{4}, 0, 1)\) generate 

\[ W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | 2x_1 + 3x_3 = 0 \text{ and } 4x_2 + x_4 = 0\} \]

Show that the set \{(3, 1, -2, -4), (3, -1, -2, 4)\} also generates \(W\).

First we notice that we can write:

\[
(-\frac{3}{2}, 0, 1, 0) = \frac{1}{4} (3, 1, -2, -4) + \frac{1}{4} (3, -1, -2, 4)
\]

\[
(0, -\frac{1}{4}, 0, 1) = -\frac{1}{8} (3, 1, -2, -4) + \frac{1}{8} (3, -1, -2, 4)
\]

How did I notice I could do that?

So let \(\vec{v} \in W\) be arbitrary, then we know that there exists \(r, s \in \mathbb{R}\) so that

\[
\vec{v} = r(-\frac{3}{2}, 0, 1, 0) + s(0, -\frac{1}{4}, 0, 1).
\]

But then we can write

\[
\vec{v} = r \left( \frac{1}{4} (3, 1, -2, -4) + \frac{1}{4} (3, -1, -2, 4) \right) + s \left( -\frac{1}{8} (3, 1, -2, -4) + \frac{1}{8} (3, -1, -2, 4) \right)
\]

So is also a linear combination of \((3, 1, -2, -4), (3, -1, -2, 4)\), hence these are also a generating set.

This process always works as explained by the next two theorems.
Suppose that $\vec{e}_1, \vec{e}_2$ are generators for a vector space $V$.
Prove that with
\[
\vec{f}_1 = \vec{e}_1 + \vec{e}_2, \quad \vec{f}_2 = \vec{e}_1 - \vec{e}_2
\]
we have $\vec{f}_1, \vec{f}_2$ are generators for $V$.

To prove that we have generators, the first line of the proof is often something like:

Let $\vec{v} \in V$ be arbitrary
we then just need to figure out how to write $\vec{v}$ in terms of $\vec{f}_i$

If you ever know you have generators, like $\vec{e}_1, \vec{e}_2$ above, and you have a vector $\vec{v}$, you probably want to use this fact:

As $\vec{v} \in V$ and $\vec{e}_1, \vec{e}_2$ are generators for a vector space $V$
we know that there are real numbers $a_1, a_2$ so that

\[
\vec{v} = a_1 \vec{e}_1 + a_2 \vec{e}_2
\]
Theorems about generating sets

**Theorem**
If $G = \{\vec{g}_1, \ldots, \vec{g}_n\}$ is a generating set for $V$, and $\vec{g}_n \in \text{Span}(\vec{g}_1, \ldots, \vec{g}_{n-1})$ then $\vec{g}_1, \ldots, \vec{g}_{n-1}$ is a generating set for $V$.

**Proof**: Let $\vec{v} \in V$ be arbitrary. Because $\vec{g}_1, \ldots, \vec{g}_n$ generate $V$ we know there are $a_1, \ldots, a_n \in \mathbb{R}$ so that
\[ \vec{v} = a_1 \vec{g}_1 + \cdots + a_n \vec{g}_n \]
Because $\vec{g}_n \in \text{Span}(\vec{g}_1, \ldots, \vec{g}_{n-1})$ we know there are $b_1, \ldots, b_{n-1} \in \mathbb{R}$ so that
\[ \vec{g}_n = b_1 \vec{g}_1 + \cdots + b_{n-1} \vec{g}_{n-1} \]
Combining these we can write:
\[ \vec{v} = a_1 \vec{g}_1 + \cdots + a_{n-1} \vec{g}_{n-1} + a_n (b_1 \vec{g}_1 + \cdots + b_{n-1} \vec{g}_{n-1}) \]
\[ = (a_1 + a_n b_1) \vec{g}_1 + \cdots + (a_1 + a_{n-1} b_1) \vec{g}_{n-1} \]
from which we see $\vec{v}$ is a linear combination of $\vec{g}_1, \ldots, \vec{g}_{n-1}$ and hence these are a generating set for $V$.

**Theorem**
If $G$ is a generating set for $V$, and $G \subset \text{Span}(S)$, then $S$ is a generating set for $V$.

This is a homework exercise!
Common Patterns in Proofs About Generators and Spans

If you are trying to prove that something is a generating set for $V$, your first line should probably be something like:

Let $\mathbf{v} \in V$ be arbitrary

If you know that $\mathbf{g}_1, \ldots, \mathbf{g}_n$ is a generating set for $V$, and you ever have a vector $\mathbf{v} \in V$, it is likely that at some point you will want to explain:

As $\mathbf{v} \in V$, and $\mathbf{g}_1, \ldots, \mathbf{g}_n$ is a generating set for $V$, we know there exist $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$\mathbf{v} = a_1 \mathbf{g}_1 + \cdots + a_n \mathbf{g}_n$$
Natural Questions to ask about generating sets

- Is the vector \( \vec{v} \) contained in \( \text{Span}(X) \)?
  
  if \( X \) is finite, this usually is solved using a system of equations as we have done.

- Does the set \( X \) generate the subspace \( W \)?
  
  This is often easier to solve using dimension (which we will see), but you can also solve it directly if you can solve the next question as we have done.

- Find a (small/finite) set \( X \) that generates the subspace \( W \).
  
  Depending on how \( W \) is described, this can be solved using a system of equations as we have done.

- What is the \emph{smallest} set \( X \) that generates \( W \)?
  
  This is an important concept, which we will come back to soon.