# Relations

Let $V$ be a vector space, and let $\vec{x}_1, \dots, \vec{x}_\ell \in V$ be a collection of vectors. We call the set of numbers $a_1, \dots, a_\ell \in \mathbb{R}$ a **relation** between $\vec{x}_1, \dots, \vec{x}_\ell$ if

$$a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}$$

**Examples**

Vectors $\vec{x}_1, \dots, \vec{x}_\ell \in V$ always have the **zero relation** (or **trivial relation**)

$$0\vec{x}_1 + \cdots + 0\vec{x}_\ell = \vec{0}$$

We also have a **non-trivial** relation:

$$(1,3,1) + 2(2,1,2) - 5(1,1,1) = (0,0,0)$$

Find a (non-trivial) relation between:

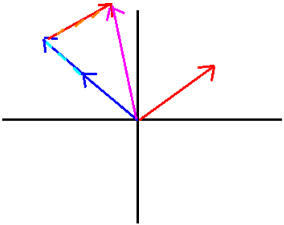
$$(1,2,3) \qquad (3,2,1) \qquad (1,1,1)$$

Solve the system $a(1,2,3) + b(3,2,1) + c(1,1,1) = (0,0,0)$.

Finding relations is always about solving a system of equations.

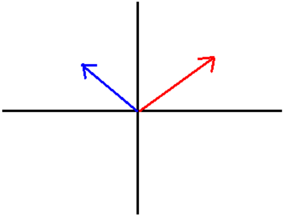
## Example - Picture

There is a relation between the vectors $(3, 2)$, $(-2, 2)$, and $(-1, 6)$



we see this because $(-1, 6) = (3, 2) + 2(-2, 2)$.

There is **no** relation between the vectors $(3, 2)$ and $(-2, 2)$



## Linear Dependence

Let $V$ be a vector space, and let $X$ be a subset. We say $X$ is **Linearly Dependent** if there exists *distinct* $\vec{x}_1, \ldots, \vec{x}_\ell \in X$ with a **non-zero** relation

$$a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}$$

Given a list of vectors $\vec{x}_1, \ldots, \vec{x}_\ell$ they are **Linearly Dependent** if

$$\exists a_1, \ldots, a_\ell \in \mathbb{R}, (a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}) \text{ and not all } a_i \text{ are } 0$$

**Note:** There is a minor distinction between the case of $X = \{\vec{x}_1, \ldots, \vec{x}_\ell\}$ and the list $\vec{x}_1, \ldots, \vec{x}_\ell$ in that in the list $\vec{x}_1, \ldots, \vec{x}_\ell$, their may be repeated vectors, but sets $X = \{\vec{x}_1, \ldots, \vec{x}_\ell\}$ do not contain repeats.
Note the inclusion of the word *distinct* in the definition when there is a set $X$.
This distinction is mostly annoying from the point of view of correctly stating theorems, not so much using/understanding them.

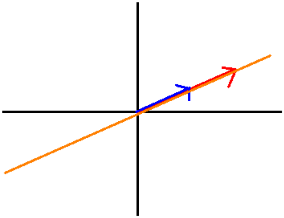
**Example**

The vectors $(1,2,3), (3,2,1), (1,1,1)$ are linearly dependent because

$$(1,2,3) + (3,2,1) - 4(1,1,1) = (0,0,0)$$

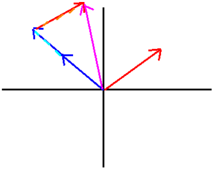
# Example - Picture

The vectors $(4, 2)$, $(2, 1)$ are linearly dependent



and we can see that they are on the same line $2y = x$.

**with more vectors, the condition is no longer the same as being on the same line**



we shall see the more general condition eventually.

# Linear Independence

Let $V$ be a vector space, and let $X$ be a subset. We say $X$ is **Linearly Independent** if it is not linearly dependent, concretely this means that for any distinct collection of vectors $\vec{x}_1, \ldots, \vec{x}_\ell \in X$ the only relation

$$a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}$$

is the zero relation $a_1 = a_2 = \cdots = a_\ell = 0$.

Likewise, given a list of vectors $\vec{x}_1, \ldots, \vec{x}_\ell$ we say it is **Linearly Independent** if it is not linearly dependent.

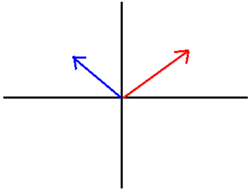
Reinterpretted in symbols:
Given a list of vectors $\vec{x}_1, \ldots, \vec{x}_\ell$ they are **Linearly Independent** if

$$\forall a_1, \ldots, a_\ell \in \mathbb{R}, (a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}) \Rightarrow (a_1 = a_2 = \cdots = a_\ell = 0)$$

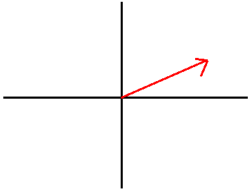
**Note:** As before there is a minor distinction between the case of $X = \{\vec{x}_1, \ldots, \vec{x}_\ell\}$ and the list $\vec{x}_1, \ldots, \vec{x}_\ell$ in that in the list $\vec{x}_1, \ldots, \vec{x}_\ell$, their may be repeated vectors, but sets $X = \{\vec{x}_1, \ldots, \vec{x}_\ell\}$ do not contain repeats.

## Example - Picture

There is **no** relation between the vectors $(3, 2)$ and $(-2, 2)$



so they are linearly independent.
There is no relation involving only the vector $(4, 2)$



so this single vector is independent.

## Examples

Show that vectors are linearly independent.

$$(1,2,3,4) \qquad (4,3,2,1) \qquad (1,-1,1,-1)$$

Solve the system $a(1,2,3,4)+b(4,3,2,1)+c(1,-1,1,-1)=(0,0,0,0)$ and show that the only solution is trivial.

If you are trying to convince someone vectors are independent then your work **must** include enough details for someone to check it, and you **must** explain why you know the only solution is the *trivial* solution.

Determine whether or not the vectors

$$(1, 2, 3, 4) \qquad (4, 3, 2, 1) \qquad (1, 1, 1, 1)$$

are linearly dependent.

Solve the system $a(1, 2, 3, 4) + b(4, 3, 2, 1) + c(1, 1, 1, 1) = (0, 0, 0, 0)$ and see if you find a non-trivial solution.

It is a very good idea when you conclude vectors are dependent to actually write down and check an explicit solution to be sure you have not made a mistake.

**For dependent, the explicit solution is the proof, for independent your work and explanation are the proof**

Suppose that $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent.
Prove that with

$$\vec{f}_1 = \vec{e}_1 + \vec{e}_2, \quad \vec{f}_2 = \vec{e}_2 + \vec{e}_3, \quad \vec{f}_3 = \vec{e}_1 + \vec{e}_3$$

that $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are linearly independent.

When asked to prove vectors are independent, you often want a first sentance to be:
*Let $a_1, a_2, a_3 \in \mathbb{R}$ be arbitrary and assume*

$$a_1\vec{f}_1 + a_2\vec{f}_2 + a_3\vec{f}_3 = \vec{0}$$

what remains to do is prove that $a_1 = a_2 = a_3 = 0$.

you will eventually have several ways to approach a question like this, though from the perspective of making a clear self contained proof, this is still typically the simplest.

## Theorems About Independence

**Theorem**

The collection, $\vec{v}_1$ is linearly indpendent if and only if $\vec{v}_1 \neq \vec{0}$.

**Proof:**

$\Rightarrow$-direction We assume $\vec{v}_1$ is linearly indpendent, we claim $\vec{v}_1 \neq \vec{0}$ and we shall prove this by contradiction, so assume $\vec{v}_1 = \vec{0}$.

Then we have a relation

$$(1)\vec{v}_1 = (1)\vec{0} = \vec{0}$$

and $1 \neq 0$, this is a contradiction.

$\Leftarrow$-direction We assume $\vec{v}_1 \neq \vec{0}$ and wish to show $\vec{v}_1$ is linearly independent.

Let $a_1 \in \mathbb{R}$ be arbitrary and assume

$$a_1\vec{v}_1 = \vec{0}$$

we claim $a_1$ must be 0, for the purpose of contradiction, suppose it is not. Then by multiplying the above by $a_1^{-1}$ we obtain:

$$(a_1^{-1})a_1\vec{v}_1 = (a_1^{-1})\vec{0}$$

but the left hand side is $(a_1^{-1})a_1\vec{v}_1 = (1)\vec{v}_1 = \vec{v}_1$ and the right hand side is $\vec{0}$, which is a contradiction.

**Corollary**

The collection $\vec{v}_1$ is linealy dependent if and only if $\vec{v}_1 = \vec{0}$.

**Note**: a direct proof of this statement avoids the proofs by contradiction above.

## Theorem

A set $X$ is linearly independent if and only if for all $\vec{v} \in \mathrm{Span}(X)$ there exists **unique** $\vec{x}_1, \ldots, \vec{x}_\ell \in X$ and **unique** $a_1, \ldots, a_\ell \in \mathbb{R}$ such that:

$$\vec{v} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell$$

**Proof Sketch:** It is automatic that for all $\vec{v} \in \mathrm{Span}(X)$ there exists a way to write it as a linear combination so the theorem can be thought of as saying

Linearly Independent $\Leftrightarrow$ Unique

$\Rightarrow$-direction, that is Linear Independent $\Rightarrow$ Unique.

Assume $X$ is linearly independent and assume for the purpose of contradiction there are two ways to write some vector $\vec{v}$. So we can write

$$\vec{v} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell \quad \text{and} \quad \vec{v} = b_1\vec{x}_1 + \cdots + b_\ell\vec{x}_\ell$$

By rearraning this we obtain:

$$(a_1 - b_1)\vec{x}_1 + \cdots + (a_\ell - b_\ell)\vec{x}_\ell = \vec{0}$$

But linear independence then implies that $a_i = b_i$ for all $i$.
This means that any two solutions, are the same solution, so the solution must be unique.

$\Leftarrow$-direction, that is Unique $\Rightarrow$ Linearly Independent

We instead prove the contrapositive, that is that Linearly Dependent $\Rightarrow$ Not Unique

By assumption the vectors are Linearly dependent, so there exists a relation: $a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell = \vec{0}$ with distinct $\vec{x}_i$ and not all $a_i = 0$, we may assume without loss of generality by relabelling that it is $a_1 \neq 0$, we then have

$$\vec{x}_1 = \frac{-a_2}{a_1}\vec{x}_2 + \cdots + \frac{-a_\ell}{a_1}\vec{x}_\ell$$

gives two ways to express $\vec{x}_1$ as a linear combination of elements from $X$. This constradicts the assumption on uniqueness.

**Theorem**

If a set $X$ is linearly independent, and $Y \subset X$ but $Y \neq X$ then $\mathrm{Span}(Y) \neq \mathrm{Span}(X)$

This says, that linearly indepentent sets are in some sense 'the smallest possible generating sets for their span'

**Proof:**

Because $Y \subset X$ and $Y \neq X$ there is an element, $\vec{x} \in Y \setminus X$.

We claim $\vec{x} \notin \mathrm{Span}(Y)$, we will do so by contradiction, so assume $\vec{x} \in \mathrm{Span}(Y)$.

Because $\vec{x} \in \mathrm{Span}(Y)$ then there $\vec{y}_1, \ldots, \vec{y}_n \in Y$ distinct, and $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$\vec{x} = a_1\vec{y}_1 + \cdots a_n\vec{y}_n$$

But then

$$\vec{0} = a_1\vec{y}_1 + \cdots a_n\vec{y}_n - \vec{x}$$

is a non-trivial relation among distinct elements of $X$. This is a contradiction as $X$ was assumed to be linearly independent.

This proves that $\mathrm{Span}(Y) \neq \mathrm{Span}(X)$

**Theorem**
If a set $X$ is linearly dependent, then there is a subset $Y \subset X$ with $Y \neq X$ such that $\mathrm{Span}(Y) = \mathrm{Span}(X)$.

**Note:** This is a rephrased version of the converse of the previous theorem.

**Proof:**

By assumption the $X$ is linearly dependent, so there exists a non-trivial relation:

$$a_1\vec{x}_1 + \cdots + a_\ell \vec{x}_\ell = \vec{0}$$

with distinct $\vec{x}_i \in X$ and not all $a_i = 0$, we may assume without loss of generality by relabelling that it is $a_1 \neq 0$, we then have

$$\vec{x}_1 = \frac{-a_2}{a_1}\vec{x}_2 + \cdots + \frac{-a_\ell}{a_1}\vec{x}_\ell$$

This proves that $x_1 \in \mathrm{Span}(X \setminus \{x_1\})$.
But we have seen, that this implies

$$\mathrm{Span}(X \setminus \{x_1\}) = \mathrm{Span}(X)$$

which proves the result.

## Example

Find a linearly independent subset $Y$ of

$$X = \{(1,2,3),(3,2,1),(1,1,1),(2,0,-2)\}$$

with $\mathrm{Span}(X) = \mathrm{Span}(Y)$.
We first should check if the vectors are already independent, if not we need to find a relation amongst the vectors so that we can identify one to delete!
We can notice that

$$(2,0,-2)-(3,2,1)+(1,2,3)=(0,0,0) \qquad \Rightarrow \qquad (2,0,-2)=(3,2,1)-(1,2,3)$$

(How do we *notice* such a thing?) so

$$\mathrm{Span}((1,2,3),(3,2,1),(1,1,1),(2,0,-2)) = \mathrm{Span}((1,2,3),(3,2,1),(1,1,1))$$

Is the new set LI?

The solution I just gave is **not** necissarily the most time efficient.

We will eventually see that the above problem always has a solution.

## Common Patterns in Patterns in Proofs About Linear Independence

If you are trying to prove that a collection $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, you will most often start your proof something like:

Let $a_1, \ldots, a_n \in \mathbb{R}$ be arbitrary, and assume

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

now your goal is to prove all the $a_i$ are zero

Many people like to phrase the proof instead as a proof by contradiction that the collection is not **linearly dependent**, this is also fine.

The key thing you need to happen, is end up with some *essentially arbitrary* numbers: $a_1, \ldots, a_n$, about which the only assumption is $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$.

If you know that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, you should expect at some point in your proof to write a line like

and because we now know that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

and because we know that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent,
we conclude that all the $a_i$ are zero.

The fact that you have the equation being equal to zero is key!

## Natural Questions to ask independence

- Find a non-zero relation among the vectors $\vec{x}_1, \ldots, \vec{x}_\ell$?
  This is pretty much always turns into a system of equations.
- Does there exists a non-zero relation among the vectors $\vec{x}_1, \ldots, \vec{x}_\ell$?
  This is pretty much always turns into a system of equations.
- Prove that the set $X$ is linearly (in)dependent.
  This is pretty much always turns into a system of equations.
- Determine/prove if the set $X$ is linearly dependent or independent.
  This is pretty much always turns into a system of equations.
- Find a linearly independent subset $Y$ of $X$ with $\mathrm{Span}(X) = \mathrm{Span}(Y)$.
  Applying the final theorem, and using what we know about spans, we can keep finding relations and deleting some elements.