Basis

Let V be a vector space, and let X be a subset. We say X is a **Basis** if it is both linearly independent and a generating set.

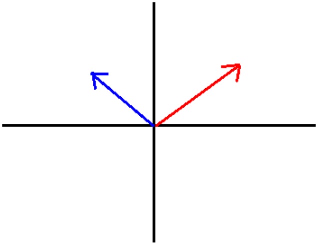
The first example of a basis is the standard basis for $\mathbb{R}^n$

$$\vec{e}_1 = (1, 0, \ldots, 0), \quad \vec{e}_2 = (0, 1, \ldots, 0), \quad \ldots, \quad \vec{e}_n = (0, 0, \ldots, 1)$$

- Why is this linearly independent?
- Why is this a generating set?

Example (Picture)

The vectors $(3, 2)$ and $(-2, 2)$ are a basis for $\mathbb{R}^2$

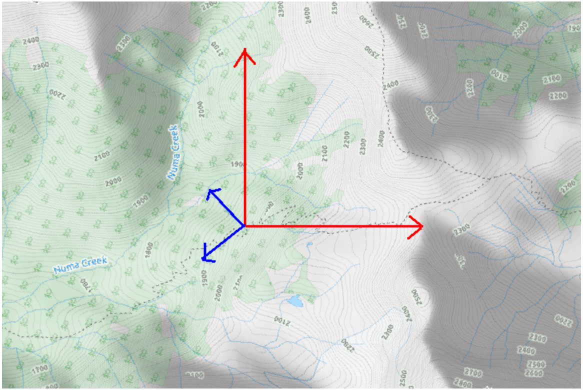


because they are generators!
and they are linearly independent.

Notice that every vector in $\mathbb{R}^2$ can be described as a linear combination of these vectors.

Why different bases, Example

Sometimes you have a map, and you want to use it to describe where to go. But the map isn't oriented with the top to the north!!



You might prefer the blue vectors, one of these points 100m north, the other 100m west.

Some people like to orient with the map, and so they might like the red vectors.

A really wierd person might pick exactly one red vector and one blue vector.

whichever option you pick, any point on the map can be described as a linear combination of those two vectors

Theorems About Basis

Theorem

A set X is a basis if and only if for all $\vec{v} \in V$ there exists **unique** $\vec{x}_1, \dots, \vec{x}_\ell \in X$ and **unique** $a_1, \dots, a_\ell \in \mathbb{R}$ such that:

$$\vec{v} = a_1\vec{x}_1 + \cdots + a_\ell\vec{x}_\ell$$

Proof Sketch:

- By definition: X is Basis $\Leftrightarrow$ X is Generating Set $+$ X is Linearly Independent.
- By definition: X is Generating Set $\Leftrightarrow$ $\mathrm{Span}(X) = V$.
- By theorem: X is Linearly Independent $\Leftrightarrow$ expressions for things in Span are unique.

$\Rightarrow$-direction

If X is a basis then it is a generating set and linearly independent, since it is a generating set there exists some way of writing each vector, since it is linearly independent these are unique.

$\Leftarrow$-direction

Since there exists a way to write every vector as a linear combination, this means by definition X is a generating set. Since the expressions are always unique, by one of our previous theorems X is linearly independent.

The Existance of a Basis

Theorem

Suppose V is any vector space. Given any linearly independent set $L \subset V$, and generating set G for V for which $L \subset G$ then there exists a basis B of V for which

$$L \subset B \subset G$$

I will give a proof of this, in the finitely generated case on the next few slides.

Corallary

Every vector space has a basis.

Theorem

Suppose V is any vector space, and B_1 and B_2 are any two basis then $|B_1| = |B_2|$, that is they have the same size.

This theorem (when V is finitely generated) is a bonus question on the assignment. We will never use this theorem in the infinite case, the proof in that case is very different than the finitely generated case..

The Existance of a Basis

Lemma

If V is any vector space. Given any linearly independent set $M \subset V$, and a generating set G for V for which $M \subset G$ then if M is not a basis for V, there exists $\vec{g} \in G \setminus M$ such that

$$M \cup \{\vec{g}\}$$

is linearly independent.

Proof:

We know M is not a basis, so it is not a generating set.

From the assignment because M is not a generating set, and G is, we know there is $\vec{g} \in G$ such that $\vec{g} \notin \mathrm{Span}(M)$.

From the assignment because M is linearly independent, and $\vec{g} \notin \mathrm{Span}(M)$, we know then that $M \cup \{\vec{g}\}$ is linearly independent.

This proves the result.

The Existance of a Basis

Theorem

Suppose V is any vector space. Given any linearly independent set $L \subset V$, and a **finite** generating set G for V for which $L \subset G$ then there exists a basis B of V for which

$$L \subset B \subset G$$

Proof: Consider the subset $\mathcal{L}$ of the power set $\mathcal{P}(G)$ of G given by

$$\mathcal{M} = \{M \subset G \mid L \subset M, \text{ and } M \text{ is linearly independent}\}$$

The set of things which could maybe be the B we want.
Consider the subset $\mathcal{N} \subset \mathbb{N}$ of natural numbers:

$$\mathcal{N} = \{n \in \mathbb{N} \mid \exists M \in \mathcal{M}\ |M| = n\}$$

We have that $\forall n \in \mathcal{N}, n \leq |G|$ because all the sets in $\mathcal{M}$ are subsets of G.
Let n be the largest element of $\mathcal{N}$, and let $B \in \mathcal{M}$ (so $L \subset B \subset G$) be such that

$$|B| = n.$$

Then there is no set $M' \neq B$ with $B \subset M' \subset G$ for which M' is linearly independent. If there was, it would be in $\mathcal{M}$ and have size larger than n.

We now know that B is a basis, for if it was not, our lemma would find such a set M.

Theorems About Basis

Theorem

If X is a basis, then no proper subset of X is a generating set, and no strictly larger set is linearly independent.

This is on the assignment.

Examples

Find a basis for

$$W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w - x - y + z = 0 \text{ and } w - 2x + y = 0\}$$

We first need to find a generating set.
We then need to find a linearly independent subset.

Solution:
We know that every vector $\vec{v} = (w, x, y, z) \in W$ satisfies the equations $w - x - y + z = 0$ and $w - 2x + y = 0$. But this is equivalent to satisfying $w - x - y + z = 0$ and $-x + 2y - z = 0$ and hence also equivalent to satisfying $w = 3y - 2z$ and $x = 2y - z$. So by taking $z = s$ and $y = t$ we can write

$$\vec{v} = (3t - 2s, 2t - s, t, s) = t(3, 2, 1, 0) + s(-2, -1, 0, 1)$$

From this we can see that every element of W can be writte in terms of $(3, 2, 1, 0)$, $(-2, -1, 0, 1)$ hence these are a generating set.
These vectors are linearly independent because if

$$\vec{0} = t(3, 2, 1, 0) + s(-2, -1, 0, 1) = (3t - 2s, 2t - s, t, s)$$

then we immediately have $t = s = 0$.
As they are generators and linearly independent, we know that $(3, 2, 1, 0)$, $(-2, -1, 0, 1)$ are a basis.

Theorem

A basis for a vector space described by a system of equations can always be found by finding the general form for the solution to the system of equations

Proof Idea: This is equivalent to the fact that writing the general form of the solution to a system of equations using gaussian elimination and back substitution is something that actually works.

If you want your solution to finding a basis for a vector space to be a proof, it is important you include enough details about the process that someone can know you have found the general solution!

Ideally you would also say why you know the vectors you find are generators and independent.

Examples

Find a basis for:

$$W = \text{Span}((1,2,3,4),(4,3,2,1),(1,1,1,1))$$

Solution: We notice that $(1,1,1,1) = \frac{1}{5}(1,2,3,4) + \frac{1}{5}(4,3,2,1)$ so that

$$(1,1,1,1) \in \text{Span}((1,2,3,4),(4,3,2,1))$$

It follows from what we know about Spans (one of our theorems) that

$$\text{Span}((1,2,3,4),(4,3,2,1)) = \text{Span}((1,2,3,4),(4,3,2,1),(1,1,1,1)) = W$$

thus any basis for $\text{Span}((1,2,3,4),(4,3,2,1))$ is a basis for $\text{Span}((1,2,3,4),(4,3,2,1),(1,1,1,1))$.

The vectors $(1,2,3,4),(4,3,2,1)$ are linearly independent, because the equation

$$a(1,2,3,4) + b(4,3,2,1) = (0,0,0,0)$$

leads $a + 4b = 0$ amd $2a + 3b = 0$ from which we conclude by subtracting twice the first equation from the second that

$$-5b = 0$$

and hence $b = 0$, plugging this into first equation gives $a = 0$. So the only solution to

$$a(1,2,3,4) + b(4,3,2,1) = (0,0,0,0)$$

is $a = b = 0$ so the vectors are linearly independent.

As $(1,2,3,4),(4,3,2,1)$ are linearly independent and span W, they are a basis.

Theorem

A basis for a vector space described by a span can always be found by finding a maximal linearly independent subset.

Proof Idea: This is basically how we proved a basis existed in the first place!!

In order to convince someone you have found a maximally linearly independent subset you must convince them it is Linearly independent, and that all the other vectors are still in the Span of these vectors.

Dimension of a Vector Space

Let V be a vector space, and let X be a basis.
The **dimension** of V is the size of X, if X is finite we say V is **finite dimensional**.
The theorem that says all basis have the same size is crucial to make sense of this.

Note: Every finitely generated vector space is finite dimensional.

Theorem

The dimension of $\mathbb{R}^n$ is n.

Proof Idea: Because we know a basis!

Theorem

If a subspace W of $\mathbb{R}^n$ is described as the solutions to a system of equations

$$A\vec{x} = \vec{0}$$

Then the dimension of W is exactly the number of parameters needed to express the general form of the solution to the system.

Proof Idea: Because we know how to write the general form to a solution of a system of equations.

Theorems About Dimension

Lemma

If $U \subset V$ is a subspace then there exists a basis B for V of the form:

$$B = B_1 \cup B_2$$

where B_1 is a basis for U and $B_1 \cap B_2 = \emptyset$.

Proof: We know that U has a basis, call it B_1.
Then as B_1 is linearly independent, V has a basis B with $B_1 \subset B$.
setting $B_2 = B \setminus B_1$ gives the result.

Corollary

If $U \subset V$ is a subspace of V then

$$\dim(U) \leq \dim(V)$$

moreover, if the dimension is finite, then equality holds if and only if $V = U$.

Proof: By the lemma, a basis B_1 for U is a subset of a basis B for V, so we have

$$|B_1| \leq |B|$$

from which the first claim follows.
For the second claim we notice that if the dimensions are the same, then $B_1 = B$ and so

$$U = \mathrm{Span}(B_1) = \mathrm{Span}(B) = V$$

and conversely, we have already seen if $\mathrm{Span}(B_1) = \mathrm{Span}(B)$ and $B_1 \subset B$ are both linearly independent, then $B_1 = B$.

Example Uses of Dimension

Consider the vector subspace $W \subset \mathbb{R}^4$ given by:

$$W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w - x - y + z = 0 \text{ and } w - 2x + y = 0\}$$

Show that

$$\mathrm{Span}((1, 2, 3, 4), (4, 3, 2, 1)) = W$$

We have already seen that $\mathrm{Span}((1, 2, 3, 4), (4, 3, 2, 1)) \subset W$.

We know that $\mathrm{Span}((1, 2, 3, 4), (4, 3, 2, 1))$ has dimension 2, because $(1, 2, 3, 4), (4, 3, 2, 1)$ are a basis.

Likewise $W = \{(w, x, y, z) \in \mathbb{R}^4 \mid w - x - y + z = 0 \text{ and } w - 2x + y = 0\}$ has dimension 2, as we have found a basis.

Therefor by the previous theorem they are equal.

Finite Dimensional Vector Spaces

Suppose V is a vector space and $\vec{e}_1, \ldots, \vec{e}_n$ is a basis so that V has dimension n. Then every vector in $\vec{v} \in V$ has a unique representation of the form:

$$\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$$

The information in $a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$ is essentially the same as the information in

$$(a_1, a_2, \ldots, a_n)$$

(provided you remember the basis $\vec{e}_1, \ldots, \vec{e}_n$ you can translate back and forth.)

Importantly, all operations in the vector space V, that is addition, scalar multiplication and confirming if vectors are equal, can be performed with representatives in the form

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$$

in a way which is equivalent to performing operations with

$$(a_1, a_2, \ldots, a_n)$$

in $\mathbb{R}^n$.
So answering questions about linear independence, generating sets, so forth will tend also to translate immediately, once we pick a basis.

What all of that is saying is summarized by the following theorem:

Theorem

Every vector space V of dimension n is isomorphic to $\mathbb{R}^n$ (as a vector space).

We will eventually make this more precise, but for now:

An **Isomorphism** is a map between two mathematical objects which

preserves the underlying structure of the objects

and has an inverse which does the same.

In the present context that means the function:

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n \leftrightarrow (a_1, a_2, \ldots, a_n)$$

on an assignment you will explain why this is a function.

An isomorphism is a map that tells you two objects are essentially the same, that is to say, in some sense equivalent. Despite the fact that they are not literally the same object.

The notion of isomorphism for vector spaces gives an equivalence relation on the category (collection of all) vector spaces.

We will define (soon) what types of maps *preserves the underlying structure* of vector spaces (they will be called Linear Transformations).

One of the main usefulnesses of basis, is that they let us do solve problems for an arbitrary vector space V, in the same way that we would solve them for $\mathbb{R}^n$.

Finite Dimensional Vector Spaces vs $\mathbb{R}^n$

The following operations are strictly equivalent between the two representations:

- Adding vectors
- Multiplying by scalars
- Checking equality (note: this is what requires linear independence)

Compare:

$$(a_1\vec{e}_1 + \cdots + a_n\vec{e}_n) + (b_1\vec{e}_1 + \cdots + b_n\vec{e}_n) = (a_1 + b_1)\vec{e}_1 + \cdots + (a_n + b_n)\vec{e}_n$$

against

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

Despite this, it is important to note that the choice of basis is a critical component of this identification. It has a significant impact on:

- How you might choose a dot product, hence on angles and lengths.
- How you might represent matricies.
- What the values of the numbers in the vectors you want to deal with are.

Finite Dimensional Vector Spaces vs $\mathbb{R}^n$ (example of the good)

If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is a basis for V then

$$a_1\vec{e}_1 + b_1\vec{e}_2 + c_1\vec{e}_3, \qquad a_2\vec{e}_1 + b_2\vec{e}_2 + c_2\vec{e}_3, \quad a_3\vec{e}_1 + b_3\vec{e}_2 + c_3\vec{e}_3$$

are linearly independent if and only if

$$(a_1, b_1, c_1), \quad (a_2, b_2, c_2), \quad (a_3, b_3, c_3)$$

are linearly indepdent.

Indeed, trying to solve

$$x(a_1\vec{e}_1 + b_1\vec{e}_2 + c_1\vec{e}_3) + y(a_2\vec{e}_1 + b_2\vec{e}_2 + c_2\vec{e}_3) + z(a_3\vec{e}_1 + b_3\vec{e}_2 + c_3\vec{e}_3) = \vec{0}$$

is the same as solving:

$$(a_1x + a_2y + a_3z)\vec{e}_1 + (b_1x + b_2y + b_3z)\vec{e}_2 + (c_1x + c_2y + c_3z)\vec{e}_3 = \vec{0}$$

Because $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are Linearly independent happens exactly when:

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0 \end{aligned}$$

But this is the same equations as for

$$x(a_1, b_1, c_1) + y(a_2, b_2, c_2) + z(a_3, b_3, c_3) = (0, 0, 0)$$

which corresponds to doing the check for independence in $\mathbb{R}^3$.

Assume that $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is a basis for V, demonstrate whether or not the following vectors are linearly independent.

$$\vec{e}_1 + 2\vec{e}_2, \quad \vec{e}_2 + \vec{e}_3$$

We translate the problem to $\mathbb{R}^3$ using the identification $a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 \leftrightarrow (a, b, c)$ so that

$$\vec{e}_1 + 2\vec{e}_2, \quad \vec{e}_2 + \vec{e}_3 \quad \leftrightarrow \quad (1, 2, 0), \quad (0, 1, 1)$$

and so we need to check if $(1, 2, 0)$, $(0, 1, 1)$ are linearly independent.
We check this by solving $a(1, 2, 0) + b(0, 1, 1) = (0, 0, 0)$ which becomes:

$$1a + 0b = 0$$
$$2a + 1b = 0$$
$$0a + 1b = 0$$

We see from the first and last equation the only solution is $a = b = 0$ which shows these vectors are linearly independent.
It is important when doing this kind of translation you explain what you are doing. If you want your solution to be a proof, it must be clear what you are doing in each step!
People often will skip steps and try to solve

$$1x + 2y + 0z = 0$$
$$0x + 1y + 1z = 0$$

there is a way to solve the problem this way... but if you don't explain what you are doing (or worse don't understand it) you can't really have a proof.

Finite Dimensional Vector Spaces vs $\mathbb{R}^n$ (example of the ugly)

For example, every subspace $U \subset \mathbb{R}^5$ has a finite basis, say it is

$$\vec{f_1} = (3, 5, 7, 19, 32),\ \vec{f_2} = (1, 2, 3, 2, 3),\ \vec{f_3} = (0, 2, 0, 1, 0),\ \vec{f_4} = (1, 1, 1, 1, 4).$$

so that $U = \mathrm{Span}(\vec{f_1}, \vec{f_2}, \vec{f_3}, \vec{f_4})$.

This means we can view:

$$U \leftrightarrow \mathbb{R}^4 \qquad \text{via} \qquad a_1\vec{f_1} + a_2\vec{f_2} + a_3\vec{f_3} + a_4\vec{f_4} \leftrightarrow (a_1, a_2, a_3, a_4)$$

With this identification $(1, 0, 0, 0) \in \mathbb{R}^4$ corresponds to $\vec{f_1}$ which is a vector in $\mathbb{R}^5$.
It is likely not a good idea to ever write:

$$(1, 0, 0, 0) = \vec{f_1} = (3, 5, 7, 19, 32)$$

as this would be confusing, and certainly you can see why these have different lengths.
And it would definitely be confusing to write $\mathbb{R}^4 = U \subset \mathbb{R}^5$.

So even when talking about finite dimensional vector spaces, it can be useful to **not automatically** identify then with $\mathbb{R}^n$.
At the same time, it **can be very useful** to do so sometimes.

Finite Dimensional vs Infinite Dimensional

Throughout this course I will state many theorems and pose many questions were we introduce a basis (or a linearly independent set, or a generating set) as

$$\vec{v}_1, \ldots, \vec{v}_n$$

when we write it like this the set is clearly finite.

It is almost always the case that the same theorems will be true if we just say

$$\{\vec{v}_i \mid i \in I\}$$

is a basis (or a linearly independent set, or a generating set) or even just say

$$B$$

is a basis (or a linearly independent set, or a generating set).

In fact, the proofs are almost always virtually identical, just with a bit of extra notation to track the index set I.
There is some extra ambiguity and clarification in proving things about the word unique in this context, because now order of things is not fixed.

Natural Questions About Basis and Dimension

- Find a basis for V.
  The proof that they exist implies an algorithm.
  The processes we used in $\mathbb{R}^n$ are much more concrete. In most cases the key is to start by finding a (finite) generating set, then find a maximal linearly independent subset.
- Is X a basis for V?
  Check the definition. Iis it linearly independent, are they generators?
- Find the dimension of V.
  Find a basis and count.