Direct Sum Decompositions

Given any vector space V, and subspaces V_1, V_2 of V we say that V is a **direct sum** of V_1 and V_2 and write

$$V = V_1 \oplus V_2$$

if every $\vec{v} \in V$ can be written *uniquely* as

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

with $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$.
In symbols we would write:

$$\forall \vec{v} \in V, \exists! \vec{v}_1 \in V_1, \exists! \vec{v}_2 \in V_2, \vec{v} = \vec{v}_1 + \vec{v}_2$$

unique here means, *if you think you have two solutions, they are the same solution*:

$$\forall \vec{v} \in V, \forall \vec{v}_1, \vec{v}_1' \in V_1, \forall \vec{v}_2, \vec{v}_2' \in V_2, (\vec{v}_1 + \vec{v}_2 = \vec{v} = \vec{v}_1' + \vec{v}_2') \Rightarrow (\vec{v}_1 = \vec{v}_1') \text{ and } (\vec{v}_2 = \vec{v}_2')$$

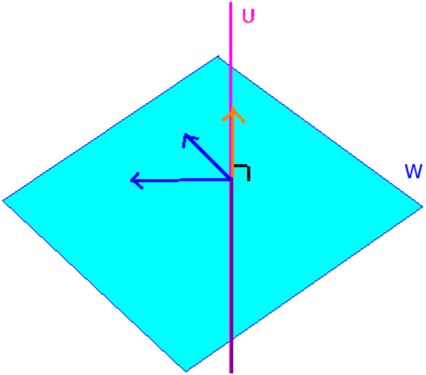
The definition extends to having more than two subspaces. Facts about the general case typically follow by induction on the case of two subspaces.

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

if

$$\forall \vec{v} \in V, \exists! \vec{v}_1 \in V_1, \ldots, \vec{v}_r \in V_r, \vec{v} = \vec{v}_1 + \cdots + \vec{v}_r$$

What we just defined is a special case of the direct sum called *an internal direct sum*.



We have a vector subspaces

$$\color{cyan}{W} = \mathrm{span}(\vec{e}_1, \vec{e}_2) \subset \mathbb{R}^3 \qquad\qquad \color{magenta}{U} = \mathrm{span}(\vec{e}_3) \subset \mathbb{R}^3$$

and since every vector in $\mathbb{R}^3$ can be written uniquely as

$$\vec{v} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 = \vec{w} + \vec{u}$$

we can see that this gives

$$\mathbb{R}^3 = U \oplus W$$

Concrete Example/Model for how to think about direct sums.

Consider $V = \mathbb{R}^5$ and

$$V_1 = \{(a, b, 0, 0, 0) \in V \mid a, b \in \mathbb{R}\} \qquad V_2 = \{(0, 0, c, d, e) \in V \mid c, d, e \in \mathbb{R}\}$$

then

$$V = V_1 \oplus V_2$$

We note that V_1 is *isomorphic* to $\mathbb{R}^2$, and V_2 is *isomorphic* to $\mathbb{R}^3$, so each of these two spaces is individually simpler to study.

Whenever we have a vector $(a, b, c, d, e) \in V$, we can think of having

$$\vec{v}_1 = (a, b, 0, 0, 0) \in V_1 \qquad \vec{v}_2 = (0, 0, c, d, e) \in V_2.$$

Given the isomorphisms V_1 with $\mathbb{R}^2$ and V_2 with $\mathbb{R}^3$ we can just think of having vectors

$$\vec{v}_1 = (a, b) \in \mathbb{R}^2 \qquad \vec{v}_2 = (c, d, e) \in \mathbb{R}^3.$$

It isn't unreasonable to think of

$$V = \{((a, b), (c, d, e)) \mid (a, b) \in \mathbb{R}^2, (c, d, e) \in \mathbb{R}^3\}$$

Note that addition/scalar multiplication all work componentwise, and that this happens in the more general case.

Writing $\mathbb{R}^5 \simeq \mathbb{R}^3 \oplus \mathbb{R}^2$ would be an *external direct sum*.

Properties of Direct Sums

Lemma

If $V = V_1 \oplus V_2$ then $V_1 \cap V_2 = \{\vec{0}\}$.

Proof:

It is obvious that $\{\vec{0}\} \subset V_1 \cap V_2$ so we only need to show the other containment.
Let $\vec{v} \in V_1 \cap V_2$ be arbitrary. So $\vec{v} \in V_1$ and $\vec{v} \in V_2$
Then we can write

$$\vec{0} \in V$$

in apparently two different ways as a sum of vectors from V_1 and V_2.

$$\vec{0} = \vec{v} - \vec{v} \qquad \text{and} \qquad \vec{0} = \vec{0} + \vec{0}$$

but, $V = V_1 \oplus V_2$, so this expression must be unique.
(in the above, we have $\vec{v} \in V_1$ and $-\vec{v} \in V_2$ and also $\vec{0} \in V_1$ and $\vec{0} \in V_2$, so the exressions are the sort which should be unique, recall unique means, *if you think you have two solutions, they are the same solution*:

$$\vec{v} \in V, \forall \vec{v_1}, \vec{v_1}' \in V_1, \forall \vec{v_2}, \vec{v_2}' \in V_2, (\vec{v_1} + \vec{v_2} = \vec{v} = \vec{v_1}' + \vec{v_2}') \Rightarrow (\vec{v_1} = \vec{v_1}') \text{ and } (\vec{v_2} = \vec{v_2}')$$

so with

$$\vec{v_1} = \vec{v}, \quad \vec{v_2} = -\vec{v}, \quad \vec{v_1}' = \vec{0}, \quad \vec{v_2}' = \vec{0}$$

We can use the uniqueness to conclude conclude $\vec{v_1} = \vec{v_1}'$.)
This gives $\vec{v} = \vec{0}$. So every vector in $V_1 \cap V_2$ is the zero vector.
This proves $V_1 \cap V_2 = \{\vec{0}\}$.

Properties of Direct Sums

Lemma

If $V = V_1 \oplus V_2$ and $\vec{e}_1, \ldots, \vec{e}_n$ are linearly independent vectors in V_1 and $\vec{f}_1, \ldots, \vec{f}_m$ are linearly independent vectors in V_2 then

$$\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$$

are linearly independent vectors in V.

Proof:

From the previous lemma we know that

$$V_1 \cap V_2 = \{\vec{0}\}$$

But because

$$\mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \subset V_1 \quad \text{and} \quad \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m) \subset V_2$$

we can conclude that

$$\mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \cap \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m) = \{\vec{0}\}$$

From the assignment because $\vec{e}_1, \ldots, \vec{e}_n$ are linearly independent, and $\vec{f}_1, \ldots, \vec{f}_m$ are linearly independent, and $\mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \cap \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m) = \{\vec{0}\}$ we get to conclude that

$$\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$$

are linearly independent.

Properties of Direct Sums

Lemma

If $V = V_1 \oplus V_2$ and $\vec{e}_1, \ldots, \vec{e}_n$ are a generating set for V_1 and $\vec{f}_1, \ldots, \vec{f}_m$ are a generating set for V_2 then

$$\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$$

are a generating set for V.

Proof: Let $\vec{v} \in V$ be arbitrary. Because $V = V_1 \oplus V_2$ we may write

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

where $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$.

As $\vec{e}_1, \ldots, \vec{e}_n$ are a generating set for V_1, and $\vec{v}_1 \in V_1$, there are $a_1, \ldots, a_n \in \mathbb{R}$ with

$$\vec{v}_1 = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$$

As $\vec{f}_1, \ldots, \vec{f}_m$ are a generating set for V_2, and $\vec{v}_2 \in V_2$, there are $b_1, \ldots, b_m \in \mathbb{R}$ with

$$\vec{v}_2 = b_1\vec{f}_1 + \cdots + b_m\vec{f}_m$$

Putting this together gives

$$\vec{v} = \vec{v}_1 + \vec{v}_2 = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n + b_1\vec{f}_1 + \cdots + b_m\vec{f}_m$$

so every vector in V is a linear combination of $\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$ and hence

$$\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$$

are a generating set for V.

Properties of Direct Sums

Theorem
If $V = V_1 \oplus V_2$ and $\vec{e}_1, \ldots, \vec{e}_n$ is a basis for V_1 and $\vec{f}_1, \ldots, \vec{f}_m$ is a basis for V_2 then

$$\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$$

is a basis for V, in particular, the dimension of V is the sum of the dimensions of V_1 and V_2.

Proof:
This is an immediate consequence of the two previous lemmas.

Constructing Direct Sums

Theorem
If $\mathrm{Span}(S) \cap \mathrm{Span}(R) = \{\vec{0}\}$ then $\mathrm{Span}(S \cup R) = \mathrm{Span}(S) \oplus \mathrm{Span}(R)$.
Proof:
The first thing to point out is that indeed, both of $\mathrm{Span}(S)$ and $\mathrm{Span}(R)$ are subsets of $\mathrm{Span}(S \cup R)$ (Why?).

From the assignment we know that every element of $\mathrm{Span}(S \cup R)$ can be written in the form:

$$\vec{s} + \vec{r}$$

where $\vec{s} \in \mathrm{Span}(S)$ and $\vec{r} \in \mathrm{Span}(R)$.

From the assignment we also know that because $\mathrm{Span}(S) \cap \mathrm{Span}(R) = \{\vec{0}\}$ this representation is unique.

This proves that

$$\mathrm{Span}(S \cup R) = \mathrm{Span}(S) \oplus \mathrm{Span}(R)$$

Constructing Direct Sums

Theorem
If $\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$ is a basis for V then

$$V = \mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \oplus \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m)$$

Proof:
From the assignment we know that because $\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m$ are linearly indepentent that

$$\mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \cap \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m) = \{\vec{0}\}$$

We then know that

$$V = \mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n, \vec{f}_1, \ldots, \vec{f}_m) = \mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n) \oplus \mathrm{Span}(\vec{f}_1, \ldots, \vec{f}_m)$$

One way to interpret a bunch of the results, is that specifying a direct sum decomposition is basically the same thing as cutting a basis into pieces and vice versa. We shall use this idea much later when we try to *change bases*.

Theorem
If $W \subset V$ is any vector subspace of a vector space V, then there exists a subspace $U \subset V$ such that

$$V = W \oplus U$$

Proof:
Let L be a basis for W, then L is linearly independent.
By our theorem on the existance of basis, we know that because L is linearly independent there exists a basis B for V such that

$$L \subset B$$

Consider the set

$$M = B \setminus L$$

so that

$$B = M \cup L$$

is a basis for V.
By the previous theorem we then know that

$$V = \mathrm{Span}(L) \oplus \mathrm{Span}(M).$$

As we know $W = \mathrm{Span}(L)$, by letting $U = \mathrm{Span}(M)$ we thus obtain

$$V = W \oplus U.$$

Abstract Example

Consider V a vector space with basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5$ and

$$V_1 = \mathrm{Span}(\vec{e}_1, \vec{e}_2) \qquad V_2 = \mathrm{Span}(\vec{e}_3, \vec{e}_4, \vec{e}_5)$$

then

$$V = V_1 \oplus V_2$$

We note that V_1 is *isomorphic* to $\mathbb{R}^2$, and V_2 is *isomorphic* to $\mathbb{R}^3$, so each of these two spaces is individually simpler to study.

Whenever we have a vector $a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5 \in V$, we can think of having

$$\vec{v}_1 = a_1\vec{e}_1 + a_2\vec{e}_2 \in V_1 \qquad \vec{v}_2 = a_3\vec{e}_3 + a_4\vec{e}_4 + a_5\vec{e}_5 \in V_2$$

It is reasonably clear that both addition/scalar multiplication work componentwise, so it isn't unreasonable to think of

$$V = \{(\vec{v}_1, \vec{v}_2) \mid \vec{v}_1 \in V_1, \vec{v}_2 \in V_2\}$$

which under the *isomorphism* V_1 with $\mathbb{R}^2$ and V_2 with $\mathbb{R}^3$ is compatible with the *isomorphism* V with $\mathbb{R}^5$ in our last example.

Note that this componentwise interpretation works in the more general case.

Benefit: lower dimensional vector spaces are simpler to think about, by cutting things into pieces we can study simpler things before combining them!

Orthogonal Direct Sums

The entire idea of direct sums generalizes the following sort of construction: Given $\mathbb{R}^n$, if $W \subset \mathbb{R}^n$ is a subspace we can define:

$$W^{\perp} = \{\vec{v} \in \mathbb{R}^n \mid \forall \vec{w} \in W, (\vec{v}, \vec{w}) = 0\}$$

this is the set of vectors perpendicular to all vectors in W.

If W was a plane (through the origin) in $\mathbb{R}^3$, then $W^{\perp}$ is the normal line (through the origin).
If W was a line (through the origin) in $\mathbb{R}^3$, then $W^{\perp}$ is the perpendicular plane (through the origin).

It is always the case that

$$\mathbb{R}^n = W \oplus W^{\perp}$$

For now, the details are an exercise.