Linear Transformations - Informal

A Linear Transformation is a function from one vector space to another that plays nice with the underlying structure that makes a vector space a vector space.
That is, a linear transformation from V to W is a function

$$L : V \to W$$

which plays nice with $+$ and $\cdot$ for the vector spaces V and W.

Linear Transformations - Formal

We call a function $L : V \to W$ a **linear transformation** if

- $\forall \vec{v}_1, \vec{v}_2 \in V, L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$.
- $\forall a \in \mathbb{R}, \forall \vec{v} \in V, L(a\vec{v}) = aL(\vec{v})$.

This basically says $+$ and $\cdot$ can be evaluated before or after the linear transformation.

Immediate consequences: If L is a linear transformation then:

- $L(\vec{0}) = \vec{0}$. $L(\vec{0}) = L(0 \cdot \vec{0}) = 0 \cdot L(\vec{0}) = \vec{0}$.
- $L(a_1\vec{v}_1 + \cdots + a_r\vec{v}_r) = a_1 L(\vec{v}_1) + \cdots + a_r L(\vec{v}_r)$. repeatedly using above rules.

Theorem

$L : V \to W$ is a linear transformation if and only if for all $\vec{v}_1, \vec{v}_2 \in V$ and $a, b \in \mathbb{R}$ we have:

$$L(a\vec{v}_1 + b\vec{v}_2) = aL(\vec{v}_1) + bL(\vec{v}_2).$$

Proof Idea: The case $a = b = 1$ gives the first rule above, the case $b = 0$ gives the second.

You can use this theorem as an alternate definition, it is often a bit faster to check this condition.

Example

Show that the map $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a linear transformation.

Proof Sketch We have that

$$\begin{aligned} L\left(a\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\left(a\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) \\ &= a\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + b\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\ &= aL\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + bL\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) \end{aligned}$$

As you can imagine, this type of thing will work for **all** matricies, we will say more about this later.

Special Transformations - the Zero map

Given any two vector spaces V and W there is a linear transformation

$$0_{V,W} : V \to W$$

It is the map

$$0_{V,W}(\vec{v}) = \vec{0}$$

which sends everything to zero.

The zero map is very useful for notational purposes, and often a good way to sanity check theorems because in many cases it will give the simplest counterexample to something you might think would be a theorem but isn't.

Theorem

Let $L : U \to V$ be a linear transformation. We have the following identities.

- $0_{V,W} \circ L = 0_{U,W}$
- $L \circ 0_{W,U} = 0_{W,V}$

All of the above is an easy excercise.

Special Transformations - the identity map

Given any vector space V, there is a linear transformation:

$$\mathrm{Id}_V : V \to V$$

it is the map

$$\mathrm{Id}_V(\vec{v}) = \vec{v}$$

which sends every vector to itself.

The identity map is very useful for notational purposes, and also helpful in many other definitions constructions we will see.

Theorem

Let $L : U \to V$ be a linear transformation. We have the following identities.

- $\mathrm{Id}_V \circ L = L$
- $L \circ \mathrm{Id}_U = L$

All of the above is an easy excercise.

Special Transformations - the inclusion map

Given any vector space V, and any subspace $U \subset V$, the inclusion map:

$$\mathrm{incl}_{\mathrm{U,V}} : U \to V$$

defined by:

$$\mathrm{incl}_{\mathrm{U,V}}(\vec{u}) = \vec{u}$$

is a linear transformation.

We often use inclusion maps implicitly, without thinking about it, and this is fine, but it can be occasionally useful when proving theorems to make an explicit reference to it.

Proof Idea The inclusion map is essentially the identity map, so the proof is identical to that case.

Special Transformations - the projection maps for direct sums

Given any vector space V, and any subspaces $W_1, W_2 \subset V$ such that

$$V = W_1 \oplus W_2$$

Recall that $\forall \vec{v} \in V$ there exists **unique** $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$ so that $\vec{v} = \vec{w}_1 + \vec{w}_2$.

We can define projection maps

$$\mathrm{Proj}_{\mathrm{V},\mathrm{W}_1} : V \to W_1 \quad \text{and} \quad \mathrm{Proj}_{\mathrm{V},\mathrm{W}_2} : V \to W_2$$

by the characteriztion

$$\vec{v} = \mathrm{Proj}_{\mathrm{V},\mathrm{W}_1}(\vec{v}) + \mathrm{Proj}_{\mathrm{V},\mathrm{W}_2}(\vec{v})$$

that is

$$\mathrm{Proj}_{\mathrm{V},\mathrm{W}_1}(\vec{v}) = \textit{the unique } \vec{w}_1 \qquad \mathrm{Proj}_{\mathrm{V},\mathrm{W}_2}(\vec{v}) = \textit{the unique } \vec{w}_2$$

are linear transformations.

Proof Sketch: If $\vec{v} = \vec{w}_1 + \vec{w}_2$ and $\vec{v}' = \vec{w}_1' + \vec{w}_2'$ then

$$a\vec{v} + b\vec{v}' = (a\vec{w}_1 + b\vec{w}_1') + (a\vec{w}_2 + b\vec{w}_2')$$

so that

$$\mathrm{Proj}_{\mathrm{V},\mathrm{W}_1}(a\vec{v} + b\vec{v}') = a\vec{w}_1 + b\vec{w}_1' = a\mathrm{Proj}_{\mathrm{V},\mathrm{W}_1}(\vec{v}) + b\mathrm{Proj}_{\mathrm{V},\mathrm{W}_1}(\vec{v}').$$

This generalizes the idea of *orthogonal projections*

Linear Transformations - Linear Combinations

Given a pair of vector spaces U, V and a pair of linear transformations:

$$L : U \to V \qquad M : U \to V$$

and any two real numbers a, b, we can define a new function

$$f : U \to V$$

according to the rule:

$$f(\vec{u}) = aL(\vec{u}) + bM(\vec{u})$$

This is definitely a function.

Theorem

With f as above f is a linear transformation.

We leave this as an exercise.

Theorem

With addition and scalar multiplication as defined above, the set:

$$\mathrm{Hom}(U, V) = \{f : U \to V \mid f \text{ is a linear transformation}\}$$

of linear transformations from U to V is a vector space.

We won't use it crucially in this course.

Linear Transformations - Composition

Given three vector spaces U, V, W and a pair of linear transformations:

$$L : V \to W \qquad M : U \to V$$

We can always write down the composition of the functions $L \circ M$:

$$L \circ M : U \to W$$

Theorem

With L and M as above the function $L \circ M : U \to V$ is a linear transformation.

Proof Sketch:

$$\begin{aligned} L \circ M(a\vec{x} + b\vec{y}) &= L(M(a\vec{x} + b\vec{y})) && \text{definition of } L \circ M \\ &= L(aM(\vec{x}) + bM(\vec{y})) && \text{linearity of } M \\ &= aL(M(\vec{x})) + bL(M(\vec{y})) && \text{linearity of } L \\ &= aL \circ M(\vec{x}) + bL \circ M(\vec{y}) && \text{definition of } L \circ M \end{aligned}$$

Theorem: With notation above, both the maps:

$$m : \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W) \qquad \text{and} \qquad \ell : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$$

given by

$$m(N) = N \circ M \qquad \text{and} \qquad \ell(N) = L \circ N$$

are linear transformations.

We won't use it for anything in this course.

Linear Combinations and Compositions

Theorem
Let A, B be linear transformations $V \to W$ and C, D be linear transformations $U \to V$ then:

$$(A + B) \circ (C + D) = A \circ C + A \circ D + B \circ C + B \circ D$$

This basically says composition works like multiplication.

Proof: We have

$$\begin{aligned}
&(A + B) \circ (C + D)(\vec{x}) && \\
=&((A + B)((C + D))(\vec{x})) && \text{definition of composition} \\
=&(A + B)(C(\vec{x}) + D(\vec{x})) && \text{definition of } (C + D) \\
=&A(C(\vec{x}) + D(\vec{x})) + B(C(\vec{x}) + D(\vec{x})) && \text{definition of } (A + B) \\
=&A(C(\vec{x})) + A(D(\vec{x})) + B(C(\vec{x})) + B(D(\vec{x})) && \text{linearity of } A \text{ and } B \\
=&A \circ C(\vec{x}) + A \circ D(\vec{x}) + B \circ C(\vec{x}) + B \circ D(\vec{x}) && \text{definition of composition}
\end{aligned}$$

this shows that the function in the LHS has the same output values as the function on the RHS for all values of the input, hence the functions are equal.

Linear Transformations - Inverses

Given a vector spaces V, W and a linear transformation $L : V \to W$ if the function L is bijective as a function, then there exists a function:

$$L^{-1} : W \to V$$

Theorem

With L as above, a bijective linear transformation, then the function L^{-1} is a linear transformation.

That L^{-1} is a function is clear, that it is a linear transformation is what we need to check!

Proof Sketch: We calculate that

$$\begin{aligned}
&L(L^{-1}(a\vec{x} + b\vec{y}) - aL^{-1}(\vec{x}) - bL^{-1}(\vec{y})) && \\
&= L(L^{-1}(a\vec{x} + b\vec{y})) - aL(L^{-1}(\vec{x})) - bL(L^{-1}(\vec{y})) && \text{linearity of } L \\
&= a\vec{x} + b\vec{y} - a\vec{x} - b\vec{y} && L \circ L^{-1}(\vec{x}) = \vec{x} \\
&= \vec{0} && \text{cancelling terms} \\
&= L(\vec{0}) && \text{because } L(\vec{0}) = \vec{0}
\end{aligned}$$

Now because L is injective, this implies

$$L^{-1}(a\vec{x} + b\vec{y}) - aL^{-1}(\vec{x}) - bL^{-1}(\vec{y}) = \vec{0}$$

but this says

$$L^{-1}(a\vec{x} + b\vec{y}) = aL^{-1}(\vec{x}) + bL^{-1}(\vec{y})$$

and so L^{-1} is linear.

Definitions about inverses

Definition

Given $L : U \to V$, a map $M : V \to U$ is the **inverse** if and only if:

$$L \circ M = \mathrm{Id}_V \qquad M \circ L = \mathrm{Id}_U$$

We generally write $M = L^{-1}$ to denote the inverse when it exists.

Definition

A bijective linear transformation is called an **isomorphism** of vector spaces.
Vector spaces U and V are said to be **Isomorphic** if there exists a bijective linear transformation between them.

Identification of Finite dimensional vector spaces with $\mathbb{R}^n$

Recall that if V is a finite dimensional vector space with basis $\vec{e}_1, \ldots, \vec{e}_n$ then we wanted to identify:

$$V \leftrightarrow \mathbb{R}^n$$

by the rules:

$$L(a_1\vec{e}_1 + \cdots + a_n\vec{e}_n) = (a_1, \ldots, a_n)$$

$$M((a_1, \ldots, a_n)) = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$$

Theorem

The expression above defines a function L that is a bijective linear transformation, that is, an isomorphism.

Consequently, every finite dimensional vector space V is isomorphic to $\mathbb{R}^n$ where n is the dimension of V.

We leave this as an exercise.

Important The isomorphisms M and L depend on a choice of basis $\vec{f}_1, \ldots, \vec{f}_n$ for V!!!

Comparing/Defining Linear Transformations

Lemma

Two linear transformations $L_1, L_2 : V \to W$ are equal, that is:

$$\forall \vec{v} \in V, L_1(\vec{v}) = L_2(\vec{v})$$

if and only if they are equal on a basis S of V, that is if

$$\forall \vec{v} \in S, L_1(\vec{v}) = L_2(\vec{v}).$$

This is on the assignment

Lemma

If $\vec{e}_1, \ldots, \vec{e}_n$ are a basis for a vector space V, and $\vec{w}_1, \ldots, \vec{w}_n$ are any n vectors in a vector space W then there is a linear transformation $L : V \to W$ such that

$$L(\vec{e}_i) = w_i \quad i \in \{1, \ldots, n\}$$

This is on the assignment

What the above results say is that linear transformations are determined by, and can be specified by, what happens to a basis.

Natural Questions About Abstract Linear Transformations

- Given some description of a function $f : V \to W$, is f a linear transformation?
  One simply needs to check that $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$ for all $a, b, \vec{x}, \vec{y}$. How one does this depends on the description of f, as we shall see on the assignment.
- Given some description of a linear transformation $L : V \to W$, what is $L(\vec{v})$?
  How one does this depends on the description of f, as we shall see on the assignment.