Choice of Basis Impacts the Matricies of Linear Transformation.

Recall Suppose V and W are finite dimensional vector spaces.
We have seen that by selecting a basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W we can identify

$$V \leftrightarrow \mathbb{R}^n \qquad W \leftrightarrow \mathbb{R}^m$$

So that a linear transformation $L : V \to W$ can be thought of as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.
In this way we associate L to a matrix A (with respect to this choice of basis), and in particular, we have seen that if

$$L(\vec{e}_j) = a_{1j}\vec{f}_1 + \cdots + a_{mj}\vec{f}_m = \sum_{i=1}^{m} a_{ij}\vec{f}_i$$

then A associated to L is the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix}$$

But the entire process involves choices of basis, how does this **choice** effect the matrix A?
Several aspects of this are looked at in A3Q3.

Suppose we pick a new basis $\vec{g}_1, \ldots, \vec{g}_m$ for W.
If we want to find the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & \\ \vdots & & \ddots & \vdots \\ x_{m1} & & \cdots & x_{mn} \end{pmatrix}$$

associated to L with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ for V and $\vec{g}_1, \ldots, \vec{g}_m$ for W, we need the c_{ij} to satisfy:

$$L(\vec{e}_j) = x_{1j}\vec{g}_1 + \cdots + x_{mj}\vec{g}_m = \sum_{i=1}^{m} x_{ij}\vec{g}_i$$

Presumably most of the time $A \neq X$! Our goal is to understand how to relate the matrix X to the matrix A

Given the two bases $\vec{f}_1, \ldots, \vec{f}_m$ and $\vec{g}_1, \ldots, \vec{g}_m$ of W we know that each $\vec{f}_i$ can be written in terms of the basis $\vec{g}_1, \ldots, \vec{g}_m$, that is there is a formula

$$\vec{f}_i = b_{1i}\vec{g}_1 + \cdots + b_{mi}\vec{g}_m = \sum_{k=1}^{m} b_{ki}\vec{g}_k$$

for some numbers b_{ki}.

Notice that we can, if we would like, arrange the numbers b_{ki} into an m by m matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & & \\ \vdots & & \ddots & \vdots \\ b_{m1} & & \cdots & b_{mm} \end{pmatrix}$$

With this in hand, we can rewrite:

$$L(\vec{e}_j) = \sum_{i=1}^{m} a_{ij}\vec{f}_i = \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{m} b_{ki}\vec{g}_k = \sum_{k=1}^{m} \left(\sum_{i=1}^{m} b_{ki}a_{ij} \right) \vec{g}_k$$

Which gives us the formula we were looking for. So the matrix for L, with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ for V and $\vec{g}_1, \ldots, \vec{g}_m$ for W, is the matrix X whose kj entry is x_{kj}, where

$$x_{kj} = \sum_{i=1}^{m} b_{ki}a_{ij} = b_{k1}a_{1j} + b_{k2}a_{2j} + \cdots + b_{km}a_{mj}$$

which is precisely the kj entry of the matrix BA

Summary

Using bases $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W, we have equation

$$L(\vec{e}_j) = a_{1j}\vec{f}_1 + \cdots + a_{mj}\vec{f}_m = \sum_{i=1}^{m} a_{ij}\vec{f}_i \quad \text{gives matrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix}$$

Using bases $\vec{e}_1, \ldots, \vec{e}_n$ for V and $\vec{g}_1, \ldots, \vec{g}_m$ for W, we have equation

$$L(\vec{e}_j) = x_{1j}\vec{g}_1 + \cdots + x_{mj}\vec{g}_m = \sum_{i=1}^{m} x_{ij}\vec{g}_i \quad \text{gives matrix} \quad X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & \\ \vdots & & \ddots & \vdots \\ x_{m1} & & \cdots & x_{mn} \end{pmatrix}$$

The relationship between A and X is

$$X = BA$$

where the equations

$$\vec{f}_i = b_{1i}\vec{g}_1 + \cdots + b_{mi}\vec{g}_m = \sum_{k=1}^{m} b_{ki}\vec{g}_k \quad \text{gives matrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & & \\ \vdots & & \ddots & \vdots \\ b_{m1} & & \cdots & b_{mm} \end{pmatrix}$$

What we just saw is:

- $L : V \to W$.
- $\vec{e}_1, \ldots, \vec{e}_n$ basis of V and $\vec{f}_1, \ldots, \vec{f}_m$ basis of W.
- $\vec{g}_1, \ldots, \vec{g}_m$ another basis for W.
- A is the matrix that describes L for the basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W.
- B is the matrix that describes how to write $\vec{f}_i$ in terms of $\vec{g}_j$.

Then the matrix for L in terms of the basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{g}_1, \ldots, \vec{g}_m$ of W is

$$BA$$

Remark: On the assignment you will prove a couple things about B, for example that it is invertible and how to describe its inverse.

Also note: left multiplication by an invertible matrix corresponds to row operations... doing Gaussian elimination...

That is, doing Gaussian elimination is the **same** as changing the basis for the codomain!!!

Example

$\mathbb{R}^3$ has a basis

$$\vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0), \quad \vec{e}_3 = (0,0,1)$$

$\mathbb{R}^2$ has two bases

$$\vec{f}_1 = (2,3), \quad \vec{f}_2 = (3,1)$$

$$\vec{g}_1 = (1,0), \quad \vec{g}_2 = (0,1)$$

We have that a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ sends

$$L(\vec{e}_1) = \vec{f}_1, \quad L(\vec{e}_2) = \vec{f}_2, \quad L(\vec{e}_3) = \vec{f}_1 + \vec{f}_2$$

What is the matrix for L with respect to the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{f}_1, \vec{f}_2$?

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

What is the matrix for L with respect to the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{g}_1, \vec{g}_2$?

$$L(\vec{e}_1) = 2\vec{g}_1 + 3\vec{g}_2 \quad L(\vec{e}_2) = 3\vec{g}_1 + 1\vec{g}_2, \quad L(\vec{e}_3) = 5\vec{g}_1 + 4\vec{g}_2$$

so

$$A' = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = BA$$

Theorem:
If A is the matrix for a linear transformation $L : V \to W$ with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W and if B is the matrix associated to writing

$$\vec{f}_j = \sum_{i=1}^{m} b_{ij} \vec{g}_i$$

in terms of the basis $\vec{g}_1, \ldots, \vec{g}_m$ for W then the matrix for L with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{g}_1, \ldots, \vec{g}_m$ of W is

$$BA$$

This is exactly what we just proved.

Theorem:
If A is the matrix for a linear transformation $L : V \to W$ with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W and if C is the matrix associated to writing

$$\vec{h}_j = \sum_{i=1}^{n} c_{ij} \vec{e}_i$$

where $\vec{h}_1, \ldots, \vec{h}_n$ is another basis for V, then the matrix for L with respect to the basis $\vec{h}_1, \ldots, \vec{h}_n$ of V and $\vec{f}_1, \ldots, \vec{f}_m$ of W is

$$AC$$

This is on the assignment.
Note: there is an asymmetry in the definition of B and C can you see it?

Corollary

With the notation as above the matrix for L with respect to the basis $\vec{h}_1, \ldots, \vec{h}_n$ of V and $\vec{g}_1, \ldots, \vec{g}_m$ of W is

$$BAC$$

where B and C are the matricies as in the previous theorems.

Proof: This directly combines the two results, by first doing the one change of basis, and then doing the other.

Recall: there is an asymmetry in the definition of B and C !

Recall B is the matrix associated to writing

$$\vec{f}_j = \sum_{i=1}^{m} b_{ij}\vec{g}_i$$

Now denote by D the matrix whose entries are d_{ij} which come from the formula

$$\vec{g}_j = \sum_{i=1}^{m} d_{ij}\vec{f}_i$$

We defining d_{ij} in the same way we defined c_{ij}, except with the bases for the codomain W rather than the domain V

Lemma
We have the formula

$$DB = BD = \mathrm{Id}_m$$

so that both B and D are invertible.
This is on the assignment
This lemma gives some insight into the asymmetry in the definition of the matricies B and C

In the special case where $L : V \to V$, so we start with a single basis $\vec{e}_1, \ldots, \vec{e}_n$.
If A is the matrix for L in the basis $\vec{e}_1, \ldots, \vec{e}_n$, so

$$L(\vec{e}_j) = a_{1j}\vec{e}_1 + \cdots + a_{nj}\vec{e}_n = \sum_{i=1}^{n} a_{ij}\vec{e}_i,$$

and if B is the matrix associated to writing

$$\vec{e}_j = \sum_{i=1}^{m} b_{ij}\vec{f}_i$$

and C is the matrix associated to writing

$$\vec{f}_j = \sum_{i=1}^{m} c_{ij}\vec{e}_i$$

then by the previous lemma $B = C^{-1}$.

Theorem

With the notation as above, the matrix for L in the basis $\vec{f}_1, \ldots, \vec{f}_n$ is

$$BAC = C^{-1}AC = BAB^{-1}$$

Follows immediately from what we have just shown.

Note: there is again hopefully clarifies the asymmetry in the definition of B and C.
It is an annoying consequence of the above, that it is often hard to remember exactly which of B or C you want to use at any given time!

Example

$\mathbb{R}^3$ has two basis
$$\vec{h}_1 = (3,2,1), \quad \vec{h}_2 = (1,2,3), \quad \vec{h}_3 = (1,0,1)$$
$$\vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0), \quad \vec{e}_3 = (0,0,1)$$

$\mathbb{R}^2$ has two bases
$$\vec{f}_1 = (2,3), \quad \vec{f}_2 = (3,1)$$
$$\vec{g}_1 = (1,0), \quad \vec{g}_2 = (0,1)$$

We have that a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ sends
$$L(\vec{h}_1) = \vec{f}_1, \quad L(\vec{h}_2) = \vec{f}_2, \quad L(\vec{h}_3) = \vec{f}_1 + \vec{f}_2$$

The matrix for L with respect to the basis $\vec{h}_1, \vec{h}_2, \vec{h}_3$ and $\vec{f}_1, \vec{f}_2$ is
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

What is the matrix for L with respect to the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{g}_1, \vec{g}_2$?
Either solve:
$$L(\vec{e}_1) = ? \quad L(\vec{e}_2) = ?, \quad L(\vec{e}_3) = ?$$

Or find:
$$A' = BAC = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}^{-1}$$

It is typically faster to directly solve for $L((1,0,0))$, $L((0,1,0))$, $L((0,0,1))$ than compute the inverse and take the product.

A few examples where we know the matrix of a transformation.

Matrix of the identity

Let V be any finite dimensional vector space, and let $\vec{e}_1, \ldots, \vec{e}_n$ be any basis.

Proposition

The matrix associated to Id_V is the identity matrix.

Proof Sketch For any basis $\vec{e}_1, \ldots, \vec{e}_n$ we will have

$$\mathrm{Id}_V(\vec{e}_j) = \vec{e}_j = \sum_{i=1}^{n} a_{ij}\vec{e}_i$$

But because $\vec{e}_1, \ldots, \vec{e}_n$ are a basis, the expressions

$$\vec{e}_j = \sum_{i=1}^{n} a_{ij}\vec{e}_i$$

must agree, and so

$$a_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

this exactly describes the identity matrix.

The above description of the identity matrix is a useful way to prove things are the identity matrix.

Matrix of the Zero Transformation

Let V and W be any finite dimensional vector space, and let $\vec{e}_1, \ldots, \vec{e}_n$, $\vec{f}_1, \ldots, \vec{f}_m$ be any basis.

Proposition

The matrix associated to $0_{V,W}$ is the n by m zero matrix.

Proof Sketch

For any basis $\vec{e}_1, \ldots, \vec{e}_n$ we will have

$$0_{V,W}(\vec{e}_j) = \vec{0} = \sum_{i=1}^{n} a_{ij} \vec{f}_i$$

Because $\vec{f}_1, \ldots, \vec{f}_m$ are a basis, they are linearly independent, which implies so $a_{ij} = 0$ for all i and j.

From which we can see that matrix A must be the zero matrix.

Matrix of Combinations

Let $L : V \to W$ and $M : V \to W$ be linear transformations and $x, y \in \mathbb{R}$ real numbers. Let $\vec{e}_1, \ldots, \vec{e}_n$ be a basis for V and $\vec{f}_1, \ldots, \vec{f}_m$ be a basis for W.

Proposition

Suppose A is the matrix for L with respect to these basis and B is the matrix for M with respect to these basis then the matrix for

$$xL + yM$$

with respect to this basis is:

$$xA + yB$$

Proof Sketch The matrix A for L comes from the identities

$$L(\vec{e}_j) = \sum_{i=1}^{m} a_{ij} \vec{f}_i$$

and the matrix B for M comes from the identities

$$M(\vec{e}_j) = \sum_{i=1}^{m} b_{ij} \vec{f}_i$$

It follows that we have the identities

$$(xL + yM)(\vec{e}_j) = xL(\vec{e}_j) + yM(\vec{e}_j) = \sum_{i=1}^{m} (xa_{ij} + yb_{ij}) \vec{f}_i$$

So that the entries of the matrix associated to $aL + bM$ are given by $(xa_{ij} + yb_{ij})$, which agree with the entries for $xA + yB$.

Matrix of Compositions

Let $L : V \to W$ and $M : U \to V$ be linear transformations. Let $\vec{e}_1, \ldots, \vec{e}_n$ be a basis for U and $\vec{f}_1, \ldots, \vec{f}_m$ be a basis for W and let $\vec{g}_1, \ldots, \vec{g}_\ell$ be a basis for V.

Proposition

Suppose A is the matrix for L with respect to these basis and B is the matrix for M with respect to these basis then the matrix for

$$L \circ M : U \to W$$

with respect to the basis $\vec{e}_1, \ldots, \vec{e}_n$ and $\vec{f}_1, \ldots, \vec{f}_m$ is

$$AB.$$

Moreover, though the matricies A and B both may depend on $\vec{g}_1, \ldots, \vec{g}_\ell$, this product AB does not depend on the choice of basis except in so far that the same basis for V must be used for both L and M.

Proof: The matrix A for L and B for M and C for $L \circ M$ come from the identities

$$L(\vec{g}_i) = \sum_{k=1}^{m} a_{ki}\vec{f}_k \qquad M(\vec{e}_j) = \sum_{i=1}^{\ell} b_{ij}\vec{g}_i \qquad L \circ M(\vec{e}_j) = \sum_{k=1}^{m} c_{kj}\vec{f}_k$$

But by observing that

$$L \circ M(\vec{e}_j) = L\left(\sum_{i=1}^{\ell} b_{ij}\vec{g}_i\right) = \sum_{i=1}^{\ell} b_{ij} L(\vec{g}_i) = \sum_{i=1}^{\ell} b_{ij} \sum_{k=1}^{m} a_{ki}\vec{f}_k = \sum_{i=1}^{\ell}\sum_{k=1}^{m} a_{ki} b_{ij}\vec{f}_k = \sum_{k=1}^{m}\left(\sum_{i=1}^{\ell} a_{ki} b_{ij}\right)\vec{f}_k$$

we conclude $c_{kj} = \sum_{i=1}^{\ell} a_{ki} b_{ij}$ and obtain the result.

Matrix of an Inverse

Let $L : U \to V$ be an invertible linear transformation and $L^{-1} : V \to U$ be its inverse. Let $\vec{e}_1, \ldots, \vec{e}_n$ be a basis for U and $\vec{f}_1, \ldots, \vec{f}_n$ be a basis for V

Proposition

Suppose A is the matrix for L with respect to these basis and B is the matrix for L^{-1}, then the matrices A and B satisfy

$$AB = \mathrm{Id}_n = BA$$

so that $B = A^{-1}$.

Proof

We know by the previous proposition that the matrix for $L \circ L^{-1}$ is AB and the matrix for $L^{-1} \circ L$ is BA.

But because they are inverses, we know

$$L \circ L^{-1} = \mathrm{Id}_V \qquad L^{-1} \circ L = \mathrm{Id}_U$$

We have seen that the matrix for Id_V and Id_U must both be Id_n.

This gives the result.

Natural Questions About Linear Transformations/Matricies

- Suppose $L : V \to W$ is a linear transformation, given the matrix for L in one basis, find the matrix for L in another basis.
  The details are on the assignment, though I have skimmed over the idea here.
- Suppose $L : V \to W$ is a linear transformation, find a basis for V and/or W so that the matrix associated to L in this basis is *nice in some way*.
  This question is open ended, we will spend a lot of time on this later.